Meta Distribution Analysis of the Downlink SIR for the Typical Cell in a Poisson Cellular Network

Praful D. Mankar, Harpreet S. Dhillon, and Martin Haenggi

***Abstract*—The stochastic geometry-based downlink analysis of a cellular network modeled as a Poisson point process (PPP) has traditionally focused on the *typical user* placed at the origin, which does not lie in the *typical cell*. In order to characterize the performance of the typical cell, one needs to explicitly consider the point process of users scheduled in a given resource block (RB), which is dependent on the base station (BS) point process. Therefore, we model the locations of the scheduled users using the so-called *Type I user process*, which places one user uniformly at random in each cell. However, this dependency in the locations of the BSs and users complicates the characterization of the point process of interferers as seen by the typical user of the Type I process. In order to overcome this challenge, we present a general approach to determine the pair correlation function (`pcf`) of stationary point processes with respect to a reference point. This approach is used to approximate the `pcf` the point process of interferers with respect to the typical user of the Type I process. With the `pcf` in hand, we provide the tightest known approximation of the point process of interfering BSs as seen by the typical user of Type I process, which is used to derive remarkably tight expressions for the moments of the downlink signal-to-interference-ratio (`SIR`) meta distribution for the typical cell.**

***Index Terms*—Cellular networks, Poisson point process, meta distribution, success probability, pair correlation function.**

I. Introduction

Owing to its unparalleled tractability, the homogeneous PPP has become a popular model to analyze cellular networks [1], [2], wherein the BS locations are modeled using a PPP independently of the user locations. Given the stationarity of the BS point process, the typical user can be simply placed at the origin for the downlink analysis. This places the typical user in the *Crofton cell* which is bigger on an average than the typical cell [3]. Naturally, the typical user in this case does not represent the performance of the typical cell. In order to characterize the performance of the typical cell, one needs to explicitly consider the point process of users scheduled in a given RB, which is not independent of the point process of BSs. It is therefore useful to start directly with the point process of scheduled users for which one can consider the Type I user point process, defined in [3], wherein a single user is distributed uniformly at random in each cell independently of all other cells. In this setting, the typical user belongs to the typical cell. This means that the link performance observed by the typical user is the same as the link performance observed by the typical cell.

Two key intermediate steps in the analysis of cellular networks are the characterization of the service link distance and the point process of interferers. The placement of the typical user independently of the BS locations allows to characterize the link distance distribution using the contact distribution of PPP and the point process of interferers as a PPP beyond the serving link distance, which makes the analysis tractable (see [1] for more details). However, the analysis is usually significantly more challenging when the user locations are dependent on the BS point process, which is the case in the Type I process considered in this paper. While this dependence has been recently captured in the cellular network models using Poisson cluster processes [4]–[6], the mathematical structure of the Type I process does not allow a similar analytical treatment. As a result, the exact analysis of the Type I process is currently a key open problem in this area. Not surprisingly, the prior art investigating the Type I process mostly resort to sensible approximations and empirical results.

In [7], we derived an exact multi-integral expression and a closed-form approximation of the service link distance distribution for the Type I process. In addition, the approximate `pcf` of the point process of interfering users as seen by the typical BS is derived in [8]. Building on this, the authors of [9] presented the *meta distribution* analysis for the uplink of the Type I process using the empirically obtained service link distance distribution and an approximation of the point process of interfering users as a non-homogeneous PPP whose density function is governed by the `pcf` derived in [8]. Therein, the meta distribution analysis for the downlink of the Type I process was also presented using the approximation of the point process of interfering BSs as a homogeneous PPP beyond the service link distance from the typical user. Similar approximations were then used in [10] for the meta distribution analysis in non-orthogonal medium access-enabled cellular networks. However, as will be evident shortly, the homogeneous PPP approximation of the point process of interfering BSs underestimates the interference power at the typical user which results in the overestimation of the moments of the meta distribution in the downlink.

Contributions: The key technical contribution of this paper is a new method to determine the `pcf` of general stationary point processes from a reference point that may not necessarily be independent of the point process. The proposed method solely relies on the knowledge of the distributions of distances of a few neighboring points of the point process from the reference point and is hence applicable to point processes for which these distributions may be known (or are easy to approximate). We then apply this method to derive the `pcf` of

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the point process of interfering BSs as seen from the typical point of the Type I user process. Since the distributions of the distances from this typical point to a few neighboring BSs is not known, we approximate these distributions using the tabulated parameters obtained through one-time simulations. Such approaches are not uncommon in the stochastic geometry literature. For instance, note that the well-known area distribution of the typical Poisson Voronoi (PV) cell is also based on a simulation-driven approach [11]. The derived pcf shows that the interfering BSs exhibit a *clustering effect* at distances slightly larger than the service link distance which cannot be captured by the homogeneous PPP approximation assumed in [9], [10]. Further, we use the pcf to approximate the point process of interferers using a non-homogeneous PPP which we then use to derive the moments of the meta distribution of the downlink SIR. We also provide an accurate beta approximation of the meta distribution. Numerical results demonstrate that the moments and beta approximation derived in this paper are more accurate compared to those derived in [9] using the homogeneous PPP approximation.

II. System Model

We consider a cellular network in which the BS locations are modeled using a homogeneous PPP Φ of density λ and the users follow the Type I user process [8] such that each BS is associated with a single user which is placed uniformly at random in its cell. The cell associated with the BS at $\mathbf{x} \in \Phi$ is the PV cell corresponding to the nucleus $\mathbf{x}$ and is given by

$$V_{\mathbf{x}} = \{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{y}-\mathbf{x}\| \leq \|\mathbf{x}'-\mathbf{y}\|,\ \forall \mathbf{x}' \in \Phi\}. \quad (1)$$

Therefore, the user point process Ψ becomes

$$\Psi \triangleq \{U(V_{\mathbf{x}}) : \mathbf{x} \in \Phi\} \quad (2)$$

where $U(A)$ is a point chosen uniformly at random from set A. This system can be thought of as an orthogonal frequency division multiple access system wherein Ψ denotes the point process of the users active in a given RB. By virtue of Slivnyak's theorem, we know that conditioning on a point at $\mathbf{x} \in \Phi$ is the same as adding $\mathbf{x}$ to Φ. Therefore, without loss of generality, we assume that the nucleus of the *typical cell* of the point process $\Phi \cup \{o\}$ is located at the origin o. By the above construction, the *typical user* of Ψ falls in the typical cell and its location is $\mathbf{y} = \Psi \cap V_o$ (or, equivalently $\mathbf{y} \sim U(V_o)$). Hence, Φ is the point process of interfering BSs for the typical user at $\mathbf{y} \in V_o$. Let $R_o = \|\mathbf{y}\|$ be the *typical link distance*, i.e., the distance between the typical user and its serving BS. Let $D_{\mathbf{x}_i} = \|\mathbf{x}_i - \mathbf{y}\|$ be the distance from the typical user at $\mathbf{y} \in V_o$ to the interfering BS at $\mathbf{x}_i \in \Phi$. Fig. 1 illustrates the network model wherein the BS of the typical cell, containing the typical user, is at the origin.

We assume that each BS transmits at fixed power P and the path loss follows the standard power law path loss model with exponent $\alpha > 2$. Assuming independent Rayleigh fading, we model the fading gains $h_{\mathbf{x}_i}$ of the links between the typical user and the BSs as i.i.d. exponential random variables with unit mean, i.e., $h_{\mathbf{x}_i} \sim \exp(1)$. The SIR at the typical user at $\mathbf{y} \in V_o$ is given by

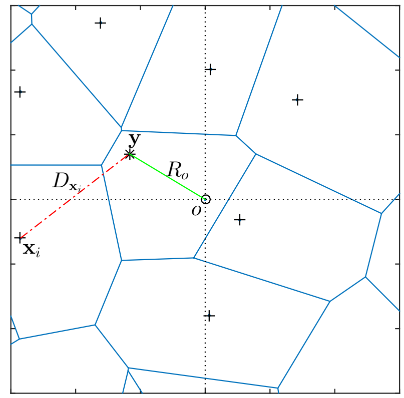

Figure 1. Illustration of the typical user at $\mathbf{y} \in V_o$. The star, circle, and plus marks denote the locations of typical user, serving BS and interfering BSs, respectively.

$$\mathtt{SIR}_{\mathbf{y}} = \frac{h_o R_o^{-\alpha}}{\sum\limits_{\mathbf{x} \in \Phi} h_{\mathbf{x}} D_{\mathbf{x}}^{-\alpha}}. \quad (3)$$

Now, we define the pcf and the meta distribution of the downlink SIR [12], which are the key metrics of interest for this paper.

Definition 1. *The* pcf *of a point process $\mathcal{P}$ w.r.t. point $\mathbf{y}$ is*

$$g(r) = \frac{1}{2\pi\lambda r}\frac{\mathrm{d}}{\mathrm{d}r}K(r), \quad (4)$$

where $K(r) = \mathbb{E}[\mathcal{P}(\mathcal{B}_{\mathbf{y}}(r))]$ is Ripley's K function and $\mathcal{B}_{\mathbf{y}}(r)$ is the disk of radius r centered at $\mathbf{y}$.

Definition 2. *The meta distribution of the downlink* SIR *is defined as*

$$\bar{F}(\beta, x) \triangleq F_{P_s}(\beta, x) = \mathbb{P}[P_s(\beta) > x], \quad (5)$$

where $x \in [0,1]$ and $P_s(\beta) = \mathbb{P}[\mathtt{SIR}_{\mathbf{y}} > \beta \mid \Phi, \mathbf{y}]$ is the conditional success probability averaged over the fading gains for given user location $\mathbf{y}$ and BS point process Φ.

In the next section, we provide an accurate approximation of the point process of interferers for the Type I process which will then be used for the meta distribution analysis.

III. Point Process of Interfering BSs

The key steps in the SIR analysis of the typical user involve the joint characterization of the service link distance distribution and the point process of interfering BSs. However, the exact characterization of the point process as seen from the typical point of the Type I user process is an open problem. In such cases, it is very useful to determine the pcf of the point process using which one can accurately approximate the original point process with more tractable point processes, such as the PPP. With this in mind, we first develop a new method to evaluate the pcf of a motion-invariant point process as seen from a reference point that may not necessarily be independent of the point process.

Lemma 1. *Let $\mathcal{P}$ be a motion-invariant point process of density λ and $\mathbf{y}$ a reference point that is a function of $\mathcal{P}$. Let R_{n-1} denote the distance from $\mathbf{y}$ to the n-th closest point in $\mathcal{P}$. Then the* pcf *of $\mathcal{P}$ w.r.t. $\mathbf{y}$ is*

$$g(r) = \frac{1}{2\pi\lambda r}\sum_{n=0}^{\infty} f_{R_n}(r) \qquad \textit{for } r > 0, \tag{6}$$

and the pcf *of* $\mathcal{P} \setminus \{\mathbf{x}_o\}$ *w.r.t.* $\mathbf{y}$ *such that* $\mathbf{x}_o = \arg\min_{\mathbf{x}\in\mathcal{P}} \|\mathbf{x}-\mathbf{y}\|$ *is*

$$g(r \mid R_0) = \frac{1}{2\pi\lambda r}\sum_{n=1}^{\infty} f_{R_n}(r \mid R_0) \qquad \textit{for } r > R_0, \tag{7}$$

where $R_0 = \|\mathbf{x}_o - \mathbf{y}\|$, $f_{R_n}(r)$ *and* $f_{R_n}(r \mid R_0)$ *are respectively the probability density function* (pdf) *of* R_n *and conditional* pdf *of* R_n *given* R_0.

Proof. Since $\mathcal{P}(\mathcal{B}_{\mathbf{y}}(r)) = \sum_{n=0}^{\infty} \mathbb{1}[R_n < r]$, Ripley's K function is given by $K(r) = \sum_{n=0}^{\infty} F_{R_n}(r)$. Thus, (6) directly follows from the definition of the pcf given in (4). The conditional pcf in (7) follows using similar arguments. □

The pcf obtained using (7) can be used to approximate the point process of interfering BSs solely using the knowledge of the distributions of R_n for the given underlying point processes of users and BSs. It is reasonable to assume that the BSs that are far from the typical user do not exhibit any coupling with the typical user location (if the point process is mixing, this is guaranteed). Thus, the point process of interferers can be approximated using a PPP beyond a certain distance from the typical user which implies that the pcf is unity at higher values of r. Therefore, the pcf can be accurately evaluated using the distributions of the distances of the N closest BSs. In the following, we apply this method to derive the best-known approximation of the pcf of the point process of interferers as seen by the typical user at $\mathbf{y} \in V_o$ of the Type I process. We will then use it for the meta distribution analysis of the Type I process in the next section. Let R_{n-1} denote the distance from the typical user at $\mathbf{y} \in V_o$ to its n-th closest BS. The approximate cumulative distribution function (CDF) of R_o, i.e., the service link distance, is [7]

$$F_{R_o}(r) = 1 - \exp\left(-\pi\rho_o\lambda r^2\right), \text{ for } r \geq 0, \tag{8}$$

where $\rho_o = \frac{9}{7}$ is the correction factor (CF), which corresponds to the ratio of the mean volumes of the Crofton and typical cells. Similar to the fact that the distribution of R_o closely follows the distribution of the closest point in the PPP of density $\rho_o\lambda$, we approximate the CDF of R_n with the CDF of $(n+1)$-th closest point in a PPP [13] by including the CF ρ_n as follows

$$F_{R_n}(r) \approx \frac{1}{\Gamma(n+1)}\gamma(n+1, \pi\lambda\rho_n r^2), \tag{9}$$

where $\Gamma(\cdot)$ and $\gamma(\cdot,\cdot)$ are the gamma function and lower incomplete gamma function, respectively. By matching the mean values, we can obtain the CF ρ_n as listed in Table I by solving

$$\rho_n^{\frac{1}{2}} = \frac{\Gamma\left(n+\frac{3}{2}\right)}{(\pi\lambda)^{\frac{1}{2}}\mu_n\Gamma(n+1)}, \tag{10}$$

where μ_n is the mean of R_n. While it is difficult to analytically determine μ_n, it is straightforward to do a one-time simulation to obtain it for a given value of λ (say $\lambda = 1$). We provide the resulting values for $\lambda = 1$ (and the corresponding CFs ρ_n) in Table I and the corresponding Matlab script in [14]. It is important to note that the scale-invariance property of the distributions of R_n (i.e. $F_{R_n}^{(\lambda)}(r) \equiv F_{R_n}^{(1)}(r\sqrt{\lambda})$) allows us to use the tabulated CFs ρ_n for $\lambda = 1$ from Table I for any BS density λ. From Table I and (8), it can be seen that the analytically approximated ($\rho_o = \frac{9}{7} \approx 1.2857$) and empirically obtained ($\rho_o = \frac{5}{4} = 1.25$) values of ρ_o are reasonably close. In order to be consistent with how we obtain ρ_n for $n > 0$, we use the empirically obtained value for ρ_o as well (from Table I) throughout our analysis. Fig. 2 illustrates the accuracy of the approximated CDF of R_n given in (9) for $n \in \{0, \ldots, 14\}$ and $\lambda = 1$. Now, in the following theorem, we approximate the pcf of the point process of interferers with respect to the typical user at $\mathbf{y} \in V_o$ of the Type I process.

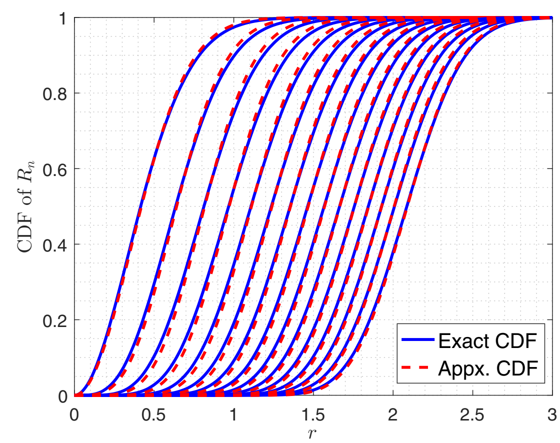

Figure 2. Approximated CDF of R_n for $\lambda = 1$ and $n \in \{0, \ldots, 14\}$.

Theorem 1. *For a given* R_o*, the* pcf *of the point process of interferers as seen by the typical user of the Type I user process can be approximated as*

$$g(r \mid R_o) = \begin{cases} 0 & \textit{for } r \leq R_o, \\ 1 + h(r, R_o) & \textit{for } r > R_o, \end{cases} \tag{11}$$

where $h(r, R_o) =$

$$\sum_{n=1}^{N} \frac{1}{\Gamma(n)} v_r^{2n-2} \exp\left(-v_r^2\right)\left[\tilde{\rho}_n^n \exp(-(\tilde{\rho}_n - 1)v_r^2) - 1\right],$$

and $\tilde{\rho}_n$ *is the solution of*

$$\frac{(\pi\lambda)^{-\frac{1}{2}}}{\Gamma(n)}\int_0^{\infty}\left(\frac{t}{\tilde{\rho}_n} + \pi\lambda R_o^2\right)^{\frac{1}{2}} t^{n-1}\exp(-t)\mathrm{d}t - \tilde{\mu}_n = 0,$$

such that $v_r^2 = \pi\lambda(r^2 - R_o^2)$, $\tilde{\mu}_n = \mathbb{E}[R_n \mid R_o]$ *and* N *is number of closest interfering BSs used to approximate the* pcf.

Proof. Please refer to Appendix A for the proof. □

Fig. 3 demonstrates the accuracy of the pcf derived in Theorem 1 using the approximated distributions of R_n (given in (16)) for $R_o \in \{0.3, 0.6, 0.9\}$ and $N = 14$. It may be noted that the conditional CF $\tilde{\rho}_n$ is obtained using the empirically obtained mean of R_n for given R_o. We have verified the accuracy of the approximation of the conditional distributions of R_n which is also apparent from the accuracy of the pcf depicted in Fig. 3. Besides, through simulation

Table I
MEAN AND CORRECTION FACTOR OF R_n

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
μ_n	0.45	0.66	0.83	0.98	1.12	1.25	1.36	1.47	1.57	1.67	1.76	1.85	1.93	2.01	2.10	2.16	2.24	2.31	2.28
ρ_n	1.25	1.30	1.27	1.24	1.20	1.18	1.15	1.14	1.12	1.11	1.10	1.09	1.08	1.08	1.07	1.06	1.06	1.05	1.04

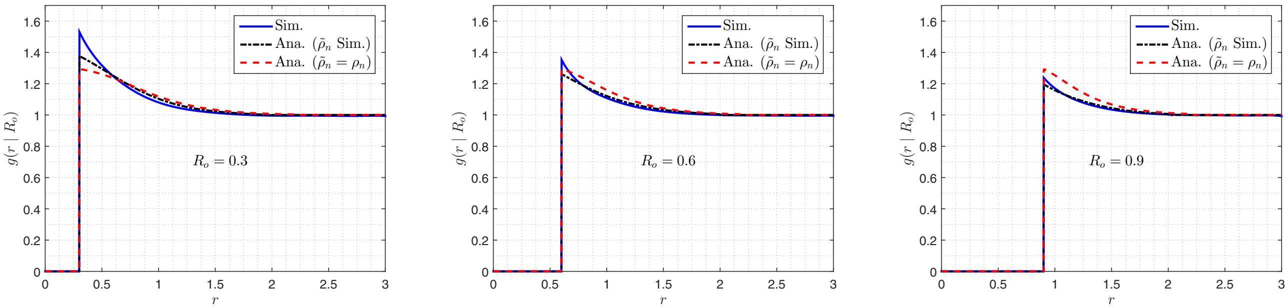

Figure 3. `pcf` of the point process of interferers observed by the typical user for given $R_o = \{0.3, 0.6, 0.9\}$.

results, we observe that the sequence of conditional CFs $\tilde{\rho}_n$ closely follows the sequence of the marginal CFs ρ_n for the values of R_o in the range of interest. Fig. 3 also depicts that the approximated `pcf` closely follows the simulation result for $\tilde{\rho}_n = \rho_n$. Thus, we will henceforth use $\tilde{\rho}_n = \rho_n$ as this will allow us to analytically analyze the performance of the typical user just by using the CFs ρ_n given in Table I. In other words, we will not have to empirically obtain $\tilde{\mu}_n = \mathbb{E}[R_n \mid R_o]$.

In the following, we approximate the point process of interferers using the `pcf` derived in Theorem 1 which will be used for the meta distribution analysis in the next section.

Approximation 1. *We approximate the point process of interferers as seen by the typical user at* $\mathbf{y} \in V_o$ *using the non-homogeneous PPP with the density function*

$$\lambda_I(\mathbf{z}) = \begin{cases} \lambda g(\|\mathbf{z}\| \mid R_o), & \text{for } \|\mathbf{z}\| \geq R_o, \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

where $\mathbf{z} = \mathbf{z}' - \mathbf{y}$, $\mathbf{z}' \in \mathbb{R}^2$, $R_o = \|\mathbf{y}\|$ *and* $g(r \mid R_o)$ *is given by Theorem 1.*

IV. META DISTRIBUTION OF THE DOWNLINK SIR

Here, we analyze the meta distribution of the downlink SIR using the point process of interferers from Approximation 1. Since it is difficult to derive the meta distribution directly [12], we derive its moments in the following theorem.

Theorem 2. *The* b*-th moment of the downlink meta distribution for the Type I user point process is given by* $M_b =$

$$2\pi\rho_o\lambda \int_0^\infty \exp\left(-\pi\lambda\rho_o v^2 - \pi\lambda \int_0^\infty f(u,v)\mathrm{d}u\right) v\mathrm{d}v, \quad (13)$$

where $f(u,v) = \tilde{g}(u)\left[1 - \left(1 + \left(\frac{v^2}{u+v^2}\right)^{\frac{1}{\delta}}\beta\right)^{-b}\right]$,

$$\tilde{g}(u) = 1 + \sum_{n=1}^{N} \frac{(\pi\lambda u)^{n-1}}{\Gamma(n)}\left[\rho_n^n \exp(-\pi\lambda\rho_n u) - \exp(-\pi\lambda u)\right],$$

$\delta = \frac{2}{\alpha}$ *and* $b \in \mathbb{C}$.

Proof. Please refer to Appendix B for the proof. □

Since the dominant interfering BS, say $\mathbf{x}_d$, contributes most of the interference power, we can explicitly consider the interference from this BS and then approximate the point process of the remaining interfering BSs using the homogeneous PPP beyond the distance $R_1 = \|\mathbf{x}_d - \mathbf{y}\|$ from the typical user at $\mathbf{y} \in V_o$. Using this insight, we present a simpler yet accurate expression for the b-th moment M_b in the following corollary.

Corollary 1. *The approximate* b*-th moment of the downlink meta distribution for the Type I user point process is given by*

$$M_b \approx \rho_o^2 \int_1^\infty \frac{1}{(1+\beta u^{-\frac{1}{\delta}})^b} \frac{1}{(\rho_o u + h(u))^2}\mathrm{d}u, \quad (14)$$

where $h(u) = \int_u^\infty \left(1 - [1+\beta t^{-\frac{1}{\delta}}]^{-b}\right)\mathrm{d}t$.

Proof. Please refer to Appendix C for the proof. □

Fig. 4 depicts the mean and variance of the conditional success probability for $\alpha \in \{3,4\}$. It can be seen that the moments derived in Theorem 2 (using the approximated `pcf`) closely match the simulation results, whereas the moments presented in Corollary 1 are still a better match compared to the moments derived in [9, Theorem 3].

Now, in the following subsection, we provide a tight approximation of the meta distribution using its first two moments obtained in Theorem 2 and Corollary 1.

A. Beta Approximation

The meta distribution can be directly obtained using the moments derived in Section IV and the Gil-Pelaez inversion theorem [15]. However, considering the complexity of the Gil-Pelaez theorem, similar to [9], [12], we provide the beta distribution approximation for the meta distribution. Thus,

$$\bar{F}(\beta, x) \approx 1 - \frac{1}{B(\theta_1,\theta_2)} \int_0^x t^{\theta_1-1}(1-t)^{\theta_2-1}\mathrm{d}t, \quad (15)$$

where $B(\cdot,\cdot)$ is the beta function and the parameters θ_1 and θ_2 are obtained by simultaneously solving

$$M_1 = \frac{\theta_1}{\theta_1+\theta_2} \text{ and } M_2 = \frac{\theta_1^2}{(\theta_1+\theta_2)^2}\left(\frac{\theta_2}{\theta_1(\theta_1+\theta_2+1)} + 1\right).$$

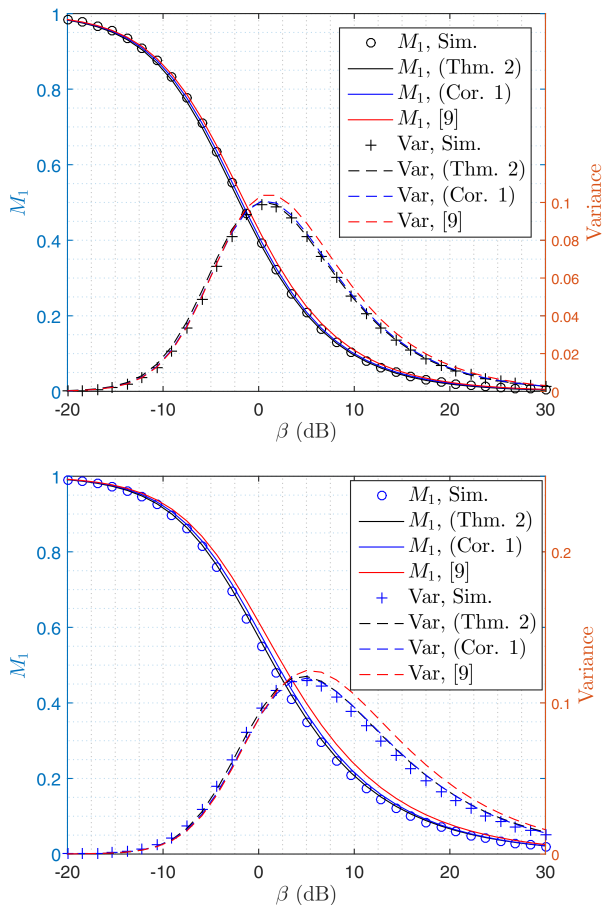

Figure 4. Mean and variance of the meta distribution for $\alpha = 3$ (top) and $\alpha = 4$ (bottom).

Let Beta1 and Beta2 denote the beta approximations evaluated using the moments obtained in Theorem 2 and Corollary 1, respectively. Fig. 5 depicts the beta approximation for $\alpha \in \{3, 4\}$. It can be observed that Beta1 closely matches the meta distribution, whereas Beta2 is not as tight as Beta1, especially for $\alpha = 4$. But Beta2 still is more accurate than the beta approximation provided in [9].

V. Conclusion

In this paper, we characterized the meta distribution of the downlink SIR for the typical cell in a cellular network whose BS locations are modeled using a PPP. The main technical contribution is the accurate derivation of the pcf of the point process of interfering BSs as seen from the typical user of the so-called Type I user point process (in which one user is placed uniformly at random in each cell). Unlike the usual analyses that focus on the typical user placed at the origin (and hence in the Crofton cell), this derivation is not straightforward because of the dependence in the locations of the users and BSs. Our characterization of the pcf demonstrates that the interfering BSs at distances slightly larger than the service link distance exhibit a clustering effect. We then use the pcf to approximate the point process of the interfering BSs using a non-homogeneous PPP which is then used to derive a tight approximation of the moments of the meta distribution of the downlink SIR. In general, the proposed approach of approximating the pcf from the distance distributions to a few neighboring points is quite general and can be applied to point processes for which these distributions are either known or can be easily approximated.

Appendix A
Proof of Theorem 1

The conditional CDF of the distance $\tilde{R}_n$ of the n-th closest point in PPP from an arbitrary point (e.g., the origin o) is $F_{\tilde{R}_n}(r \mid \tilde{R}_o) = \frac{1}{\Gamma(n-1)}\gamma(n-1, \pi\lambda(r^2 - \tilde{R}_o^2))$ for $r \geq \tilde{R}_o$ where $\tilde{R}_o$ is the distance to the closest point. Thus, similar to (9), we approximate the CDF of R_n given R_o as follows

$$F_{R_n}(r \mid R_o) = \frac{\gamma(n, \tilde{\rho}_n v_r^2)}{\Gamma(n)}, \quad \text{for } v_r \geq 0, \tag{16}$$

where $v_r^2 = \pi\lambda(r^2 - R_o^2)$ and $\tilde{\rho}_n$ is the conditional CF given R_o. By matching the means, we obtain $\tilde{\rho}_n$ as the solution of

$$\frac{(\pi\lambda)^{-\frac{1}{2}}}{\Gamma(n)} \int_0^\infty \left(\frac{t}{\tilde{\rho}_n} + \pi\lambda R_o^2\right)^{\frac{1}{2}} t^{n-1}\exp(-t)\mathrm{d}t = \tilde{\mu}_n,$$

where $\tilde{\mu}_n = \mathbb{E}[R_n \mid R_o]$. The pdf of R_n given R_o becomes

$$f_{R_n}(r \mid R_o) = \frac{2\pi\lambda r}{\Gamma(n)}\tilde{\rho}_n^n v_r^{2n-2}\exp(-\tilde{\rho}_n v_r^2), \text{ for } v_r > 0. \tag{17}$$

Now, using Lemma 1 and (17), we can write the pcf of the point process of interferers of the typical user at $\mathbf{y} \in V_o$ as

$$g(r \mid R_o) = \begin{cases} \sum_{n=1}^\infty \frac{1}{\Gamma(n)}\tilde{\rho}_n^n v_r^{2n-2}\exp(-\tilde{\rho}_n v_r^2), & \text{for } r > R_o \\ 0, & \text{for } r \leq R_o. \end{cases}$$

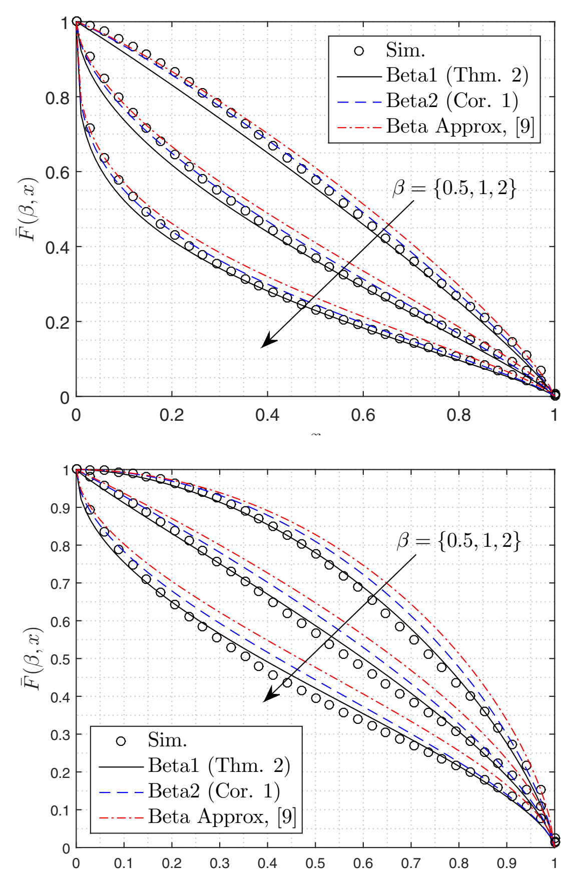

Figure 5. Beta approximation for $\alpha = 3$ (top) and $\alpha = 4$ (bottom). Beta1 and Beta2 denote the beta approximations evaluated using the moments obtained using Theorem 2 and Corollary 1, respectively.

From Table I, it is reasonable to deduce that the conditional CF $\tilde{\rho}_n$ approaches unity for large values of n. Therefore, we set $\tilde{\rho}_n = 1$ for all $n > N$. Hence, for $r > R_o$, we have

$$\begin{aligned} g(r \mid R_o) &= \sum_{n=1}^{N} \frac{1}{\Gamma(n)} \tilde{\rho}_n^n v_r^{2n-2} \exp(-\tilde{\rho}_n v_r^2) \\ &\quad + \sum_{n=N+1}^{\infty} \frac{1}{\Gamma(n)} v_r^{2n-2} \exp(-v_r^2) \\ &= \sum_{n=1}^{N} \frac{1}{\Gamma(n)} v_r^{2n-2} \exp\left(-v_r^2\right) \left[\tilde{\rho}_n^n \exp(-(\tilde{\rho}_n - 1)v_r^2) - 1\right] \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n)} v_r^{2n-2} \exp(-v_r^2) \\ &= 1 + \sum_{n=1}^{N} \frac{1}{\Gamma(n)} v_r^{2n-2} \exp\left(-v_r^2\right) \left[\tilde{\rho}_n^n \exp(-(\tilde{\rho}_n - 1)v_r^2) - 1\right]. \end{aligned}$$

This completes the proof.

Appendix B
Proof of Theorem 2

The downlink success probability of the user at $\mathbf{y} \in V_o$, conditioned on Φ and $\|\mathbf{y}\| = R_o$, is given by

$$P_s(\beta) = \mathbb{P}\left[\frac{h_o R_o^{-\alpha}}{\sum_{\mathbf{x}\in\Phi} h_{\mathbf{x}} D_{\mathbf{x}}^{-\alpha}} > \beta \mid \Phi\right] \stackrel{(a)}{=} \prod_{\mathbf{x}\in\Phi} \frac{1}{1 + \beta R_o^{\alpha} D_{\mathbf{x}}^{-\alpha}},$$

where step (a) follows directly as $h_{\mathbf{x}}$ are i.i.d. exponential random variables. Thus, the b-th moment of the downlink meta distribution becomes

$$\begin{aligned} M_b &= \mathbb{E}_{R_o}\mathbb{E}_{\Phi}\left[\prod_{\mathbf{x}\in\Phi} \frac{1}{\left(1 + \beta R_o^{\alpha} D_{\mathbf{x}}^{-\alpha}\right)^b} \mid R_o\right] \\ &\stackrel{(a)}{=} \mathbb{E}_{R_o} \exp\left(-\int_{\mathbb{R}^2 \setminus \mathcal{B}_{\mathbf{y}}(R_o)} \left(1 - \left(1 + \beta R_o^{\alpha} D_{\mathbf{x}}^{-\alpha}\right)^{-b}\right) \Lambda(\mathrm{d}\mathbf{x})\right) \\ &\stackrel{(b)}{=} \mathbb{E}_{R_o} \exp\left(-2\pi\lambda \int_{R_o}^{\infty} \frac{g(r \mid R_o)}{\left(1 - (1 + \beta R_o^{\alpha} r^{-\alpha})^{-b}\right)^{-1}} r \mathrm{d}r\right), \end{aligned}$$

where step (a) follows from the fact that all interfering BSs to the typical user at $\mathbf{y} \in V_o$ are outside the disk $\mathcal{B}_{\mathbf{y}}(R_o)$ and the application of the probability generating functional (PGFL). Step (b) follows from the non-homogeneous PPP approximation of the point process of the interfering BSs (given in Approximation 1) and the Cartesian-to-polar coordinate conversion after the substitution of $\mathbf{x} - \mathbf{y} = \mathbf{z}$. Finally, using the distribution of R_o given in (8) and some simplifications, we obtain the b-th moment M_b as given in (13). This completes the proof.

Appendix C
Proof of Corollary 1

Let $\mathbf{x}_d \in \Phi$ be the dominant interfering BS to the typical user at $\mathbf{y} \in V_o$ and $R_1 = \|\mathbf{x}_d\|$. The success probability for user at $\mathbf{y} \in V_o$, conditioned on Φ and $\|\mathbf{y}\| = R_o$, is given by

$$P_s(\beta) = \frac{1}{1 + \beta R_o^{\alpha} R_1^{-\alpha}} \prod_{\mathbf{x}\in\tilde{\Phi}} \frac{1}{1 + \beta R_o^{\alpha} D_{\mathbf{x}}^{-\alpha}},$$

where $\tilde{\Phi} = \Phi \setminus \{\mathbf{x}_d\}$. Thus, the b-th moment of the meta distribution becomes

$$\begin{aligned} M_b &= \mathbb{E}_{R_o,R_1}\mathbb{E}_{\tilde{\Phi}}\left[\prod_{\mathbf{x}\in\tilde{\Phi}} \frac{\left(1 + \beta R_o^{\alpha} R_1{}^{-\alpha}\right)^{-b}}{\left(1 + \beta R_o{}^{\alpha} D_{\mathbf{x}}{}^{-\alpha}\right)^b} \mid R_o, R_1\right] \\ &\stackrel{(a)}{\approx} \mathbb{E}_{R_o,R_1}\left[\frac{1}{\left(1 + \beta R_o^{\alpha} R_1{}^{-\alpha}\right)^b} \times \right. \\ &\quad \left. \exp\left(-2\pi\lambda \int_{R_1}^{\infty} \left(1 - \left(1 + \beta R_o^{\alpha} r^{-\alpha}\right)^{-b}\right) r \mathrm{d}r\right)\right], \quad (18) \end{aligned}$$

where step (a) follows using the homogeneous PPP approximation of $\tilde{\Phi}$ beyond R_1 from the typical user. Now, using (8) and (16) along with $\rho_1 = \rho_o$, we obtain the joint distribution of R_o and R_1 as

$$f_{R_o,R_1}(r_o, r_1) = (2\pi\rho_o^2\lambda) r_o r_1 \exp(-\pi\lambda\rho_o r_1^2). \quad (19)$$

Finally, using (19) and (18) along with some mathematical simplifications, we obtain (14). This completes the proof.

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