# Asymptotic Deployment Gain: A New Approach to Characterize Coverage Probability

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***Abstract*—In cellular network models, the base stations are usually assumed to form a lattice or a Poisson point process (PPP). In reality, however, they are deployed neither fully regularly nor completely randomly. Accordingly, in this paper, we consider the very general class of motion-invariant models and analyze the behavior of the coverage probability (the probability that the signal-to-interference-plus-noise-ratio (SINR) exceeds a threshold) as the threshold goes to zero. We show that, surprisingly, the slope of the coverage probability as a function of the threshold is the same for essentially all motion-invariant point processes. The slope merely depends on the fading statistics.**

**Using this result, we introduce the notion of the *asymptotic deployment gain* (ADG), which characterizes the horizontal gap between the coverage probability of the PPP and another point process in the high-reliability regime (where the coverage probability is near $1$).**

**To demonstrate the usefulness of the ADG for the characterization of coverage probabilities, we investigate the coverage properties and the ADGs for different point processes and fading statistics by simulations.**

## I. Introduction

The topology of the base stations (BSs) in cellular networks depends on many natural or man-made factors, such as the landscape, topography, bodies of water, population densities, and traffic demands. Despite, base stations were usually modeled deterministically as triangular or square lattices until recently, when it was shown in [1], [2] that a completely irregular point process, the Poisson point process (PPP), may be used without any loss in accuracy but significant gain in analytical tractability. Real deployments fall somewhere in between these two extremes of full regularity (the triangular lattice) and complete randomness (the PPP), as investigated in [3] using base station data from the UK. They exhibit some degree of repulsion between the BSs, since the operators do not place them closely together. Such repulsion can be modeled using point processes with a hard minimum distance between BSs (hard-core processes) or a high likelihood that BSs are a certain distance apart (soft-core processes). At a larger scale, at the level of a state of country, BSs may appear clustered due to the high density of BSs in cities and low density in rural regions. The analysis of such non-Poisson processes is significantly more difficult than the analysis of the PPP, since dependencies exist between the locations of base stations.

The coverage probability, defined as the probability that the signal-to-interference-plus-noise ratio (SINR) exceeds a certain threshold $\theta$, denoted as $P_{\mathrm{c}}(\theta)$ is one of the main performance metrics in cellular networks. It depends on many factors, such as the fading, the path loss and the distribution of the BSs. In [1], the authors derive the expressions for the coverage probability and the mean achievable rate for networks whose BSs form a homogeneous PPP. For general models, it is much harder to compute the coverage probability due to the dependence between BS locations mentioned above, and we are not aware of any tractable analytical methods that are applicable in general.

In this paper, we provide an indirect approach to the coverage analysis in which the coverage probability of an arbitrary motion-invariant (isotropic and stationary[1]) point process [4, Ch. 2] by comparing its coverage probability to the coverage probability of the PPP. To validate this approach, we establish that *the outage probability $1 - P_{\mathrm{c}}(\theta)$ of essentially all motion-invariant (m.i.) point processes, expressed in dB, as a function of the SINR threshold $\theta$, also in dB, has the same slope as* $\theta \to 0$. The slope depends on the fading statistics. This result shows that asymptotically the coverage curves $P_{\mathrm{c}}(\theta)$ of all m.i. models are just (horizontally) shifted versions of each other in a log-log plot, and the shift can be quantified in terms of the horizontal difference $\hat{G}$ along the $\theta$ (in dB) axis. Since the coverage probability of the PPP is known analytically, the PPP is a sensible choice as a reference model, which then allows to express the coverage of an arbitrary m.i. model as a gain relative to the PPP. This gain is denoted as the *asymptotic deployment gain* (ADG).

We introduced the concept of the *deployment gain* (DG) in our previous work [3]. It measures how close a point process or a point set is to the PPP at a given target coverage probability. Here we extend the DG to include noise and then, to obtain a quantity that does not depend on a target coverage probability, formally define its asymptotic counterpart—the ADG.

The paper makes the following contributions:

- We introduce the asymptotic deployment gain.
- We formally prove its existence for a large class of m.i. point processes.
- We show how the asymptotic slope of the outage probability depends on the fading statistics.

[1] Stationarity implies that the coverage probability does not depend on the location of a mobile user.

- We demonstrate through simulations how the ADG is employed to quantify the coverage probability of several non-Poisson models, even if the SINR threshold $\theta$ is not small.

## II. System Model and Asymptotic Deployment Gain

### A. System Model

We consider a cellular network that consists of BSs and mobile users. The BSs are modeled as a general m.i. point process $\Phi$ of intensity $\lambda$ on the plane. We assume that $\Phi$ is mixing [4, Def. 2.31], which implies that $\rho^{(2)}(x) \to \lambda^2$ as $\|x\| \to \infty$, where $\rho^{(2)}(x)$ is the second-order density. Intuitively, $\rho^{(2)}(x)$ is the probability that there are two points at distance $\|x\|$. Rigorously, the second-order density is the density pertaining to the second factorial moment measure [4, Def. 6.4], which is given by

$$\alpha^{(2)}(A\times B) = \mathbb{E}\Big(\sum_{x,y\in\Phi}^{\neq} \mathbf{1}_A(x)\mathbf{1}_B(y)\Big) = \int_{A\times B} \rho^{(2)}(y-x)dxdy,$$

where $A, B$ are two compact subsets of $\mathbb{R}^2$ and the $\neq$ symbol indicates that the sum is taken only over distinct point pairs. Since the point processes considered are isotropic, $\rho^{(2)}(y-x)$ only depends on $\|y-x\|$.

We assume all BSs are always transmitting and the transmit power is fixed to 1. Each mobile user receives signals from its nearest BS, and all other BSs act as interferers (the frequency reuse factor is 1). Every signal is assumed to experience path loss, fading and additive thermal noise. The shadowing effect is neglected. The path loss model we consider is $\ell(x) = (1 + \|x\|^\alpha)^{-1}$, where $\alpha > 2$. We assume that the fading is independent and identically distributed (i.i.d.) for signals from all BSs. We mainly focus on Nakagami-$m$ fading, which includes Rayleigh fading as a special case. The thermal noise power is assumed to be additive and constant with $W$. We define the mean SNR as the received SNR at a distance of $r = 1$ and $\mathrm{SNR} = 1/(2W)$.

Under the above assumptions, to formulate the SINR and the coverage probability, we first define the nearest-point operator $\mathrm{NP}_\varphi$ for a countable point pattern $\varphi \subset \mathbb{R}^2$ as

$$\mathrm{NP}_\varphi(x) \triangleq \arg\min_{y\in\varphi}\{\|y-x\|\}, x \in \mathbb{R}^2. \tag{1}$$

If the nearest point is not unique, the operator picks one of the nearest points uniformly at random. The SINR at location $z \in \mathbb{R}^2$ has the form

$$\mathrm{SINR}_z = \frac{h_u \ell(u-z)}{W + \sum_{x\in\Phi\setminus\{u\}} h_x \ell(x-z)}, \tag{2}$$

where $u = \mathrm{NP}_\Phi(z)$ and $h_x$ denotes the i.i.d. fading variable for $x \in \Phi$ with CDF $F_h$ and PDF $f_h$. For a m.i. point process, the coverage probability $\mathbb{P}(\mathrm{SINR}_z > \theta)$ does not depend on $z$, and we define

$$P_\mathrm{c}(\theta) = \mathbb{P}(\mathrm{SINR} > \theta). \tag{3}$$

Hence, without any loss of generality, we focus on the coverage probability at the origin $o$. Since each user communicates with its nearest BS, the interference at $o$ only comes from the BSs outside the open disk $b(o, \|u\|) \triangleq \{x \in \mathbb{R}^2 : \|x\| < \|u\|\}$, where $u = \mathrm{NP}_\Phi(o)$. The total interference, denoted by $I(\Phi)$, is

$$I(\Phi) = \sum_{x\in\Phi\setminus\mathrm{NP}_\Phi(o)} h_x \ell(x). \tag{4}$$

### B. Asymptotic Deployment Gain

In [3], we introduced the deployment gain (DG). Here we redefine the DG, since the thermal noise is not included in [3].

*Definition 1 (Deployment gain):* The deployment gain, denoted by $G(p_\mathrm{t})$, is the SINR ratio between the coverage curves of the given point process (or point set) and the PPP at a given target outage probability $p_\mathrm{t}$.

$$G(p_\mathrm{t}) = \frac{P_\mathrm{c}^{-1}(p_\mathrm{t})}{(P_\mathrm{c}^{\mathrm{PPP}})^{-1}(p_\mathrm{t})} \tag{5}$$

where $P_\mathrm{c}^{\mathrm{PPP}}(\theta)$ and $P_\mathrm{c}(\theta)$ are, respectively, the coverage probabilities of the PPP and the given point process $\Phi$.

This definition is analogous to the notion of the coding gain commonly used in coding theory [5, Ch. 1].

Fig. 1 shows the coverage probability of the PPP, the Matérn cluster process (MCP) [4, Ch. 3] and the triangular lattice. The intensities of all the three point processes are the same. We observe that for $p_\mathrm{t} > 0.6$, the DG is approximately constant, e.g. the DG of the MCP is about $-3$ dB. In Fig. 1, the coverage curves of the PPP that are shifted by $G(0.6)$ of the MCP and the triangular lattice are also drawn. We see that the shifted curves overlap quite exactly with the curves of the MCP and the triangular lattice, respectively, for all $p_\mathrm{t} > 0.6$. It is thus sensible to study the DG as $p_\mathrm{t} \to 1$ and find out whether the DG approaches a constant. To do so, analogous to the notion of the asymptotic coding gain, we define the asymptotic deployment gain (ADG).

*Definition 2 (Asymptotic deployment gain):* The ADG, denoted by $\hat{G}$, is the deployment gain $G(p_\mathrm{t})$ when $\theta \to 0$, or equivalently, when $p_\mathrm{t} \to 1$.

$$\hat{G} = \lim_{p_\mathrm{t}\to 1} G(p_\mathrm{t}). \tag{6}$$

Similar to the DG, the ADG also measures the coverage probability but characterizes the difference between the coverage of the PPP and a given point process as the coverage probability approaches 1 instead of for a target coverage probability, and by observation from Fig. 1, the ADG closely approximates the DG for all practical values of the coverage probability. Hence, given the ADG of a point process, we can evaluate its coverage probability by shifting the corresponding PPP results.

## III. Asymptotic Property of the Coverage

### A. General Case and Main Result

In this subsection, we derive an important asymptotic property of the coverage probability in Theorem 1, given some

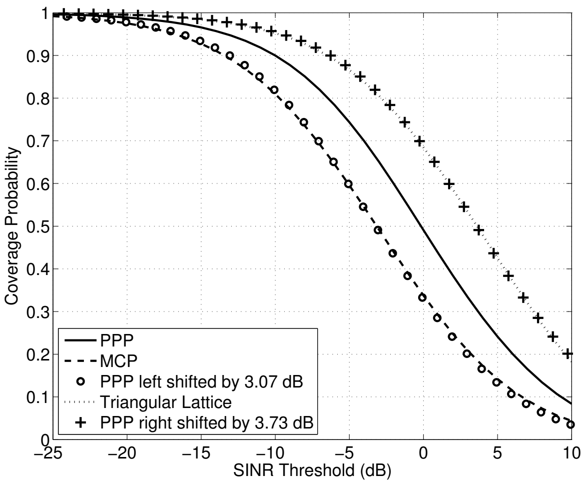


Fig. 1. The coverage probability of the PPP with intensity $\lambda = 0.1$, the MCP with $\lambda_{\mathrm{p}} = 0.01$, $\bar{c} = 10$ and $r_{\mathrm{c}} = 5$ and the triangular lattice with density $\lambda = 0.1$ (see Section III-B for an explanation of these parameters) for Rayleigh fading, path loss exponent $\alpha = 4$ and noise $W = 0$, which are simulated on a $100 \times 100$ square. The lines are the coverage curves of the three point processes, while the markers indicate the coverage curves of the PPP shifted by the deployment gains of the MCP and the triangular lattice at $p_{\mathrm{t}} = 0.6$.

general assumptions about the point process and the CDF of the fading variable.

First we give several notations, based on which we introduce the precise class of point processes we focus on. We define $\xi \triangleq \|\mathrm{NP}_\Phi(o)\|$, and define the supremum of $\xi$ as

$$\xi_{\max} \triangleq \sup_{x \in \mathbb{R}^2} \min_{y \in \Phi} \{\|x - y\|\}. \tag{7}$$

Due to the ergodicity [4, Ch. 2] of the point process (which follows from the mixing property), $\xi_{\max}$ does not depend on the realization of $\Phi$. $\xi_{\max} = \infty$ in many mixing point processes.

We define $\Phi_o^\zeta \triangleq (\Phi \mid \mathrm{NP}_\Phi(o) = \zeta)$. Note that for this point process, $\zeta \in \Phi_o^\zeta$ and $\Phi_o^\zeta(b(o, \|\zeta\|)) = 0$. We compare the interference in $\Phi_o^\zeta$ with the interference from a point process where the desired BS $\zeta$ is not necessarily the closest one. To this end, we define $\Phi^\zeta \triangleq (\Phi \mid \zeta \in \Phi)$ and consider its interference except for a disk of radius $\|\zeta\|/2$ around the origin:

$$\hat{I}(\Phi^\zeta) = \sum_{x \in \Phi^\zeta \bigcap b(o, \|\zeta\|/2)^c \setminus \{\zeta\}} h_x \ell(x), \tag{8}$$

where $b(o, \|\zeta\|/2)^c = \mathbb{R}^2 \setminus b(o, \|\zeta\|/2)$.

To better understand the above notations, we give an illustration of them in Fig. 2. Both $\Phi_o^\zeta$ and $\Phi^\zeta$ have a point at $\zeta$ and $\|\zeta\| = y$. All points of $\Phi_o^\zeta$ are located in the striped region (outside $b(o, y)$) and $I(\Phi_o^\zeta)$ is the interference from all these points except $\zeta$. While, $\Phi^\zeta$ may have points throughout the whole plane, but $\hat{I}(\Phi^\zeta)$ is the interference only from the points of $\Phi^\zeta$ in the shaded region (outside $b(o, y/2)$) except $\zeta$.

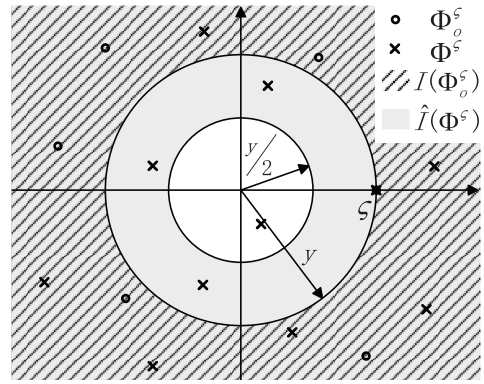


Fig. 2. An illustration of $\Phi_o^\zeta$, $\Phi^\zeta$, $I(\Phi_o^\zeta)$ and $\hat{I}(\Phi^\zeta)$, where $\|\zeta\| = y$.

Using the above notations, we define a general class of point process distributions that we use to rigorously state our main result on the coverage probability.

*Definition 3 (Set $\mathcal{A}$):* The set $\mathcal{A} = \{P_\Phi\}$ is the set of all point process distributions $P_\Phi$ that are mixing and that satisfy the following four conditions. If a point process $\Phi$ is distributed as $P_\Phi \in \mathcal{A}$,

1) $\xi_{\max} = \infty$;
2) for all $y > 0$, $\exists \zeta \in \mathbb{R}^2$ with $\|\zeta\| = y$, such that $\mathbb{P}(\Phi^\zeta(b(o, y)) = 0) \neq 0$;
3) $\exists y_0 > 0$, such that for all $y > y_0$ and $\|\zeta\| = y$, $\hat{I}(\Phi^\zeta)$ stochastically dominates $I(\Phi_o^\zeta)$, i.e., $\mathbb{P}(I(\Phi_o^\zeta) > z) \leq \mathbb{P}(\hat{I}(\Phi^\zeta) > z)$, for all $z \geq 0$;
4) $\forall n \in \mathbb{N}$, the $n$-th moment of $\xi$ is bounded, i.e., $\exists b_n \in (0, +\infty)$, s.t. $\mathbb{E}(\xi^n) < b_n$.

The four conditions in Def. 3 are quite mild; they are satisfied by most point processes that are usually considered in wireless networks and in stochastic geometry, such as the PPP, the MCP and the Matérn hard-core process (MHP) [4, Ch. 3]. The triangular lattice is not included, since it is not mixing. We will prove that the laws of the PPP, the MCP and the MHP belong to $\mathcal{A}$ in Section III-B.

Before introducing the main theorem, we show a property of the distribution of $I(\Phi_o^\zeta)$ under certain assumptions.

*Lemma 1:* For $P_\Phi \in \mathcal{A}$, if the fading has at most an exponential tail, i.e., $-\log F_h^c(x) = \Omega(x)$, where $F_h^c(x)$ is the CCDF of the fading variable $h$, then the interference tail is bounded by an exponential, i.e., $-\log F^c_{I(\Phi_o^\zeta)}(x) = \Omega(x)$, where $F^c_{I(\Phi_o^\zeta)}(x)$ is the CCDF of $I(\Phi_o^\zeta)$.

*Proof:* See Appendix A. ■

A similar property has been derived in [6], namely, that in ad hoc networks modeled by m.i. point processes, an exponential tail in the fading distribution implies an exponential tail in the interference distribution. The result cannot be directly applied to cellular networks. Because in the cellular network that we consider, each user communicates with its nearest BS $u$ and thus no interferers can be nearer than $u$, while the authors in [6] assume the receiver communicates with a transmitter with a fixed location and there can be some interferers nearer to the receiver than the transmitter.

*Theorem 1:* For $P_\Phi \in \mathcal{A}$, if $F_h^c(x)$ has at most an exponential tail and $\exists m \in (0, +\infty)$, s.t. $F_h(t) \sim at^m$, as $t \to 0$, where $a > 0$ is a constant, then

$$\frac{1 - P_{\rm c}(\theta)}{\theta^m} \to \kappa, \text{ as } \theta \to 0, \tag{9}$$

where $\kappa > 0$ does not depend on $\theta$ and given by

$$\kappa = \int_0^\infty \mathbb{E}_{I(\Phi_o^\zeta)}\Big[a\ell(\zeta)^{-m}\big(I(\Phi_o^\zeta) + W\big)^m\Big] f_\xi(y) dy \tag{10}$$

($\|\zeta\| = y$) and $f_\xi$ is the PDF of $\xi$.

*Proof:* See Appendix B. ■

Theorem 1 shows that the ADG exists and, given the type of fading, how it depends on the other network parameters.

*Corollary 1:* For $P_\Phi \in \mathcal{A}$ and a fading type that satisfies the assumption in Theorem 1, the ADG of $\Phi$ is given by

$$\hat{G} = \Big(\frac{\kappa^{\rm PPP}}{\kappa}\Big)^{\frac{1}{m}}, \tag{11}$$

where $\kappa^{\rm PPP}$ is the value for the PPP and $\kappa$ is the value for $\Phi$.

*Proof:* This follows directly from Theorem 1. ■

One point process has different ADGs with respect to the fading types with different $m$. So it is sensible to compare the ADGs of different point process models only under the same fading assumption.

In the following subsection, we will present several special cases regarding the fading types and point processes.

### B. Special Cases

We assume the fading type is Nakagami-$m$ fading. When $m = 1$, the fading reduces to Rayleigh fading. As to the point processes, we specifically concentrate on the PPP, the MCP and the MHP.

*Poisson Point Process*: The PPP is the simplest model of point processes, which exhibits complete spatial randomness. The points in the PPP are stochastically independent, which makes the PPP the most tractable point process.

*Matérn Cluster Process*: As a class of clustered point processes on the plane built on a PPP, the MCPs are doubly Poisson cluster processes, where the parent points form a uniform PPP $\Phi_{\rm p}$ of intensity $\lambda_{\rm p}$ and the daughter points are uniformly scattered on the ball of radius $r_{\rm c}$ centered at each parent point $x_{\rm p}$ with intensity

$$\lambda_0(x) = \frac{\bar{c}}{\pi r_{\rm c}^2}\mathbf{1}_{B(x_{\rm p}, r_{\rm c})}(x), \tag{12}$$

where $B(x_{\rm p}, r_{\rm c}) \triangleq \{x \in \mathbb{R}^2 : \|x - x_{\rm p}\| \le r_{\rm c}\}$ is the closed disk of radius $r_{\rm c}$ centered at $x_{\rm p}$. The mean number of daughter points in one cluster is $\bar{c}$. So the intensity of the process is $\lambda = \lambda_{\rm p}\bar{c}$.

*Matérn Hard-core Process*: The MHPs are a class of regular point process, where points are forbidden to be closer than a certain minimum distance. The MHPs have many types. Here we only consider the MHP of type I [4, Ch. 3], which is generated by starting with a basic uniform PPP $\Phi_{\rm b}$ of intensity $\lambda_{\rm b}$ and removing all points that have a neighbor within the hard-core distance $r_{\rm h}$ simultaneously. The intensity of the MHP is $\lambda = \lambda_{\rm b}\exp(-\lambda_{\rm b}\pi r_{\rm h}^2)$. The highest density it can achieve is $\lambda_{\max} = 1/(e\pi r_{\rm h}^2)$.

*Lemma 2:* The distributions of the PPP, the MCP and the MHP belong to the set $\mathcal{A}$.

*Proof:* See Appendix C. ■

By Lemma 2, we have the following corollary to Theorem 1.

*Corollary 2:* When the fading is Nakagami-$m$ fading, for the PPP, the MCP and the MHP,

$$1 - P_{\rm c}(\theta) \sim \kappa\theta^m, \text{as } \theta \to 0, \tag{13}$$

where $\kappa > 0$ is given in (10).

*Proof:* To prove this theorem, we only need to verify the assumption in Theorem 1 for Nakagami-$m$ fading.

As the fading parameter $h \sim \text{Gamma}(m, 1/m)$, $F_h^c(x)$ has an exponential tail and

$$\lim_{t\to 0}\frac{F_h(t)}{t^m} = \lim_{t\to 0}\frac{(mt)^{m-1}\exp(-mt)}{\Gamma(m)t^{m-1}} = \frac{m^{m-1}}{\Gamma(m)} < +\infty. \tag{14}$$

Therefore, by Lemma 2 and Theorem 1, this corollary is proved. ■

## IV. Simulations

In this section, we give some simulations of the coverage probability for the PPP, the MCP, and the MHP under Nakagami-$m$ fading. We perform the simulations on a $100 \times 100$ square and fix the path loss exponent to $\alpha = 4$ and the intensity of the point processes to $\lambda = 0.1$. More precisely, for the MCP, we let $\lambda_{\rm p} = 0.01$, $\bar{c} = 10$ and $r_{\rm c} = 5$; for the MHP, we let $\lambda_{\rm b} = 0.532$ and $r_{\rm h} = 1$.

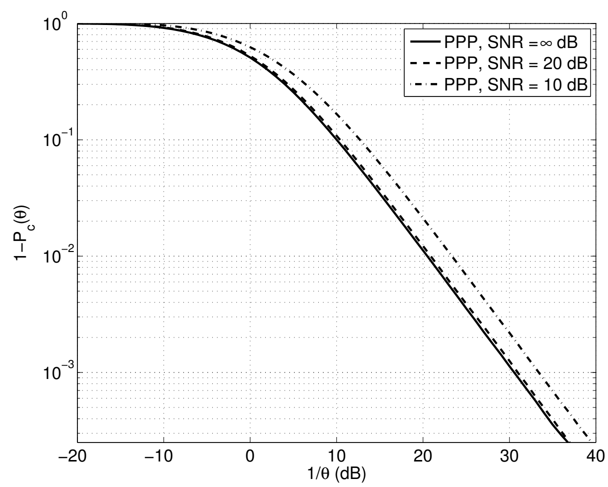


Fig. 3. The outage probability $1 - P_{\rm c}(\theta)$ vs. $1/\theta$ for the PPP when $m = 1$ under different SNR settings.

Fig. 3 shows the outage curves $1 - P_{\rm c}(\theta)$ of the PPP for Rayleigh fading and different SNR values. The slopes of the curves are all $-1$, as $\theta$ approaches 0. We also observe that there is only a quite small gap between the cases of SNR $= 20$

dB and SNR $= \infty$ dB, thus the thermal noise is not a very important factor that affects the asymptotic performance of the coverage. We will neglect noise in the rest part of this section.

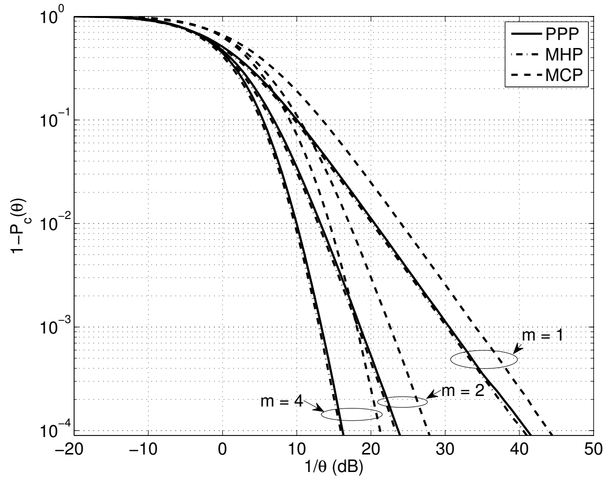


Fig. 4. The outage probability $1 - P_{\rm c}(\theta)$ vs. $1/\theta$ for the PPP, the MCP and the MHP when $m \in \{1, 2, 4\}$ (no noise).

In Fig. 4, we find that for the same point process, different $m$ implies different asymptotic slopes. In fact, the slope is $-m$. For the same $m$, different point processes have the same asymptotic slope as predicted by Theorem 1. Besides, we also observe that for any $m$, the coverage probability of the MCP is always smaller than that of the PPP and the coverage probability of the MHP is always larger than that of the PPP. Intuitively, the MHP has a better coverage because it is more regular than the PPP. Similarly, the MCP has a poorer coverage because it is more clustered than the PPP.

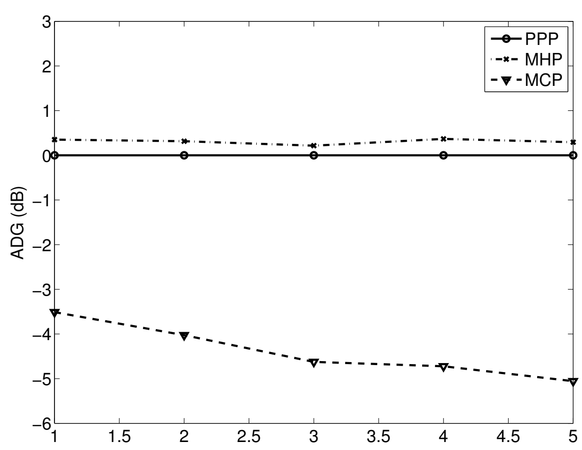


Fig. 5. The ADG for the different point processes and $m$ values (approximated by the DG value at $p_{\rm t} = 1 - 10^{-3}$) for $\alpha = 4$.

Fig. 5 shows the empirical values of the ADG for the PPP, the MCP and the MHP when $m$ takes different values. The ADG is approximated by the DG value at $p_{\rm t} = 1-10^{-3}$, since from Fig. 4, the DG seems unchanged for $p_{\rm t} > 1 - 10^{-2}$. We see that the ADGs of the MHP and the MCP vary for different $m$.

TABLE I
THE ADGS FOR DIFFERENT $\alpha$ (RAYLEIGH FADING, NO NOISE).

| ADG (in dB) | $\alpha = 2.5$ | $\alpha = 3.0$ | $\alpha = 3.5$ | $\alpha = 4$ |
|---|---|---|---|---|
| MCP | $-3.5601$ | $-3.5273$ | $-3.7824$ | $-3.5107$ |
| MHP | 0.1857 | 0.2258 | 0.2163 | 0.3523 |

Table I shows the ADGs of the MCP and the MHP for Rayleigh fading, $W = 0$ and different values of path loss exponent $\alpha$. The ADG is a function of $\alpha$ and does not monotonically increase as $\alpha$ increases.

## V. CONCLUSIONS

In this paper, we show the asymptotic property of the coverage probability that $1 - P_{\rm c}(\theta) \sim \kappa\theta^m$, as $\theta \to 0$, and the ADG exists for a variety of motion-invariant point processes given some general assumptions on the point process and the fading type. Note that the existence of the ADG for all cases remains to be validated. The assumptions on the point process are satisfied by many commonly used point processes, e.g. the PPP, the MHP and the MCP. The triangular lattice is an extreme case, which has the largest ADG of all point processes.

Under the same system configurations on the fading and path loss, different point processes with the same intensity have different ADGs. Thus, the ADG can be used as a new metric to characterize the coverage probability in the high-reliability regime. Given the ADG of a point process, we can obtain the precise CDF of the coverage probability near 1 by shifting the coverage curve of the PPP with the same intensity by the ADG.

## APPENDIX A
## PROOF OF LEMMA 1

*Proof:* Due to space constraints we only provide the sketch of the proof. The basic idea is to consider the worst case of the fading, i.e. $F_h^c(x) \sim \exp(-ax)$, $x \to \infty$ for some $a > 0$ and show that there exists $\tau < 0$, such that the Laplace transform of the $I(\Phi_o^\zeta)$, denoted by $\mathcal{L}_{I(\Phi_o^\zeta)}(s)$, converges for $s > \tau$. By Theorem 3 in [7], it follows that the interference has an exponential tail. To show such $\tau$ exists, we first evaluate $\mathcal{L}_{I(\Phi_o^\zeta)}(s)$ for $s < 0$ and derive an upper bound that only depends on $\Phi^\zeta$, so that we can take advantage of the reduced Palm distribution [4, Ch. 8]. Using the assumptions on the point process, we prove that $\forall s < 0$, there exists a positive $K_s < \infty$, s.t. $\mathcal{L}_{I(\Phi_o^\zeta)}(s) \le K_s \mathcal{L}_{\hat{I}(\Phi^\zeta)}(s)$. Then we only need to show $\mathcal{L}_{\hat{I}(\Phi^\zeta)}(s)$ converges for some $\tau_0 < 0$, which can be proved using a similar method described in the proof of Theorem 3 in [6]. ■

## APPENDIX B
## PROOF OF THEOREM 1

*Proof:* We first consider the case when the noise power $W = 0$. Let $\hat{\ell}(x) = 1/\ell(x)$ and $\|\zeta\| = y$. The coverage probability is

$$P_{\rm c}(\theta) = \mathbb{E}_\xi[\mathbb{P}(\text{SINR} > \theta \mid \xi)] = \int_0^\infty \mathbb{P}(h_\zeta > \theta\hat{\ell}(\zeta)I(\Phi_o^\zeta))f_\xi(y)dy. \quad (15)$$

To calculate $\mathbb{P}(h_\zeta > \theta\hat{\ell}(\zeta)I(\Phi_o^\zeta))$, we condition on $I(\Phi_o^\zeta)$ and have

$$\mathbb{P}(h_\zeta > \theta\hat{\ell}(\zeta)I(\Phi_o^\zeta)) = \mathbb{E}_{I(\Phi_o^\zeta)}[F_h^c(\theta\hat{\ell}(\zeta)I(\Phi_o^\zeta))]. \quad (16)$$

Thus,

$$\lim_{\theta\to 0}\frac{1-P_{\rm c}(\theta)}{\theta^m} = \lim_{\theta\to 0}\int_0^\infty \mathbb{E}_{I(\Phi_o^\zeta)}\left[\frac{F_h(\theta\hat{\ell}(\zeta)I(\Phi_o^\zeta))}{\theta^m}\right]f_\xi(y)dy. \quad (17)$$

Assume $G(t) = F_h(t)/t^m$ and $G(0) = \lim_{t\to 0}F_h(t)/t^m$. We can easily prove that $\exists A > 0$, such that $G(t) < A$, for all $t > 0$.

By Lemma 1, we have that $\forall n \in \mathbb{N}$, $\exists c_n < +\infty$, such that $\mathbb{E}(I(\Phi_o^\zeta)^n) < c_n$.

Thus, we have

$$\begin{aligned} H(y) &\triangleq \mathbb{E}_{I(\Phi_o^\zeta)}\left[\frac{F_h(\theta\hat{\ell}(\zeta)I(\Phi_o^\zeta))}{\theta^m}\right] \\ &< \mathbb{E}_{I(\Phi_o^\zeta)}\left[A(\hat{\ell}(\zeta)I(\Phi_o^\zeta))^m\right] \\ &< Ac_m\hat{\ell}(\zeta)^m < +\infty, \end{aligned} \quad (18)$$

and

$$\int_0^\infty H(y)f_\xi(y)dy < Ac_m\mathbb{E}_\xi\big((1+\xi^\alpha)^m\big) < +\infty. \quad (19)$$

We apply the Dominated Convergence Theorem twice to (17), then we have

$$\begin{aligned} &\lim_{\theta\to 0}\frac{1-P_{\rm c}(\theta)}{\theta^m} \\ &= \int_0^\infty \mathbb{E}_{I(\Phi_o^\zeta)}\left[\lim_{\theta\to 0}\frac{F_h(\theta\hat{\ell}(\zeta)I(\Phi_o^\zeta))}{\theta^m}\right]f_\xi(y)dy \\ &= \int_0^\infty \mathbb{E}_{I(\Phi_o^\zeta)}\left[a(\hat{\ell}(\zeta)I(\Phi_o^\zeta))^m\right]f_\xi(y)dy. \end{aligned} \quad (20)$$

Now, we consider the case when $W > 0$. In (17), we only need to replace $I(\Phi_o^\zeta)$ with $(I(\Phi_o^\zeta) + W)$ in the expectation and (18) becomes

$$\begin{aligned} &\mathbb{E}_{I(\Phi_o^\zeta)}\left[\frac{F_h(\theta\hat{\ell}(\zeta)(I(\Phi_o^\zeta)+W))}{\theta^m}\right] \\ &< \mathbb{E}_{I(\Phi_o^\zeta)}\left[A\hat{\ell}(\zeta)^m(I(\Phi_o^\zeta)+W)^m\right]. \end{aligned} \quad (21)$$

By expanding $(I(\Phi_o^\zeta)+W)^m$, we observe that the right-hand side of (21) is still finite. Analogous to the case when $W = 0$, we can prove that Theorem 1 also holds for $W > 0$. ■

## APPENDIX C
## PROOF OF LEMMA 2

*Proof:* Due to the limited space, we only provide the basic idea of the proof for the three point processes. Conditions 1) and 2) in Definition 3 hold for all the three point processes obviously. For Conditions 3) and 4), we treat the three point processes separately.

For the PPP, Condition 3) holds, because the points in $\Phi$ are independent; Condition 4) holds, because $\mathbb{P}(\xi > x) = \mathbb{P}(\Phi(b(o,x)) = 0) = \exp(-\lambda\pi x^2)$.

For the MCP, since the parent process is the PPP, clusters are independent. We only need to consider the region $B(o, y + r_{\rm c})$ for large $y$. As in each cluster, the daughter points are also independent, we can prove that the part of $\hat{I}(\Phi^\zeta)$ that comes from the clusters with centers located in $B(o, y+r_{\rm c})\backslash b(o, y-r_{\rm c})$ stochastically dominates the corresponding part of $I(\Phi_o^\zeta)$. Thus, Condition 3) holds. For large $y$, assume $\mathcal{S}_y = \{x \in \Phi_{\rm p} : x \in B(o, y-r_{\rm c})\}$ and let $\tilde{\Phi}_x$ be the daughter process for the cluster centered at $x \in \Phi_{\rm p}$. We have $\mathbb{P}(\xi > y) \le \mathbb{P}(\tilde{\Phi}_x(B(x, r_{\rm c})) = 0$, for all $x \in \mathcal{S}_y) = \exp\big(-(1-\exp(-\bar{c}))\lambda_{\rm p}\pi(y-r_{\rm c})^2\big)$. So, Condition 4) holds.

For the MHP, to prove Condition 3), we consider $\Phi_o^\zeta$ and $\Phi^\zeta$ in view of the base PPP $\Phi_{\rm b}$ and only need to consider the region $B(o, y+2r_{\rm h})$ for large $y$ if we condition on $\Phi_{\rm b}\bigcap(B(o, y+2r_{\rm h})\backslash B(o, y+r_{\rm h}))$. We can prove that the portion of $\hat{I}(\Phi^\zeta)$ that comes from the retained points in $B(o, y+2r_{\rm h})$ stochastically dominates the corresponding portion of $I(\Phi_o^\zeta)$. Hence, Condition 3) holds. For Condition 4), we use the CCDF of $\xi$ of the hard-core process $F_\xi^c(x)$ expressed in the form (15.1.5) in [8] and can prove that its tail follows the one of the PPP. Because the difference of the factorial $k$-th moment measure from the PPP to the MHP grows no faster than some $a^k$ $(a > 0)$, which makes the difference bounded and, in fact, arbitrarily small as $x$ increases. ■

## REFERENCES


[1] J. G. Andrews, F. Baccelli, and R. K. Ganti, "A Tractable Approach to Coverage and Rate in Cellular Networks," *IEEE Transactions on Communications*, Vol. 59, No. 11, Nov. 2011.

[2] H. S. Dhillon, R. K. Ganti, F. Baccelli and J. G. Andrews, "Modeling and Analysis of K-Tier Downlink Heterogeneous Cellular Networks," *Selected Areas in Communications, IEEE Journal on*, Vol. 30, No. 3, pp. 550-560, Apr. 2012.

[3] A. Guo and M. Haenggi, "Spatial Stochastic Models and Metrics for the Structure of Base Stations in Cellular Networks," *IEEE Transactions on Wireless Communications*, 2013. Accepted. http://www.nd.edu/~mhaenggi/pubs/twc13b.pdf

[4] M. Haenggi, *Stochastic Geometry for Wireless Networks*, Cambridge University Press, 2012.

[5] S. Lin and D. J. Costello, *Error Control Coding*, 2nd ed., Englewood Cliffs, NJ: Prentice-Hall, 2004.

[6] R. K. Ganti and M. Haenggi, "Interference in Ad Hoc Networks with General Motion-Invariant Node Distributions," *in 2008 IEEE International Symposium on Information Theory (ISIT'08)*, (Toronto, Canada), Jul. 2008.

[7] K. Nakagawa, "Application of Tauberian theorem to the exponential decay of the tail probability of a random variable," *Information Theory, IEEE Transactions on*, Vol. 53, pp. 3239-3249, Sep. 2007.

[8] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes: Volume II: General Theory and Structure (Vol. 2)*, Springer, second edition, 2007.