Delay Analysis in Static Poisson Network

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***Abstract*—We investigate the delay of the discrete-time slotted ALOHA network where the sources are distributed as a Poisson point process. Each of the sources is paired with a destination at a given distance and a buffer of infinite capacity. The network is assumed to be static, i.e., the sources and the destinations are generated at first and remain static during all the time slots. Employing tools from queueing theory as well as point process theory, we obtain upper bounds and lower bounds for the cumulative distribution function (cdf) of the conditional mean delay. Numerical results show that the gap between the upper and lower bounds is small, and the results also reveal how these bounds vary with system parameters.**

***Index Terms*—ALOHA, delay, dominant system, interacting queues, Poisson bipolar model, queueing theory, static network.**

I. Introduction

In the literature, the protocol of slotted ALOHA is studied extensively. However, most works concentrate on the capacity analysis and assume that terminals are backlogged. To make the model more practically relevant, each terminal should provide a buffer for queueing. This problem is complex because it involves interacting queues, i.e., the serving rate of each queue depends on the sizes of queues, the analysis of which should combine queueing theory and multi-access information theory and is notoriously difficult to cope with.

Previous analyses of interacting queues are mostly based on an oversimplified physical layer and consider a discrete-time slotted ALOHA system. Each terminal attempts to transmit the head-of-line packet in each time slot with a certain probability if its buffer is not empty. If two or more terminals transmit in the same time slot, a collision occurs. Only the stability region has been studied [1] for this simplified ALOHA system. In practical networks, concurrent transmissions lead to interference, which cannot be accurately modeled as collisions. Moreover, the randomness in the deployment makes accurate analysis complicated.

In this work, we model a large-scale network by point process theory, which is widely used to analyze the performance of wireless networks [2], [3]. A common and meaningful model is the Poisson point process (PPP), in which each transmitter is modeled as one point of the PPP. We combine queueing theory and stochastic geometry to analyze the delay in a static ALOHA network, i.e., the transmitters and the receivers are generated at first and remain static during all the time slots. Although ALOHA is a very simple MAC, the stability and delay analysis is still an open problem, and studying over it helps us understand more complex protocols. If each transmitter maintains a buffer to store the generated packets, the analysis becomes complex since the serving rate of each queue depends on the status of other queues, the channel status and the ALOHA protocol. We derive upper bounds and lower bounds for the cdf of the delay, and by slightly relaxing the results, we obtain the closed-form results.

Previous analyses have yielded only bounds of arrival rate for which the system is stable [4]–[6]. In [7], the stability and delay of high-mobility networks are analyzed by combining queueing theory and stochastic geometry. In high-mobility networks, the sizes of queues and the serving rates are decoupled, which, however, does not hold in static networks. In [8], we derived the sufficient conditions and necessary conditions for stability of static Poisson networks, and in this paper, we extend that work and analyze the delay of such networks.

II. System Model

We consider a discrete-time slotted ALOHA network with transmitters and receivers distributed as a Poisson bipolar process [3, Def. 5.8], i.e., we model the locations of the transmitters as a PPP $\Phi = \{x_i\} \subset \mathbb{R}^d$ of intensity λ. Each transmitter is attached with a receiver at a fixed distance r_0 and a random orientation. In the analysis, we condition on $x_0 \in \Phi$ at which a typical transmitter under consideration is located, where $r_0 = |x_0|$ is the distance between this point and the origin at which the corresponding receiver is located (see Fig. 1). The time is divided into discrete slots with equal duration, and each transmission attempt occupies exactly one time slot. The network is assumed to be static, i.e., the locations of the nodes are generated once at the beginning and then kept unchanged during all the time slots.

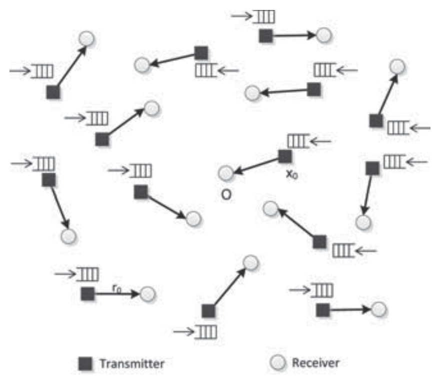

Fig. 1. A snapshot of the bipolar model with ALOHA.

We assume the Rayleigh block fading model, in which the power fading coefficients remain constant over each time slot and are spatially and temporally independent with exponential distribution of mean 1. Let α be the path loss exponent and $h_{k,x}$ be the fading coefficient between transmitter x and the considered receiver located at origin o in time slot k. All transmitters are assumed to transmit at unit power. The power spectral density of the thermal noise is set as N_0 and the bandwidth is W. We assume that if its SINR is above a threshold θ, a link can be successfully used for information transmission.

Each transmitter has a buffer of infinite capacity to store the generated packets. The packets are generated at each transmitter according to a Bernoulli process with arrival rate λ_a $(0 \leq \lambda_a \leq 1)$ packets per time slot. The arrival processes at different transmitters are independent, and each transmitter attempts to send its head-of-line packet with probability p if its buffer is not empty. We assume that the feedback of each transmission attempt, either successful or failed, is instantaneous so that each transmitter is aware of the outcome. If the transmission attempt fails, the transmitter attempts to retransmit the packet at the next time slot with probability p; otherwise, it deletes the packet from the buffer.

For any time slot $k \in \mathbb{N}^+$, let Φ_k be the set of transmitters that are transmitting in time slot k. The interference at the typical receiver located at the origin o in time slot k is

$$I_k = \sum_{x \in \Phi \setminus \{x_0\}} h_{k,x} |x|^{-\alpha} \mathbf{1}(x \in \Phi_k). \tag{1}$$

When the typical transmitter is allowed to transmit, the SINR of the typical receiver in time slot k is

$$\mathrm{SINR}_k = \frac{h_{k,x_0} r_0^{-\alpha}}{W N_0 + \sum_{x \in \Phi \setminus \{x_0\}} h_{k,x} |x|^{-\alpha} \mathbf{1}(x \in \Phi_k)}. \tag{2}$$

The relevant probability measure of the point process in this paper is the Palm probability $\mathbb{P}^{x_0}$. Correspondingly, the expectation $\mathbb{E}^{x_0}$ is taken with respect to the measure $\mathbb{P}^{x_0}$. Whether the transmission of the typical transmitter x_0 is successful or not is uncertain because of the four sources of randomness, i.e., the bursty arrival of traffic, the fading, the ALOHA protocol and the realization of PPP. Let $\mathcal{C}_\Phi^k$ be the event that the typical transmission succeeds conditioned on the realization of the PPP Φ and the status of each queue in time slot k, i.e., $\mathcal{C}_\Phi^k$ consists of two events: that the transmission is scheduled by the ALOHA and also successful in time slot k. The success event $\mathcal{C}_\Phi^k$ relies on the realization of the PPP since the locations of serving transmitter and the interferers are different for different realizations of the PPP. Meanwhile, the success event $\mathcal{C}_\Phi^k$ relies on the index of the time slot k because the statuses of the queues at the interferers change over the time slot. When the queue at an interfering transmitter is empty, no interference will be caused by the said transmitter. Even if the realization of the PPP Φ and the time slot index k are given, whether the typical link is successful or not is still uncertain because of the effect of ALOHA and fading. Let $\mathbb{P}^{x_0}(\mathcal{C}_\Phi^k)$ be the success probability conditioned on the PPP Φ in time slot k.

III. Delay Statistics

For the typical link in the network, the delay considered in this paper includes two parts, one is the queueing delay, which measures the delay between the time when a packet arrives at the queue and the time when it starts to be served, the other is the service time, which is the time to transmit said packet. In the static network, given the locations of the transmitters and receivers, the success probabilities are different for different transmissions; thus, the serving rates are also different, which results in different mean delays for different transmissions. Therefore, the mean delay has a statistical distribution for the overall network. To avoid confusion, we denote the mean delay of the typical transmitter conditioning on the realization of the PPP as the *conditional mean delay*.[1] In the following discussions, we analyze the statistics of the conditional mean delay. Specifically, we derive upper bounds and lower bounds for the cdf of the conditional mean delay, which is quite suitable to characterize the delay performance. Unless otherwise specified, we assume that the delay mentioned in this paper is evaluated in terms of time slots. Let D_Φ be the conditional mean delay conditioned on the realization of the PPP; thus it is a random variable uniquely defined by the PPP and formally defined as follows.

Definition 1. *Let $A(t)$ be the number of packets that arrived at the transmitter x_0 in $[0, t]$. Let $T_{i,\Phi}$ be the time in number of time slots between the arrival of the ith packet and the moment of successfully transmitting it to the receiver at the origin conditioned on the realization of the PPP Φ. Then, the conditional mean delay is defined as*

$$D_\Phi \triangleq \lim_{t \to \infty} \frac{\sum_{i=1}^{A(t)} T_{i,\Phi}}{A(t)}. \tag{3}$$

A. Lower Bound

In order to derive a lower bound for the cdf of the mean delays in the overall network, we consider a dominant system. Assume that in the dominant system the typical transmitter behaves exactly the same as that in the original system. However, for the other transmissions in the dominant system, we assume that when the queues at the transmitters become empty, the transmitters continue to transmit "dummy" packets with the ALOHA probability p, thus continuing to cause interference to the other transmissions no matter whether their queues are empty or not. The queue size at each transmitter in the dominant system will never be smaller than that in the original system, resulting in larger delay. Therefore, the cdf we obtain under these assumptions will be a lower bound for the cdf of the mean delay in the original system. The following theorem gives the lower bound for the cdf of the conditional mean delay in the overall network.

Theorem 1. *Given a slotted ALOHA system with the transmitters distributed as a PPP and with a packet arrival being Bernoulli processes, a lower bound of the cdf of the conditional mean delay in the overall network is*

$$\begin{aligned}\mathbb{P}^{x_0}(D_\Phi \leq T) \geq \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{1}{\omega} \mathrm{Im}\Big\{ p^{j\omega} \exp\Big(- j\omega\theta r_0^\alpha W N_0 \\ - j\omega \ln\Big(\frac{\sqrt{(2 + 2\lambda_a T - \lambda_a)^2 - 8T\lambda_a} - \lambda_a + 2}{4T} + \frac{\lambda_a}{2} \Big) \\ - 2\pi\lambda \int_0^\infty \Big(1 - \Big(\frac{p}{1 + \theta r_0^\alpha r^{-\alpha}} + 1 - p \Big)^{j\omega} \Big) r \mathrm{d}r \Big) \Big\} \mathrm{d}\omega. \end{aligned} \tag{4}$$

Proof: The success probability for the typical link given Φ in the dominant system, denoted by $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$, is the same for each time slot, which is given by (7). Given the realization of the point process Φ, the queueing system at the typical transmitter is equivalent to a Geo/G/1 queue, or a discrete-time single server retrial queue [9], [10]. In the equivalent Geo/G/1 queue, the time is slotted and the packets arrive according to a Bernoulli process

[1] Strictly speaking, there is no typical transmitter in a realization of the PPP. Since there is no commonly accepted terminology, we still call the link from x_0 to o the *typical link*, even when conditioning on Φ (and on $x_0 \in \Phi$). When taking the expectation w.r.t. Φ later, the notion of typicality has its usual meaning.

with intensity λ_a packets per time slot. The arrival process is also called geometric arrival process since the probability that a packet arrives in a time slot is λ_a, and the number of time slots between two adjacent arrivals is a geometric random variable. The success probability is $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$, thus the service times of packets are independent and identically distributed with geometric distribution. From [9], we get the conditional mean delay D_Φ as

$$D_\Phi = \begin{cases} \frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_\Phi)} + \frac{\lambda_a - \lambda_a \mathbb{P}^{x_0}(\mathcal{C}_\Phi)}{2(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)-\lambda_a)\mathbb{P}^{x_0}(\mathcal{C}_\Phi)} & \text{if } \mathbb{P}^{x_0}(\mathcal{C}_\Phi) > \lambda_a \\ \infty & \text{if } \mathbb{P}^{x_0}(\mathcal{C}_\Phi) \le \lambda_a. \end{cases} \quad (5)$$

Noticing that the queue is stable only when $\mathbb{P}^{x_0}(\mathcal{C}_\Phi) > \lambda_a$, we get the cdf of the conditional mean delay as

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) = \mathbb{P}^{x_0}\Big(\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \ge \frac{\sqrt{(2+2\lambda_a T - \lambda_a)^2 - 8T\lambda_a} - \lambda_a + 2}{4T} + \frac{\lambda_a}{2}\Big). \quad (6)$$

The success probability for the typical link conditioned on Φ in the dominant system is evaluated as

$$\begin{aligned} &\mathbb{P}^{x_0}(\mathcal{C}_\Phi) = p\mathbb{P}^{x_0}(\mathrm{SINR} > \theta \mid \Phi) \\ &= p\mathbb{P}^{x_0}\left(h_{k,x_0} r_0^{-\alpha} > \theta\left(WN_0 + I_k\right) \mid \Phi\right) \\ &= p\mathbb{E}^{x_0}\Big(\exp\Big(-\theta r_0^\alpha W N_0 \\ &\quad - \sum_{x\in\Phi\setminus\{x_0\}} \theta r_0^\alpha h_{k,x}|x|^{-\alpha}\mathbf{1}(x \in \Phi_k)\Big) \mid \Phi\Big) \\ &= p\exp\left(-\theta r_0^\alpha W N_0\right) \prod_{x\in\Phi\setminus\{x_0\}} \Big(\frac{p}{1+\theta r_0^\alpha |x|^{-\alpha}} + 1 - p\Big). \end{aligned} \quad (7)$$

The moment generating function of $Y \triangleq \ln\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right)$ is

$$\begin{aligned} M_Y(s) = p^s \exp\Big(&-s\theta r_0^\alpha W N_0 \\ &-2\pi\lambda \int_0^\infty \Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^s\Big) r\mathrm{d}r\Big). \end{aligned} \quad (8)$$

The cdf of Y, denoted by $F_Y(y) = \mathbb{P}\left(Y \le y\right)$, follows from the Gil-Pelaez Theorem [11] as

$$F_Y(y) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\{e^{-j\omega y} M_Y(j\omega)\}}{\omega}\mathrm{d}\omega. \quad (9)$$

The cdf of $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$ is evaluated as

$$\begin{aligned} &\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \le \lambda_a\right\} = F_Y(\ln(\lambda_a)) \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\{e^{-j\omega \ln(\lambda_a)} M_Y(j\omega)\}}{\omega}\mathrm{d}\omega. \end{aligned} \quad (10)$$

Therefore, we have

$$\begin{aligned} \mathbb{P}^{x_0}\left(D_\Phi \le T\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\mathrm{Im}\Big\{p^{j\omega}\exp\Big(&-j\omega\theta r_0^\alpha W N_0 \\ -j\omega \ln\Big(\frac{\sqrt{(2+2\lambda_a T - \lambda_a)^2 - 8T\lambda_a} - \lambda_a + 2}{4T} + \frac{\lambda_a}{2}\Big) \\ -2\pi\lambda\int_0^\infty \Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{j\omega}\Big) r\mathrm{d}r\Big)\Big\}\mathrm{d}\omega. \end{aligned} \quad (11)$$

Thus, for the original system, we get the result in Theorem 1. ■

In order to simplify the lower bound, we derive a lower bound by using the Markov inequality.

Corollary 1. *Given a slotted ALOHA system with the transmitters distributed as a PPP and with a packet arrival being Bernoulli processes, for all $t > 0$, a lower bound of the cdf of the conditional mean delay in the overall network is*

$$\begin{aligned} \mathbb{P}^{x_0}\left(D_\Phi \le T\right) \ge \max\Big\{0, 1 - p^{-t}\exp\Big(&t\theta r_0^\alpha W N_0 \\ -2\pi\lambda\int_0^\infty \Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{-t}\Big) r\mathrm{d}r \\ +t\ln\Big(\frac{\sqrt{(2+2\lambda_a T - \lambda_a)^2 - 8T\lambda_a} - \lambda_a + 2}{4T} + \frac{\lambda_a}{2}\Big)\Big)\Big\}. \end{aligned} \quad (12)$$

When t is a positive integer, denoted by n, we get a closed-form lower bound

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) \ge p^{\mathrm{lb}}(T, n) \quad (13)$$

where

$$\begin{aligned} p^{\mathrm{lb}}(T,n) = \max\Big\{0, 1 - p^{-n}\exp\Big(&n\theta r_0^\alpha W N_0 \\ +\pi\lambda\delta n(1-p)^\delta\theta^\delta r_0^2 \sum_{i=1}^{n}((1-p)^{-i} - 1)\frac{\Gamma(i-\delta)\Gamma(n-i+\delta)}{\Gamma(i+1)\Gamma(n-i+1)} \\ +n\ln\Big(\frac{\sqrt{(2+2\lambda_a T - \lambda_a)^2 - 8T\lambda_a} - \lambda_a + 2}{4T} + \frac{\lambda_a}{2}\Big)\Big\}. \end{aligned} \quad (14)$$

When n is chosen as the optimal value for given T, i.e., $n_{\max}(T) = \mathrm{argmax}_{n\in\mathbb{N}^+} p^{\mathrm{lb}}(T,n)$, we get a lower bound that is superior to other lower bounds corresponding to other n as

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) \ge p^{\mathrm{lb}}(T, n_{\max}(T)), \quad (15)$$

Proof: For all $t > 0$, the cdf of $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$ is

$$\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) < \lambda_a\right\} = \mathbb{P}^{x_0}\left\{e^{-t\ln(\mathbb{P}^{x_0}(\mathcal{C}_\Phi))} > e^{-t\ln(\lambda_a)}\right\}. \quad (16)$$

By applying the Markov inequality, we obtain

$$\begin{aligned} &\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) < \lambda_a\right\} < \frac{1}{e^{-t\ln(\lambda_a)}}\mathbb{E}\left(e^{-t\ln(\mathbb{P}^{x_0}(\mathcal{C}_\Phi))}\right) \\ &= p^{-t}\exp\Big(t\ln(\lambda_a) + t\theta r_0^\alpha W N_0 \\ &\quad -2\pi\lambda\int_0^\infty \Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{-t}\Big) r\mathrm{d}r\Big). \end{aligned} \quad (17)$$

Therefore, the cdf of the conditional mean delay is

$$\begin{aligned} \mathbb{P}^{x_0}\left(D_\Phi \le T\right) \ge \max\Big\{0, 1 - p^{-t}\exp\Big(&t\theta r_0^\alpha W N_0 \\ -2\pi\lambda\int_0^\infty \Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{-t}\Big) r\mathrm{d}r \\ +t\ln\Big(\frac{\sqrt{(2+2\lambda_a T - \lambda_a)^2 - 8T\lambda_a} - \lambda_a + 2}{4T} + \frac{\lambda_a}{2}\Big)\Big)\Big\}. \end{aligned}$$

By setting $t = n \in \mathbb{N}^+$, we get

$$\begin{aligned} &\int_0^\infty \Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{-n}\Big) r\mathrm{d}r \\ &\overset{(a)}{=} -\sum_{i=0}^{n} C_n^i (1 - (1-p)^i) \int_0^\infty \frac{(\theta r_0^\alpha r^{-\alpha})^i r}{(1 + (1-p)\theta r_0^\alpha r^{-\alpha})^n}\mathrm{d}r \\ &\overset{(b)}{=} -\frac{n}{2}\delta(1-p)^\delta\theta^\delta r_0^2 \sum_{i=1}^{n}((1-p)^{-i} - 1)\frac{\Gamma(i-\delta)\Gamma(n-i+\delta)}{\Gamma(i+1)\Gamma(n-i+1)}. \end{aligned}$$

where $C_n^i = n!/(i!(n-i)!) = \Gamma(n+1)/(\Gamma(i+1)\Gamma(n-i+1))$ is the binomial coefficient and $\delta = 2/\alpha$. (a) holds from the binomial expansion and the exchange of summation and integral. (b) follows from the relationship between the beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$ and the gamma function and from the fact that the term for $i = 0$ equals to zero. Thus, we get the results in Corollary 1. ■

B. Upper Bound

To derive an upper bound of the cdf of the conditional mean delay, we consider a modified system as follows: if a packet in an interfering transmitter is not scheduled by the ALOHA or failed for the transmission, it will be dropped rather than retransmitted. In this way, since the interference in the modified system is always smaller than that in the original system and the packets will not accumulate at the interfering transmitters, the cdf of the conditional mean delay for the modified system will be an upper bound of that for the original system.

Theorem 2. *An upper bound of the cdf of the conditional mean delay of the slotted ALOHA system with the transmitters distributed as a PPP and with a packet arrival being Bernoulli processes is*

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) \le \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\mathrm{Im}\Big\{p^{j\omega}\exp\Big(-j\omega\theta r_0^\alpha W N_0 -j\omega\ln\Big(\frac{\sqrt{(2+2\lambda_a T-\lambda_a)^2-8T\lambda_a}-\lambda_a+2}{4T}+\frac{\lambda_a}{2}\Big) -2\pi\lambda\int_0^\infty\Big(1-\Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}}+1-\lambda_a p\Big)^{j\omega}\Big)r\mathrm{d}r\Big)\Big\}\mathrm{d}\omega. \quad (18)$$

Proof: In the modified system, the interfering transmitter is active if there is a packet arriving at the queue and the transmission is scheduled by ALOHA, thus the probability is $\lambda_a p$. From (6), we get the cdf of the conditional mean delay as

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) = \mathbb{P}^{x_0}\Big(\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \ge \frac{\sqrt{(2+2\lambda_a T-\lambda_a)^2-8T\lambda_a}-\lambda_a+2}{4T}+\frac{\lambda_a}{2}\Big). \quad (19)$$

By introducing the modified system, packets at the interfering transmitters are dropped if the transmitters are silenced due to ALOHA or if the transmission fails due to the SINR condition, thus an interfering transmitter is active with probability $\lambda_a p$. Similar to the derivation of (7), we get the success probability for the typical link conditioned on Φ in the modified system as

$$\mathbb{P}^{x_0}(\mathcal{C}_\Phi) = p\exp\left(-\theta r_0^\alpha W N_0\right)\prod_{x\in\Phi\setminus\{x_0\}}\Big(\frac{\lambda_a p}{1+\theta r_0^\alpha |x|^{-\alpha}}+1-\lambda_a p\Big). \quad (20)$$

The moment generating function of $Y \triangleq \ln\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right)$ is

$$M_Y(s) = p^s\exp\Big(-s\theta r_0^\alpha W N_0 -2\pi\lambda\int_0^\infty\Big(1-\Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}}+1-\lambda_a p\Big)^{s}\Big)r\mathrm{d}r\Big). \quad (21)$$

The cdf of Y can be derived as follows by applying the Gil-Pelaez Theorem given by (9).

$$F_Y(y) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\{e^{-j\omega y}M_Y(j\omega)\}}{\omega}\mathrm{d}\omega. \quad (22)$$

Therefore, we have

$$\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \ge x\right\} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\mathrm{Im}\Big\{p^{j\omega}\exp\Big(-j\omega\theta r_0^\alpha W N_0 - j\omega\ln(x) -2\pi\lambda\int_0^\infty\Big(1-\Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}}+1-\lambda_a p\Big)^{j\omega}\Big)r\mathrm{d}r\Big)\Big\}\mathrm{d}\omega. \quad (23)$$

Therefore, the cdf of the conditional mean delay for the modified system is evaluated as

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\mathrm{Im}\Big\{p^{j\omega}\exp\Big(-j\omega\theta r_0^\alpha W N_0 -j\omega\ln\Big(\frac{\sqrt{(2+2\lambda_a T-\lambda_a)^2-8T\lambda_a}-\lambda_a+2}{4T}+\frac{\lambda_a}{2}\Big) -2\pi\lambda\int_0^\infty\Big(1-\Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}}+1-\lambda_a p\Big)^{j\omega}\Big)r\mathrm{d}r\Big)\Big\}\mathrm{d}\omega. \quad (24)$$

Since the cdf of the conditional mean delay of the modified system is only an upper bound of the original system, we get the result in Theorem 2. ■

In the following, we use the Markov inequality to derive an upper bound that is slightly easier to evaluate.

Corollary 2. *An upper bound of the cdf of the conditional mean delay in the slotted ALOHA system with the transmitters distributed as a PPP and with a Bernoulli packet arrival is*

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) \le p^t\exp\Big(-t\theta r_0^\alpha W N_0 -2\pi\lambda\int_0^\infty\Big(1-\Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}}+1-\lambda_a p\Big)^{t}\Big)r\mathrm{d}r -t\ln\Big(\frac{\sqrt{(2+2\lambda_a T-\lambda_a)^2-8T\lambda_a}-\lambda_a+2}{4T}+\frac{\lambda_a}{2}\Big)\Big), \quad (25)$$

for all $t > 0$. When t is a positive integer, denoted by n, we get a lower bound in closed form as

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) \le p^n\exp\Big(-n\theta r_0^\alpha W N_0 -\pi\lambda n\delta\theta^\delta r_0^2\sum_{i=1}^{n}(1-(1-\lambda_a p)^i)\frac{\Gamma(i-\delta)\Gamma(n-i+\delta)}{\Gamma(i+1)\Gamma(n-i+1)} -n\ln\Big(\frac{\sqrt{(2+2\lambda_a T-\lambda_a)^2-8T\lambda_a}-\lambda_a+2}{4T}+\frac{\lambda_a}{2}\Big)\Big). \quad (26)$$

When the optimum value is chosen for n, we have

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) \le \min_{n\in\mathbb{N}^+}\Big(p^n\exp\Big(-n\theta r_0^\alpha W N_0 -\pi\lambda n\delta\theta^\delta r_0^2\sum_{i=1}^{n}(1-(1-\lambda_a p)^i)\frac{\Gamma(i-\delta)\Gamma(n-i+\delta)}{\Gamma(i+1)\Gamma(n-i+1)} -n\ln\Big(\frac{\sqrt{(2+2\lambda_a T-\lambda_a)^2-8T\lambda_a}-\lambda_a+2}{4T}+\frac{\lambda_a}{2}\Big)\Big)\Big). \quad (27)$$

Proof: For all $t > 0$, by applying the Markov inequality, we obtain the following inequality

$$\begin{aligned}&\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) < \lambda_a\right\}\\ &= \mathbb{P}^{x_0}\left\{\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right)^t < \lambda_a^t\right\}\\ &> 1-\lambda_a^{-t}\mathbb{E}\left(\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right)^t\right)\\ &= 1-p^t\exp\Big(-t\ln(\lambda_a)-t\theta r_0^\alpha W N_0 -2\pi\lambda\int_0^\infty\Big(1-\Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}}+1-\lambda_a p\Big)^{t}\Big)r\mathrm{d}r\Big). \quad (28)\end{aligned}$$

Therefore,

$$\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \ge x\right\} \le p^t\exp\Big(-t\ln(x)-t\theta r_0^\alpha W N_0 -2\pi\lambda\int_0^\infty\Big(1-\Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}}+1-\lambda_a p\Big)^{t}\Big)r\mathrm{d}r\Big). \quad (29)$$

From (6), we get an upper bound for the cdf of the conditional mean delay for the modified system as follows

$$\mathbb{P}^{x_0}\left(D_\Phi \le T\right) \le p^t \exp\Bigg(- t\theta r_0^\alpha W N_0$$
$$-2\pi\lambda \int_0^\infty \Big(1 - \Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - \lambda_a p\Big)^t\Big) r \mathrm{d}r$$
$$-t \ln\Bigg(\frac{\sqrt{(2+2\lambda_a T - \lambda_a)^2 - 8T\lambda_a} - \lambda_a + 2}{4T} + \frac{\lambda_a}{2}\Bigg)\Bigg). \quad (30)$$

Since $\mathbb{P}^{x_0}\left(D_\Phi \le T\right) \le 1$, we give an upper bound in Corollary 2 for the cdf of the conditional mean delay for the original system. By the same derivations as Corollary 1, we get the results. ■

C. Comparison of Lower and Upper Bounds

In this section, we compare the lower and upper bounds for the cdf of conditional mean delay based on their numerical evaluation.

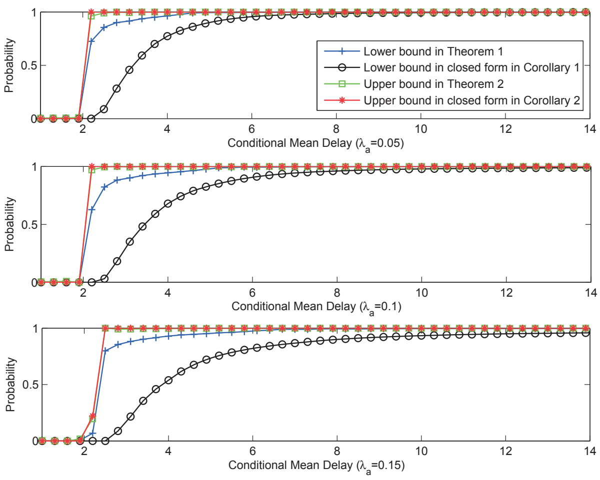

Fig. 2. Comparison of lower bound and upper bound for the cdf of conditional mean delay for different packet arrival rates λ_a. The parameters are set as $\theta =$ 10dB, $r_0 = 1$, $N_0 = -173$dBm, $W = 20$MHz, $\alpha = 4$ and $\lambda = 0.01$.

Fig. 2 plots the lower and upper bounds for the cdf of the conditional mean delay for different packets arrival rate λ_a. It is observed that the lower and upper bounds are tight for different λ_a. However, the closed form lower bounds become loose when the packet arrival rate increases, which is caused by the fact that the Markov inequality becomes loose when λ_a gets large.

Fig. 3 plots the lower and upper bounds for the cdf of the conditional mean delay for different λ. It is observed that the bounds become loose when the intensity of transmitters increases. This is because as λ increases, the effect of interference is non-negligible, thus increasing the difference between the dominant/modified system and the original system. Although the closed form lower bounds (black lines) are loose for large λ, the non-closed form lower bounds (blue lines) are still tight.

IV. Conclusions

In this paper, we investigated the delay of the discrete-time slotted ALOHA network with the transmitters and receivers distributed as a Poisson bipolar process. We employed tools from queueing theory as well as point process theory and proposed several novel approaches to derive the upper and lower bounds for the cdf of the conditional mean delay. The numerical results

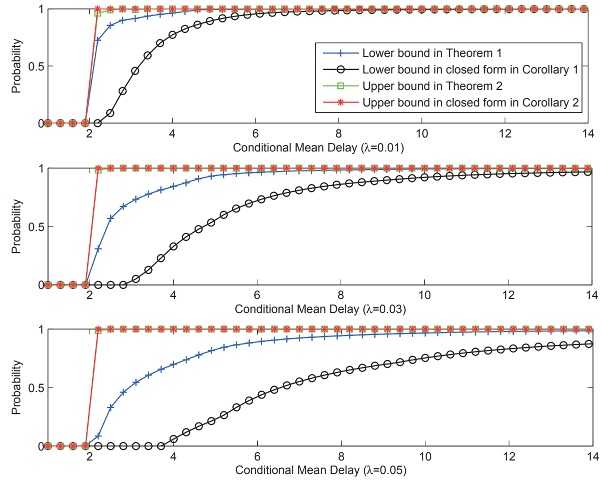

Fig. 3. Comparison of lower bound and upper bound for the cdf of conditional mean delay for different transmitter intensities λ. The parameters are set as $\theta =$ 10dB, $r_0 = 1$, $N_0 = -173$dBm, $W = 20$MHz, $\alpha = 4$ and $\lambda_a = 0.1$.

show that the gap between the non-closed form upper and lower bounds is small. As for the closed form upper and lower bounds, the gap is small for small arrival rate λ_a and small intensity of transmitters λ.

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