A Geometry-Inclusive Fading Model for Random Wireless Networks

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***Abstract*— A new fading model is proposed and discussed that combines the uncertainties in the transmission distance as well as small-scale fading. If nodes are assumed to be distributed according to a Poisson point process and the fading is Rayleigh, the joint fading distribution is particularly simple. Interpreting fading as a stochastic mapping, we show that a node cannot infer on the presence of fading by measuring link qualities. Other applications of the fading model include connectivity, opportunistic communication, and probabilistic progress.**

I. Introduction

In wireless networks, distances have a strong impact on the signal strengths and the signal-to-noise-and-interference ratios (SINRs), and, consequently, on the quality of the links. In addition, given a transmitter-receiver distance d, the path loss may deviate significantly from the expected value obtained from a large-scale path loss model, usually of the form d^α, a phenomenon referred to as fading. While it is widely acknowledged that *small-scale* fading should be modeled stochastically (at least until a proper training sequence is received in the slow fading case), the distance d is usually assumed to be known. However, in an ad hoc network, only statistical information on the nodes' positions may be available. Consequently, the *geometry of the network*, in particular the internode distances, should also be modeled stochastically.

We propose and investigate such a geometry-inclusive fading model for networks whose nodes are distributed according to a Poisson point process (PPP) and whose small-scale fading is Rayleigh. This assumption has two advantages; it is analytically tractable on the one hand, and it constitutes a worst or extreme case on the other hand, in the sense that most fading models are more benign than Rayleigh fading, and all (homogeneous) point processes have a smaller entropy than the PPP. For the large-scale path loss, we employ the common power law mentioned above, well aware of its shortcoming at small distances [1].

To complete the link model, we assume a transmission is successful if the signal-to-noise ratio exceeds a certain threshold, or, equivalently, the path gain between a transmitter and a receiver exceeds a certain value s. Then, assuming the path gain is given by Q and the link distance by D, the probability of successful reception is

$$p_r(s) = \mathbb{P}[Q(D) \geqslant s] = \mathbb{E}[e^{-D^\alpha s}]\,. \tag{1}$$

Notation. We use capital symbols (*e.g.*, Q, L, D) or sans-serif lower-case symbols (*e.g.*, x, n) to denote random variables. As for the distributions $\mathcal{E}(a)$ and $\mathcal{P}o(b)$ refer to the exponential distribution with parameter a (mean $1/a$) and the Poisson distribution with mean b, respectively, and $\mathcal{U}[0,a]$ refers to the uniform distribution in the interval $[0,a]$. For probability distributions and densities, we use F and f, respectively.

Node Distribution: The Poisson point process. A well accepted model for the node distribution[1] is the homogeneous *Poisson point process* of intensity λ. For the simplicity of our exposition, we will focus on infinite networks, and without loss of generality, we can assume $\lambda = 1$ (scale-invariance). From the Poisson property, the following result can immediately be derived [2]: For an m-dimensional network, the distance D_n between a node and its n-th neighbor has the generalized gamma probability density function (pdf)

$$f_{D_n}(r) = e^{-c_m r^m} \frac{m\,(c_m r^m)^n}{r(n-1)!}\,, \tag{2}$$

where $c_m := \pi^{m/2}/\Gamma(1+m/2)$ such that $c_m r^m$ is the volume of the m-dimensional ball of radius r. In particular, in two dimensions, the distance to the nearest neighbor is Rayleigh distributed with mean $1/2$, and the squared ordered distances D_n^2 are Erlang with parameter $1/\pi$, *i.e.*, $\mathbb{E}[D_n^2] = n/\pi$.

Fig. 1 shows a PPP of intensity 1 in a 16×16 square, with the nodes marked that can be reached from the center, assuming a path gain threshold of $s = 0.1$. The disk shows the maximum transmission distance in the non-fading case.

II. A Fading Model for n-th Nearest-Neighbor Communication

A. Distribution of path gain

Theorem 1 *Consider a node in a Rayleigh fading network whose nodes are distributed according to a Poisson point process in $\mathbb{R}^2$ with intensity 1. Let Q_n denote the (power) path gain between the node and its n-th nearest neighbor for a path loss exponent of 2. The cdf of Q_n is*

$$F_{Q_n}(x) = 1 - \frac{\pi^n}{(\pi + x)^n}\,. \tag{3}$$

[1] In particular, if nodes move around randomly and independently, or if sensor nodes are deployed from an airplane in large quantities.

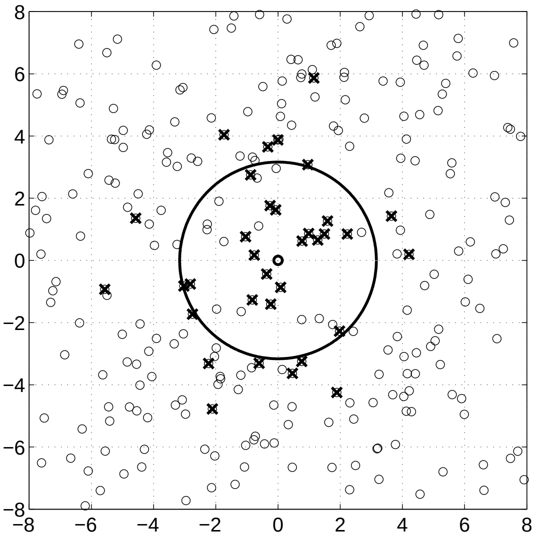

Fig. 1. A Poisson point process of intensity 1 in a 16×16 square. The reachable nodes by the center node are indicated by a bold $\times$ for a path gain threshold of $s = 0.1$. The disk indicates the transmission in the non-fading case, *i.e.*, its radius is $1/\sqrt{s} \approx 3.16$.

Proof: Given the distance d_n, the received power Q_n is exponentially distributed with mean D_n^{-2} due to the Rayleigh fading assumption, and D_n^2 is Erlang as mentioned previously. Let $A := D_n^2$. We obtain for the cdf $\mathbb{P}[Q_n < x]$

$$\begin{aligned} F_{Q_n}(x) &= \mathbb{E}_A[1 - e^{-Ax}] \\ &= \int_0^\infty (1 - e^{-ax}) \left(\frac{\pi^n a^{n-1}}{\Gamma(n)} e^{-\pi a} \right) \mathrm{d}a \\ &= 1 - \frac{\pi^n}{(\pi + x)^n} \,. \end{aligned}$$

□

Note that $Q_n = Q_n^f / Q_n^d$, where Q_n^f (the fading part) is iid exponential with mean 1 and $Q_n^d := D_n^2$ (the distance part) is Erlang. In particular, for $n = 1$, (3) is the cdf of the ratio of two exponential random variables whose means have a ratio π. Also, $1 - F_{Q_n}(s)$ is the moment-generating function of $-D_n^2$ (see (1)). The pdf of Q_n is

$$f_{Q_n}(x) = \frac{n\pi^n}{(\pi + x)^{n+1}} \,, \tag{4}$$

and the first and second moments of Q_n are

$$\mathbb{E}Q_n = \frac{\pi}{n-1} \quad \text{for } n > 1 \tag{5}$$

$$\mathbb{E}Q_n^2 = \frac{2\pi^2}{(n-1)(n-2)} \quad \text{for } n > 2 \,. \tag{6}$$

Generally, given n, the highest existing (finite) moment is $\mathbb{E}Q_n^{n-1} = \pi^{n-1}$. The variance is decreasing quickly: $\mathrm{Var}(Q_n) = O(1/n^2)$.

Entropy. The differential entropy $h(Q_n) := \mathbb{E}[-\ln f_{Q_n}(Q_n)]$ is

$$h(Q_n) = \frac{n+1}{n} + \ln\left(\frac{\pi}{n}\right) \,, \tag{7}$$

which is (as expected due to the decreasing variance) monotonically decreasing with increasing n.

B. Distribution of path loss

Instead of considering the path *gain*, we may be interested in the path *loss*. Let $L_n = Q_n^d / Q_n^f = Q_n^{-1}$ be the path loss to the n-th nearest neighbor. For the cdf we obtain

$$F_{L_n}(x) = \frac{(\pi x)^n}{(\pi x + 1)^n} \,, \tag{8}$$

and the pdf is

$$f_{L_n} = \frac{n}{x} \frac{(\pi x)^n}{(\pi x + 1)^{n+1}} \,. \tag{9}$$

So $\mathbb{E}L_n$ does not exist for any n. In particular, for $n = 1$, both the mean path gain and the mean path loss are infinite. The differential entropy is

$$h(L_n) = \ln \pi - \gamma - \Psi n \,, \tag{10}$$

where γ is the Euler-Mascheroni constant and Ψ is the digamma function (*i.e.*, the logarithmic derivative of the gamma function). Since $\Psi n \approx \ln n$ for $n \gg 1$, this is essentially the same as (7).

C. Dependence

The RVs Q_n (or L_n) are *not* independent. Consider the joint pdf of the Q_d part. Let $\mathsf{x}_i = D_i^2$, *i.e.*, consider the squared ordered distances. Since x_i forms a one-dimensional PPP of intensity π, the differences $\mathsf{x}_{i+1} - \mathsf{x}_i$ are $\mathcal{E}(\pi)$, thus the squared ordered distances of the first n nodes have the joint pdf

$$f_{\mathsf{x}_1 \ldots \mathsf{x}_n}(x_1, \ldots, x_n) = \pi^n e^{-\pi x_n} \mathbf{1}_{0 < x_1 < \ldots < x_n} \,, \tag{11}$$

where $\mathbf{1}_{0 < x_1 < \ldots < x_n}$ denotes the (positive) order cone (or hyperoctant) in n dimensions. For $n = 2$, the joint cdf is

$$F_{\mathsf{x}_1 \mathsf{x}_2}(x_1, x_2) = 1 - e^{-\pi \min(x_1, x_2)} - \pi \min(x_1, x_2) e^{-\pi x_2} \,.$$

For $x_1 > x_2$, this reduces to the cdf of the Erlang distribution, since x_1 is smaller than x_2. Clearly, dividing the x_i by f_i iid $\mathcal{E}(1)$ does not make them independent.

So, the ordering creates dependence. To obtain a set of *independent* RVs, we may condition the PPP on having a certain number of nodes $\mathsf{n} = n$ within a given interval $[0, a]$. Equivalently, we may fix the position of the $n{+}1$-th node, *i.e.*, set $\mathsf{x}_{n+1} = a$. Then, the n nodes inside $[0, a]$ are iid $\mathcal{U}[0, a]$. With $\mathsf{x} \sim \mathcal{U}[0, a]$ and $\mathsf{f} \sim \mathcal{E}(1)$, the cdf of x/f is

$$F_{\mathsf{x}/\mathsf{f}}(x) = \frac{x}{a} \left(1 - e^{-a/x}\right) . \tag{12}$$

Again, due to the division by an exponential, this RV does not have any finite moments. To obtain results for the actual PPP, the expectation over $\mathsf{n} \sim \mathcal{P}o(a\pi)$ is to be taken.

III. Connectivity

Assume that a path gain of at least s is needed for two nodes to be connected, *i.e.*, to be able to communicate (at a given rate with a certain desired reliability). Here we focus on the node at the origin and denote by n^c the number of nodes that are connected to the origin. Let C_k be the event that the origin is connected to its k-th nearest neighbor, *i.e.*,

$\mathbb{P}[C_k] = \mathbb{P}[Q_k > s] = \pi^k/(\pi+s)^k$. Since this is a geometric series,

$$\mathbb{E}n^c = \sum_{k=1}^{\infty} \mathbb{P}[C_k] = \frac{\pi}{s}\,. \tag{13}$$

Note that this is valid despite the dependence of the events C_k, and that this expected node degree is exactly the same as in the disk graph model. More can be said:

Lemma 1 *Let $\{x_i^c\} \subset \{x_i\}$ be the set of connected nodes, with $x_i := D_i^2$ the squared ordered distances. Then $\{x_i^c\}$ is an inhomogeneous PPP with intensity $\lambda(x) = \pi e^{-xs}$ $(x \geqslant 0)$, and the probability of the origin being isolated is $e^{-\pi/s}$.*

Proof: Let $f \sim \mathcal{E}(1)$. The probability that a point at position x has a path loss smaller than $1/s$ is

$$\mathbb{P}[x/f < 1/s] = \mathbb{P}[f > sx] = e^{-xs}\,. \tag{14}$$

So the set of connected nodes is obtained by *thinning* the original homogeneous PPP by e^{-xs}. It follows that the number n^c of connected nodes is Poisson distributed with parameter

$$\mathbb{E}n^c = \pi \int_0^{\infty} e^{-xs}\mathrm{d}x = \pi/s\,, \tag{15}$$

and thus $\mathbb{P}[n^c = 0] = e^{-\pi/s}$. $\square$

The fact that Rayleigh fading does not change the connectivity of a single node in a network with $\alpha = 2$ was also derived in [3]. They also showed that for $\alpha > 2$, Rayleigh fading is harmful, although not significantly.

We may also use (12) to derive this result: Given $n = n$ nodes in the interval $[0, a]$, they are iid $\mathcal{U}[0, a]$, and each one is disconnected with probability

$$1 - F_{x/f}(1/s) = 1 - \frac{1}{as}(1 - e^{-as})\,. \tag{16}$$

independently of the other $n - 1$. So the (unconditioned) probability that all nodes in the interval $[0, a]$ are disconnected is

$$\mathbb{P}[n_a^c = 0] = \mathbb{E}_n\left[\left(1 - \frac{1 - e^{-as}}{as}\right)^n\right] \tag{17}$$

$$= e^{-\frac{\pi}{s}(1-e^{-sa})}\,. \tag{18}$$

As expected, $\lim_{a\to\infty} \mathbb{P}[n_a^c = 0] = e^{-\pi/s}$. Further, setting $a = 1/s$ yields $e^{-\pi/s(1-1/e)}$, which is the probability that all nodes that are connected under the disk model become disconnected under the fading model.

IV. Fading as a Stochastic Mapping

In this section, we interpret the Rayleigh fading process as a stochastic mapping of the points $x_i := D_i^2$ to $\xi_i := x_i/f_i$, where $f_i \sim \mathcal{E}(1)$.[2] So, $\{x_i\}$ are the points in the geographical domain (they indicate distance), whereas $\{\xi_i\}$ are the points in the path loss domain, since ξ_i is the actual path loss including fading. This mapping is visualized in Fig. 2. In the path loss domain, the connected nodes are simply given by

[2]Hence the ξ_i correspond to the L_i in Section II.

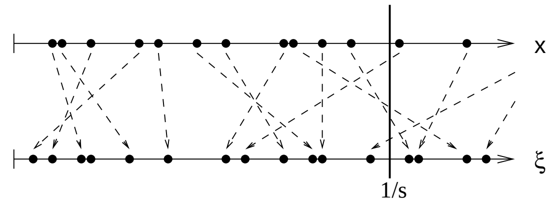

Fig. 2. The points of a Poisson point process x_i are mapped and reordered according to $\xi_i := x_i/f_i$, where f_i is iid $\mathcal{E}(1)$. In the lower axis, the nodes to the left of the threshold $1/s$ are connected to the origin (path loss smaller than $1/s$).

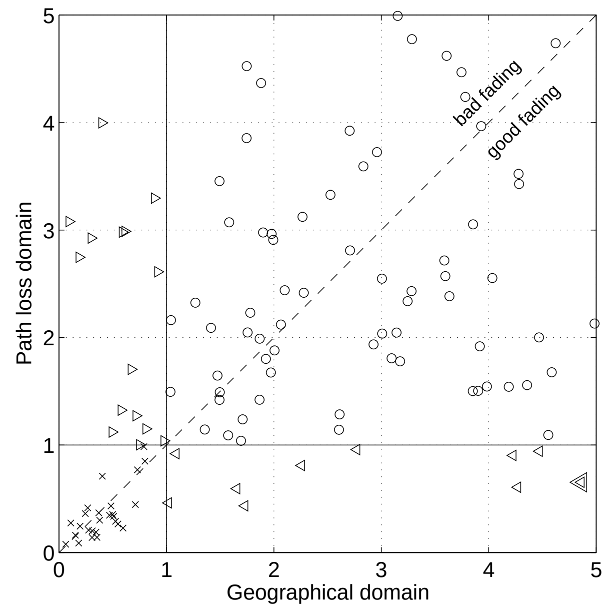

Fig. 3. Illustration of the Rayleigh mapping. 200 points x_i are chosen uniformly randomly in $[0, 5]$. Plotted are the points $(x_i, x_i/f_i)$, where the f_i are drawn iid $\mathcal{E}(1)$. Consider the interval $[0, 1]$ (*i.e.*, assume a threshold $s = 1$). Points marked by $\times$ are points that remain inside $[0, 1]$, those marked by o remain outside, the ones marked with left- and right-pointing triangles are the ones that moved in and out, respectively. The node marked with a double triangle is the furthest reachable node. On average the same number of nodes move in and out. Note that not all points are shown, since a fraction e^{-1} is mapped outside of $[0, 5]$.

$\{\xi_i^c\} = \{\xi_i\} \cap [0, 1/s]$. Note that while we assumed x_i to be ordered, this no longer holds for ξ_i, since fading is quite likely to reorder the nodes.

A. Path loss distribution of connected nodes

How are the connected nodes $\{\xi_i^c\}$ distributed in the path loss domain? We have established that $n^c = |\{\xi_i^c\}| \sim \mathcal{P}o(\pi/s)$. Since this is true for arbitrary s, we may conjecture that the connected nodes are homogeneous in the path loss domain. This is indeed the case:

Theorem 2 *The connected nodes $\{\xi_i^c\}$ in the path loss domain form a* homogeneous *PPP of intensity π on $[0, 1/s]$.*

Proof: We need to show homogeneity of ξ_i^c inside $[0, 1/s]$. To that end we determine the conditional distribution

$\mathbb{P}[\xi < x \,|\, \xi < 1/s]$ for a node $\xi := x/f$ with $x \sim \mathcal{U}[0,a]$. Applying (12),

$$\mathbb{P}[\xi < x \,|\, \xi < 1/s] = sx\frac{1-e^{-a/x}}{1-e^{-as}}\,, \quad 0 \leqslant x \leqslant 1/s\,. \quad (19)$$

As $a \to \infty$, this is sx, so indeed the distribution is uniform. $\square$

Remark. The Mapping Theorem [4, Sect. 2.3] states that an inhomogeneous PPP in one dimension can be transformed to a homogeneous PPP by means of a continuous monotonic transformation. If the inhomogeneous PPP has an exponentially decreasing density, then the total number of nodes is finite, and the support of the "homogenized" process is necessarily finite. In our case, where the support is $[0,1/s]$, the transformation is $\frac{1}{s}e^{-sx}$.

An immediate consequence is the following:

Corollary 1 *The path loss to the connected nodes is uniform in* $[0,1/s]$*, irrespective of whether there is Rayleigh fading or not. Thus the node at the origin cannot decide whether the network is subject to Rayleigh fading or not.*

B. Impact of fading

First we address the question about the probability that a node from inside $[0,1/s]$ (geographical domain) is mapped to the outside (path loss domain), and vice versa, *i.e.*, the probability that a node is (dis)connected only due to fading. A node at position x will be mapped outside the interval with probability $1-e^{-xs}$. Conditioned on having n nodes inside, these n nodes are uniformly randomly placed. The probability that one of these nodes ends up outside $[0,1/s]$ is

$$\mathbb{E}_x[1-e^{-xs}] = e^{-1}\,. \quad (20)$$

So, out of n nodes, ne^{-1} will become disconnected due to fading. Unconditioning on n, we see that the number of nodes moving out (*i.e.*, becoming disconnected by fading), is $\mathcal{P}o(\pi/(se))$.

On the other hand, we expect the same number of nodes moving *into* the interval $[0,1/s]$ from outside, *i.e.*, getting connected by fading. The probability of a node at x to end up inside is e^{-xs}. Consider the interval $[1/s,(1+a)/s]$, and assume the number of nodes in this interval be fixed to n. These n nodes are again uniformly randomly distributed, and the probability of a node moving inside $[0,1/s]$ is

$$\mathbb{E}_x[e^{-xs}] = \int_{1/s}^{(1+a)/s}\frac{s}{a}e^{-xs}\mathrm{d}x \quad (21)$$

$$= \frac{1}{ae}(1-e^{-a}). \quad (22)$$

$\mathbb{E}n = \pi a/s$, so that on average $\pi(1-e^{-a})/(se)$ nodes move in. As $a \to \infty$, this is exactly compensating for the nodes moving out. Note that this holds for any threshold $1/s$.

Fig. 3 illustrates the situation for 200 nodes randomly chosen from $[0,5]$ with a threshold $s=1$. Before fading, we expect 40 inside. From these, a fraction e^{-1} is moving out (right triangles), the rest stays in (marked by $\times$). From the ones outside, a fraction $(1-e^{-4})(ae) \approx 9\%$ moves in (left triangles), the rest stays out (circles).

What is the probability that all nodes within $[0,1/s]$ move out? Given n nodes, that probability is e^{-n}. So

$$\mathbb{P}[\xi_i > 1/s \;\forall i \in \{i \,|\, x_i < 1/s\}] = \mathbb{E}_n[e^{-n}] = e^{-\pi/s(1-1/e)}\,. \quad (23)$$

What is the probability that no node outside $[0,1/s]$ moves in? Consider an interval $[1/s,(1+a)/s]$. Given n nodes, the probability of none moving inside $[0,1/s]$ is $(1-(1-e^{-a})/(ae))^n$. Since n is Poisson with parameter $\pi a/s$, we have

$$\mathbb{P}[\{i \,|\, \xi_i < 1/s\} \cap \{i \,|\, 1/s < x_i \leqslant (1+a)/s\} = \varnothing] \quad (24)$$

$$= \mathbb{E}_n\left[\left(1-\frac{1-e^{-a}}{ae}\right)^n\right] = e^{-\frac{\pi}{se}(1-e^{-a})}\,. \quad (25)$$

So, as $a \to \infty$, this probability is $e^{-\pi/(se)}$. The probability of both events, all nodes inside the interval moving out and no node outside moving in, is the product, which equals $e^{-\pi/s}$, as expected.

C. Reordering

What is the probability that node $n+m$ has a smaller path loss than node n?

$$\mathbb{P}[x_n/f_n > x_{n+m}/f_{n+m}] = \mathbb{P}\left[\frac{x_n}{x_n+y_m} > \frac{f_n}{f_{n+m}}\right] \quad (26)$$

x_n is Erlang with parameters n and π, y_m is the distance from x_n to x_{n+m} and thus Erlang with parameters m and π, and the cdf of $z := f_n/f_{n+m}$ is $F_z(x) = x/(x+1)$. Hence

$$P_{n,m} = \mathbb{E}_{n,m}\left[\frac{x_n}{2x_n+y_m}\right] = \int_0^\infty\int_0^\infty \frac{x}{2x+y}\frac{\pi^{n+m}x^{n-1}y^{m-1}}{\Gamma(n)\Gamma(m)}e^{-\pi(x+y)}\mathrm{d}x\mathrm{d}y\,.$$

Closed-form expressions include $P_{1,1} = 1-\ln 2 \approx 0.307$, and $P_{1,2} = 3-4\ln 2 \approx 0.227$. Generally $P_{k,k}$ can be determined analytically. For $k=1,2,3,4$, we obtain $1-\ln 2,\; 12\ln 2-8,\; 167/2-120\ln 2,\; 1120\ln 2-776$. For $k=10$, this is 0.3298. Further, $\lim_{k\to\infty} P_{k,k} = 1/3$, which is the probability that an RV $\sim \mathcal{E}(1)$ is larger than a RV $\sim \mathcal{E}(2)$.

In the limit, as $n \to \infty$, we have $P_{n,m} = 1/(m+1)$, which is the probability that a node has the largest fading coefficient among $m+1$ nodes that are at the same distance. Indeed, as $n \to \infty$, $x_{n+m} < x_n(1+\epsilon)$ a.s. for any $\epsilon > 0$ and finite m.

V. Maximum Transmission Distance and Probabilistic Progress

In this section we explore the benefits of fading in terms of transmission distance and address the question which node to transmit to.

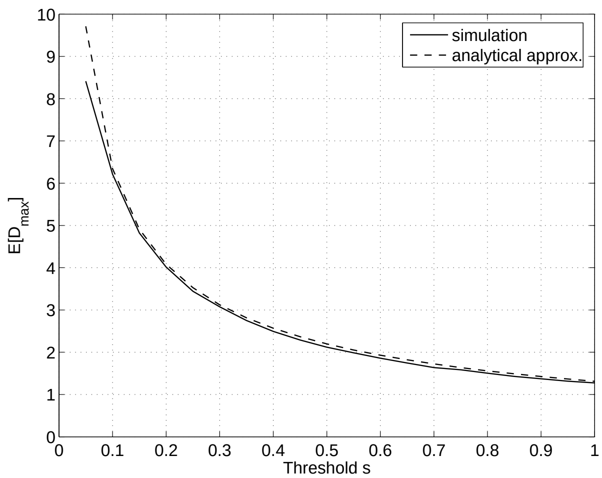

Fig. 4. Expected maximum transmission distances for $s \in [0.05, 1.00]$.

A. Maximum transmission distance

The probability that a node outside $[0, 1/s]$ can be reached is $1-e^{-\pi/(se)}$. But how far is the furthest node that is connected, on average?

The maximum of $\hat{x}$ of n exponential RVs with $n \sim \mathcal{P}o(\pi/s)$ is given by the Gumbel distribution $F_{\hat{x}}(x) = \exp(-\frac{\pi}{s}e^{-sx})$. This is not 0 at $x = 0$, however, since there is no guarantee that there is at least one connected node. Conditioning on $n > 0$, we obtain

$$F_{\hat{x}}(x) = \frac{\exp(\frac{\pi}{s}\,(1-e^{-sx}))-1}{\exp(\frac{\pi}{s})-1}\,, \tag{27}$$

which is a proper distribution on $[0, \infty)$. Since the expectation can only be evaluated numerically, we resort to finding a tractable approximation.

From Lemma 1 follows that the squared distance of an arbitrarily chosen connected node is $\mathcal{E}(1/s)$. Let $\hat{x} = \max_i\{x_i^c\}$. Then the cdf of $\hat{x}$ is $\mathbb{P}[\hat{x} < x] = (1-e^{-xs})^n$, and the mean (given n) is

$$\mathbb{E}\hat{x}|n = \frac{1}{s}\Big(\Psi(n+1)+\gamma\Big) \gtrapprox \frac{1}{s}\Big(\ln n+\gamma\Big)\,. \tag{28}$$

We replace n by its expectation π/s (invoking Jensen's inequality) to obtain the approximation

$$\mathbb{E}\hat{x} \approx \frac{1}{s}\Big(\ln\Big(\frac{\pi}{s}\Big)+\gamma\Big)\,. \tag{29}$$

The expected maximum distance of transmission is approximately the square root of this quantity. It turns out that this approximation is actually a tight upper bound, see Fig. 4. Also compare with Fig. 1, where the most distant node is quite exactly 6 units away ($s = 0.1$).

B. Probabilistic progress

Define the probabilistic progress as the product of expected link distance and success probability. The expected link distance when transmitting to node n is $\sqrt{n}/2$.

$$P_n = \frac{\sqrt{n}}{2}\,\left(\frac{\pi}{\pi+s}\right)^n \tag{30}$$

What is the optimum n? If n were real, this would be maximized for $2\log(1+s/\pi))^{-1}$. Rounding and lower bounding by 1 yields the estimate

$$\hat{n}_{\rm opt} = \max\left\{1, \lceil(2\log(1+s/\pi))^{-1}\rfloor\right\}. \tag{31}$$

For small s and large s, we have

$$n_{\rm opt} = \lceil\frac{\pi}{2s}\rceil\,. \tag{32}$$

In particular, for $s > \pi(\sqrt{2}-1)$, the nearest neighbor has the largest probabilistic progress. For $s = \pi(\sqrt{2}-1)$, $P_1 = P_2 = 1/\sqrt{2}$. For smaller s, the optimum n is larger than 1. In general, $P_n = P_{n+1}$ for $s = \pi(\sqrt{1+1/n}-1)$. So

$$n_{\rm opt} = n \quad\Longleftrightarrow\quad \frac{n+1}{n} < \left(\frac{s}{\pi}+1\right)^2 < \frac{n}{n-1}\,. \tag{33}$$

Finally,

$$n_{\rm opt} = \lceil\big((s/\pi+1)^2-1\big)^{-1}\rceil\,. \tag{34}$$

VI. Concluding Remarks

We have proposed a novel fading model that incorporates the two main types of uncertainty in the channels of wireless ad hoc networks, namely the fading state and the link distance. The model is characterized by the distribution of the path gain. We discussed several applications that demonstrate the analytical tractability of the model, at least for a path loss exponent of 2. For other path loss exponents, the cdf can be written in pseudo-closed-form, as it includes hypergeometric functions.

The effect of fading is *thinning* in the geographical domain. In the path loss domain, the distribution of connected nodes is uniform. Interpreting fading as a stochastic mapping yields additional insights on its effect.

We expect the proposed model to provide better insight into the behavior of large ad hoc networks and to provide a tool to derive new analytical results, *e.g.*, in throughput and outage analyses, connectivity, the design of flooding algorithms, and RSSI-based localization.

Acknowledgments

The support of the NSF (ECS03-29766, CAREER CNS 04-47869 and DMS 505624) is gratefully acknowledged.

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