Delay Scaling in Poisson Networks

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***Abstract*—The local delay, defined as the mean time it takes a node to connect to its nearest neighbor, is a fundamental performance metric in a wireless network. For a network with Poisson distributed nodes, we find its scaling behavior (as a function of the rate of transmission) for different types of nearest-neighbor and the two extreme cases of mobility (infinite mobility and no mobility). Remarkably, it turns out that the level of mobility has hardly any effect on the scaling behavior of the local delay; it affects the pre-constant only.**

I. Introduction

The delay it takes a node to successfully communicate to its nearest neighbor, denotes as the *local delay*, lower bounds all other types of delays such as single-hop, end-to-end, or round-trip delays, which makes it a fundamental quantity to study. If it is infinite, there is little hope that the network provides any useful service to its users. Furthermore, the local delay is a sensitive indicator of the diversity present in a network model; in particular, it measures the interference correlation in network models with interference.

Focusing on the case where nodes are distributed on the two-dimensional plane as a homogeneous Poisson point process (PPP), we analyze the scaling behavior of the local delay in the high- and low-rate regime for the cases of infinite mobility (where a new realization of the PPP is drawn in each time slot) and no mobility (where only a single realization is drawn and the nodes stay fixed forever).

A mathematical framework for the analysis of the local delay in Poisson networks is provided in [1, Sect. 17.5] and [2]. We build on this framework and our earlier work in [3] to obtain concrete results for the scaling of the local delay for all four basic types of nearest-neighbor transmission in the interference-limited case. Complete proofs are available in [4].

II. Network Model

We consider a marked Poisson point process (PPP) $\hat{\Phi} = \{(x_i, t_{x_i})\} \subset \mathbb{R}^2 \times \{0,1\}$, where $\Phi = \{x_i\}$ is a homogeneous PPP of intensity λ, and the marks t_{x_i} are iid Bernoulli with $\mathbb{P}(t=1) = p = 1-q$. A mark of 1 indicates that the node transmits whereas a 0 indicates listening. The large-scale path loss is assumed to be r^α over distance r. A transmission from a node x to a node y is successful if the signal-to-interference ratio (SIR) exceeds a threshold θ. For a transmission from $x \in \Phi$ to $y \in \Phi$, the SIR is

$$\mathrm{SIR}_{xy} \triangleq \frac{S_{xy}}{I_{xy}}\,,$$

where $S_{xy} \triangleq t_x h_{xy} \|x-y\|^{-\alpha}$ and

$$I_{xy} \triangleq \sum_{(z,t_z)\in\hat{\Phi}\setminus\{(x,t_x)\}} t_z h_{zy} \|z-y\|^{-\alpha}\,.$$

This definition implies that the transmit powers are normalized to 1, that $I = \infty$ if $t_y = 1$ (y is itself transmitting), and $\mathrm{SIR} = 0$ if $t_x = 0$. The power fading coefficients h_{xy} are exponential with mean 1 and iid for all $x, y \in \Phi$ and over time (block Rayleigh fading). Time is slotted, and transmission attempts are synchronized.

The (normalized) rate of transmission (or spectral efficiency) $\mathcal{R}$ is, slightly optimistically, assumed to be related to the threshold θ by $\mathcal{R} = \log_2(1+\theta)$.

We consider the two extremes cases of mobility, *infinite mobility*, where a new realization of $\hat{\Phi}$ is drawn in each time slot, and the *static case*, where Φ (the node locations) stays fixed forever. The main event of interest is the event that the typical node, situated at the origin $o \triangleq (0,0) \in \mathbb{R}^2$, successfully connects to its nearest neighbor in a single time slot. In the infinite mobility case, we denote this event by $\mathcal{C}$. In the static case, we first focus on the success event *conditioned on the point process* Φ, which we call $\mathcal{C}_\Phi$. Success events in different time slots are independent, so there is no need to add a time index to this event. Conditioning on Φ having a point at the origin o implies that the relevant probability measure is the Palm probability $\mathbb{P}^o$, and that expectations that involve the point process are taken with respect to $\mathbb{P}^o$ and denoted by $\mathbb{E}^o$ [5]. The partner node y of the origin will be chosen according to one of the four basic cases of nearest-neighbor communication: nearest-receiver transmission (NRT), nearest-neighbor transmission (NNT), nearest-transmitter reception (NTR), and nearest-neighbor reception (NNR).

In the infinitely mobile case, we have $\mathbb{P}^o(\mathcal{C}) = \mathbb{P}^o(\mathrm{SIR}_{uv} > \theta)$ and in the static case, $\mathbb{P}^o(\mathcal{C}_\Phi) = \mathbb{P}^o(\mathrm{SIR}_{uv} > \theta \mid \Phi)$, where $u = o$, $v = y$ for NRT and NNT, and $u = y$, $v = o$ for NTR and NNR. The link distance $R = \|u-v\|$ is itself a (Rayleigh distributed) random variable. The local delay D is the mean number of slots needed until success. Formally,

$$\text{NRT, NNT:} \quad D \triangleq \mathbb{E}^o\left(\min\left\{k \in \mathbb{N}\colon \mathbf{1}_k\big(o \to \mathrm{NN}(o)\big)\right\}\right)$$

$$\text{NTR, NNR:} \quad D \triangleq \mathbb{E}^o\left(\min\left\{k \in \mathbb{N}\colon \mathbf{1}_k\big(\mathrm{NN}(o) \to o\big)\right\}\right),$$

where $\mathbf{1}_k(x \to y) = 1$ if $\mathrm{SIR}_{xy} > \theta$ in time slot k, and 0 otherwise. $\mathrm{NN}(o)$ denotes the origin's nearest node (for NNT and NNR), its nearest receiver (NRT), or its nearest transmitter (NTR).

In the high-mobility case, the local delay is simply $\mathbb{P}^o(\mathcal{C})^{-1}$; in the static case, the success events are only conditionally independent, hence the conditional local delay is geometric with mean $\mathbb{P}^o(\mathcal{C}_\Phi)^{-1}$, and the expectation with respect to the point process yields the local delay:

$$\text{Infinite mobility:} \quad D = \mathbb{P}^o(\mathcal{C})^{-1} \tag{1}$$

$$\text{Static:} \quad D = \mathbb{E}^o_\Phi\left(\frac{1}{\mathbb{P}^o(\mathcal{C}_\Phi)}\right) \tag{2}$$

In our approach for the static case, we will decondition on Φ in two steps, first with respect to the interferers and then with respect to the link distance. This method can be used whenever conditioning on Φ also fixes the link distance. The static NRT and NTR cases as described above do not meet this requirement, as the link distance would also depend on who is transmitting. So we will make a small amendment to the network model in these cases, namely a fixed partitioning of the point process into point processes of potential transmitters and receivers of the appropriate densities.

Considering D as a function of the transmit probability p, we define the *minimum delay* as

$$D_{\min} \triangleq \min_p\{D(p)\}.$$

An important parameter that will be used throughout the paper is the *spatial contention* γ, introduced in [6] and generalized in [7], [8], which measures a network's capability of spatial reuse by quantifying how quickly the success probability of a transmission (over fixed distance) decreases when the density of interfering nodes is increased. It is defined as the slope of the outage probability of a transmission over unit distance as a function of the interferer density at density zero [7, Def. 2]. It depends on the path loss exponent α, the SIR threshold θ, and the network geometry. For a transmission over distance R in a Poisson field of interferers with Rayleigh fading, the success probability is [5]

$$p_{s|R} = \exp(-C(\alpha)\theta^{2/\alpha}p\lambda R^2)\,, \tag{3}$$

where $C(\alpha) \triangleq 2\pi^2/(\alpha\sin(2\pi/\alpha))$. Asymptotically, as the transmitter density $\lambda p \to 0$, $p_{s|1} \sim 1 - C(\alpha)\theta^{2/\alpha}p\lambda$, thus the spatial contention is

$$\gamma = \theta^\delta C(\alpha) = \theta^\delta \pi\frac{\pi\delta}{\sin(\pi\delta)} = \theta^\delta\pi\Gamma(1+\delta)\Gamma(1-\delta)\,, \tag{4}$$

where $\delta \triangleq 2/\alpha$. For $\alpha = 4$, $\gamma = \sqrt{\theta}\pi^2/2$, and for $\alpha = 3$, $\gamma = \theta^{2/3}4\pi^2/(3\sqrt{3})$. As $\alpha \downarrow 2$, $\gamma \to \infty$, since the interference is infinite a.s. for $\alpha \le 2$.

The asymptotic regimes considered are $\theta \to 0$ and $\theta \to \infty$, or, equivalently, $\mathcal{R} \to 0$ and $\mathcal{R} \to \infty$. Since γ increases monotonically with θ, we may also write $\gamma \to 0$ and $\gamma \to \infty$.

III. Infinitely Mobile Networks

The analysis of the four cases of nearest-neighbor transmission is based on [4, Lemma 1].

A. Nearest-receiver transmission (NRT)

In this case, the destination node is always listening, so

$$\mathbb{P}^o(\mathcal{C}^{\mathrm{NRT}} \mid R) = p\exp(-\gamma p\lambda R^2)\,.$$

Since the point process of receivers has intensity λq, the link distance R is is Rayleigh distributed with mean $1/(2\sqrt{q\lambda})$ [9], *i.e.*, $f_R(r) = 2q\lambda\pi r\exp(-q\lambda\pi r^2)$. Hence

$$\mathbb{P}^o(\mathcal{C}^{\mathrm{NRT}}) = \mathbb{E}\big(\mathbb{P}^o(\mathcal{C}^{\mathrm{NRT}} \mid R)\big) = \frac{p\pi}{\pi + \gamma p q^{-1}}$$

and

$$D^{\mathrm{NRT}} = \frac{1}{\mathbb{P}^o(\mathcal{C}^{\mathrm{NRT}})} = \frac{1}{p} + \frac{\gamma}{\pi q}\,. \tag{5}$$

Hence the minimum delay only depends on the spatial contention:

$$D^{\mathrm{NRT}}_{\min} = 1 + 2\sqrt{\frac{\gamma}{\pi}} + \frac{\gamma}{\pi}\,. \tag{6}$$

B. Nearest-neighbor transmission (NNT)

Let y be the typical node's nearest neighbor and $R = \|y\|$. In this case R is distributed as $f_R(r) = 2\lambda\pi r\exp(-\lambda\pi r^2)$, and having the nearest neighbor at distance R implies that *there is no interferer in the ball $B_o(R)$ centered at o with radius* R. So y sees the conditional interference, conditioned on the disk $B_o(R)$ being empty, and the interference observed at the receiver is smaller than at a typical node, *i.e.*, $\gamma^{\mathrm{NNT}} < \gamma$.

Theorem 1 *The success probability of nearest-neighbor transmission given R is*

$$\mathbb{P}^o(\mathcal{C} \mid R) = pq\exp(-\gamma^{\mathrm{NNT}}p\lambda R^2)\,, \tag{7}$$

with γ^{NNT} denoting the spatial contention for nearest-neighbor transmission. γ^{NNT} is bounded as follows:

(a) $\gamma > \gamma^{\mathrm{NNT}} > \gamma - \pi$, *where γ is the unconditioned spatial contention given in* (4). *Also,* $\lim_{\theta\to\infty} \gamma - \gamma^{\mathrm{NNT}} = \pi$.

(b) Letting $\delta = 2/\alpha$ and denoting by $H_\delta(x)$ the Gauss hypergeometric function

$$H_\delta(x) \triangleq {}_2F_1(1,\delta;1+\delta;x)\,,$$

we have

$$\gamma - \gamma^{\mathrm{NNT}} < \frac{2\pi}{3}H_\delta(-2^\alpha/\theta) + \frac{\pi}{4}H_\delta(-3^{\alpha/2}/\theta) + \frac{\pi}{6}H_\delta(-2^{\alpha/2}/\theta) + \frac{\pi}{6}H_\delta(-1/\theta) \tag{8}$$

$$\gamma - \gamma^{\mathrm{NNT}} > \frac{\pi}{2}H_\delta(-3^{\alpha/2}/\theta) + \frac{\pi}{6}H_\delta(-2^{\alpha/2}/\theta) + \frac{\pi}{12}H_\delta(-1/\theta) \tag{9}$$

The proof is omitted due to space constraints—see [4]. Essentially it requires careful bounding of the Laplace transform of the interference at the origin o stemming from transmitters outside $B_o(R)$.

While the absolute gain in the spatial contention increases with θ, the *relative gain* decreases with θ and approaches 1 as $\theta \to \infty$. In fact, the upper bound on the difference (8) results

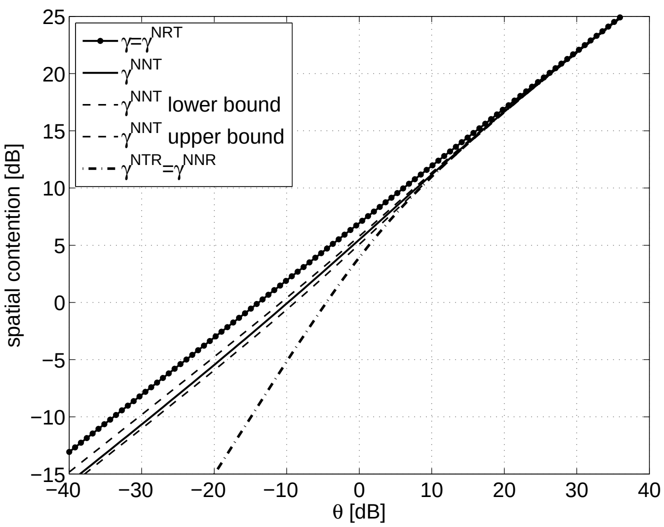

Fig. 1. Spatial contention in the infinitely mobility case for $\alpha = 4$. The bounds are obtained from (8) and (9).

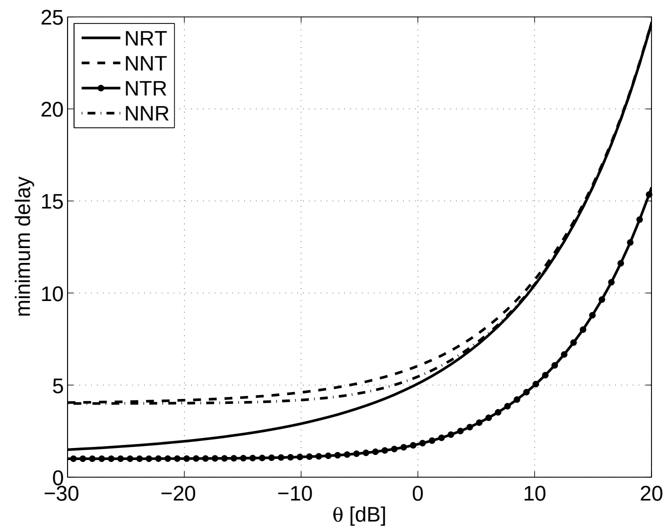

Fig. 2. Minimum achievable local delay in the infinite-mobility case for $\alpha = 4$.

in a lower bound on the ratio $\gamma^{\rm NNT}/\gamma$ that approaches $1/2$ as $\theta \to 0$. This follows from

$$H_\delta(-x) \sim x^{-\delta} \frac{\pi\delta}{\sin(\pi\delta)}\,, \quad x \to \infty\,. \tag{10}$$

In particular, for $\alpha = 4$ ($\delta = 1/2$),

$$\lim_{\theta\downarrow 0} \frac{H_{1/2}(-t/\theta)}{\sqrt{\theta}} = \frac{\pi}{2\sqrt{t}}\,.$$

Using this limit in (8) yields $\gamma^{\rm NNT}/\gamma = \gamma^{\rm NNT}/(\sqrt{\theta}\pi^2/2) > 1/2$. Applied to (9), we obtain $\gamma^{\rm NNT}/\gamma < 2/3$.

The local delay follows from integration with respect to R, which is Rayleigh with mean $1/(2\sqrt{\lambda})$ in this case:

$$D^{\rm NNT} = \frac{p\gamma^{\rm NNT} + \pi}{\pi p q} = \frac{1}{pq} + \frac{\gamma^{\rm NNT}}{\pi q} \tag{11}$$

The delay is composed of two parts, the *access delay* $1/(pq)$, which is the time it takes for the transmitter to transmit and the receiver to listen, and the *service time*, which is proportional to the spatial contention $\gamma^{\rm NNT}$. Compared with the nearest-receiver case, we observe the following:

Corollary 1 *For a fixed p and finite θ, $D^{\rm NRT} < D^{\rm NNT}$. Asymptotically, the delays are identical,* i.e., $D^{\rm NRT} \uparrow D^{\rm NNT}$ *as* $\theta \to \infty$.

Proof: The maximum difference $\gamma - \gamma^{\rm NNT}$ is π, achieved as $\theta \to \infty$. Since $p(\gamma - \pi) = p\gamma + \pi q$, the two delays are then identical. For finite θ, the difference is smaller and thus $D^{\rm NRT} < D^{\rm NNT}$. ∎

So at high rates, the gain in the spatial contention in the NNT case is exactly offset by the fact that the nearest neighbor is only listening with probability q. The minimum delay is

$$D^{\rm NNT}_{\rm min} = \frac{g^2}{g + 2(1 - \sqrt{1+g})}\,, \tag{12}$$

where $g = \gamma^{\rm NNT}/\pi$. As $\theta \to 0$,

$$D^{\rm NNT}_{\rm min} \sim 4 + \frac{2\gamma^{\rm NNT}}{\pi} \sim 4 + c\gamma/\pi\,, \tag{13}$$

where $c \in (1, 4/3)$. Fig. 2 shows the optimum transmit probability p and the minimum delay. As $\theta \to 0$, the optimum transmit probability for NRT approaches 1, whereas for NNT, it approaches $1/2$. The difference is due to the fact that the *receiver* density is less critical in NRT. In the delay plot, it is observed that $D^{\rm NNT} > 4$, since $1/(pq)$ is at least 4.

C. Nearest-transmitter reception (NTR)

Next we consider the case where the typical node at o receives from its nearest transmitter, say y. This implies that there are no interferers in the disk of radius $R = \|y\|$ around the receiver. Using the hypergeometric function defined in Thm. 1, we obtain $\gamma^{\rm NTR} = \gamma - \pi H_\delta(-1/\theta)$ and

$$\mathbb{P}^o(\mathcal{C}^{\rm NTR} \mid R) = q \exp\left(-\lambda p \pi R^2(\gamma - \pi H_\delta(-1/\theta))\right)\,.$$

As $\theta \to \infty$, the gain in the spatial contention approaches π, as in the NNT case, hence $\gamma^{\rm NTR} \sim \gamma - \pi \sim \gamma$. This is to be expected, since for large θ, an area much larger than the disk of radius R needs to be free of interferers, so it does not matter whether the disk is centered at the receiver or translated by R. As $\theta \to 0$, it follows from (10) that $\pi H_\delta(-1/\delta) \to \gamma$, which indicates that the spatial contention vanishes faster than θ^δ. In fact,

$$\gamma^{\rm NTR} \sim \frac{2\pi}{\alpha - 2}\theta = \frac{\delta\pi}{1-\delta}\,, \quad \theta \to 0\,. \tag{14}$$

The two asymptotic regimes are clearly visible in Fig. 1. For $\theta < 1$, the slope is about one (or 10dB/decade), whereas for $\theta > 1$ it is about 5dB/decade.

In the NTR case, R is distributed as $f_R(r) = 2\pi p\lambda r \exp(-p\lambda\pi r^2)$. It follows that

$$D^{\rm NTR} = \frac{1}{q} + \frac{\gamma^{\rm NTR}}{\pi q}\,, \tag{15}$$

which is monotonically decreasing as $p \downarrow 0$. This indicates that, without noise, the benefit of reducing the interferer density compensates for the increased transmission distance. (For $p = 0$, the delay is undefined since there is no nearest transmitter in this case.) For small θ, we have the particularly simple result

$$D_{\min}^{\mathrm{NTR}} \sim 1 + \frac{2}{\alpha - 2}\theta = 1 + \frac{\delta}{1-\delta}\theta\,, \quad \theta \to 0\,.$$

D. Nearest-neighbor reception (NNR)

This case is quite similar to NTR, with the difference that the nearest neighbor is at distance $1/(2\sqrt{\lambda})$ on average and that the delay increases by a factor $1/p$ since the nearest neighbor only transmits with probability p. So $\gamma^{\mathrm{NNR}} = \gamma^{\mathrm{NTR}}$, and

$$D^{\mathrm{NNR}} = \frac{1}{pq} + \frac{\gamma^{\mathrm{NNR}}}{\pi q}\,.$$

The expression has the same form as the one for NNT, the only difference being the spatial contention. So the minimum delay follows from (12), with γ^{NNR} instead of γ^{NNT}. As $\theta \to 0$,

$$D_{\min}^{\mathrm{NNR}} \sim 4 + \frac{4}{\alpha - 2}\theta = 4 + \frac{2\delta}{1-\delta}\theta\,. \tag{16}$$

The results for all four cases are shown in Fig. 2.

IV. Static Networks

In the static case, only a single realization of the point process is drawn. Comparing (1) and (2), we obtain a bound on the local delay in the static case by Jensen's inequality: $D > \mathbb{P}(\mathcal{C})^{-1}$. Not surprisingly, this bound is often very loose. In particular, the actual delay may be infinite while the lower bound is always finite. The reason is the correlation of the interference in the static case [10]. The analysis of the static case makes use of [4, Lemma 2], which provides the expected inverse conditional Laplace transform of the interference $\mathbb{E}^o\left(\frac{1}{\mathcal{L}_I(s|\Phi)}\right)$ that is needed to calculate (2).

A. Nearest-receiver transmission (NRT)

Here we consider the case where the partitioning into potential transmitters and receivers is fixed, *i.e.*, the transmitters are chosen from Φ with probability p, as before, but there exists another, independent PPP of receivers Φ_{r} of intensity $\lambda_{\mathrm{r}} = q\lambda$. So, in this model, the nodes in Φ that do not transmit are not available as receivers.

In this case, the local delay as a function of the transmit probability p is [4, Thm. 2]

$$D^{\mathrm{NRT}} = \frac{1}{p}\,\frac{\pi}{\pi - \gamma p q^{\delta-2}}\,, \quad pq^{\delta-2} < \pi/\gamma. \tag{17}$$

At $pq^{\delta-2} = \pi/\gamma$, the local delay undergoes a *phase transition*, *i.e.*, the local delay becomes infinite, as first observed in [2].

Eqn. (17) cannot be minimized in closed form. However, it can be shown that asymptotically,

$$D_{\min}^{\mathrm{NRT}} \sim 4\left(1 - \frac{1}{\alpha} + \frac{\gamma}{\pi}\right)\,, \quad \gamma \to \infty\,. \tag{18}$$

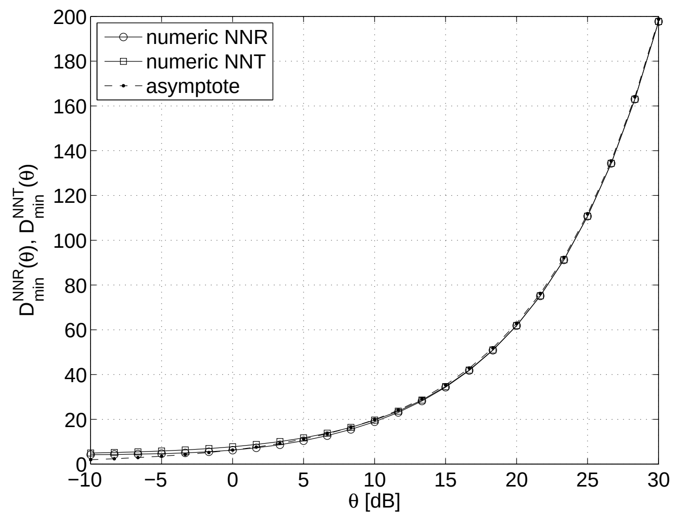

Fig. 3. The minimum local delay for the static NNT and NNR cases as a function of the SIR threshold θ for $\alpha = 4$. The curves are very close; at small θ, the delay for NNT is slightly larger. The asymptote is $4\gamma/\pi$, per (20), in both cases. It is quite tight as soon as $\theta > 1$.

On the other hand, as $\gamma \to 0$, a careful examination of the asymptotic behavior of upper and lower bound yields

$$D_{\min}^{\mathrm{NRT}} = 1 + O(\gamma^{\max\{1/3,1/\alpha\}})\,, \quad \gamma \to 0\,. \tag{19}$$

More precisely,

$$\lim_{\theta\to 0}(D_{\min}^{\mathrm{NRT}} - 1)\gamma^{-1/\alpha} < \pi^{1/\alpha}$$
$$\lim_{\theta\to 0}(D_{\min}^{\mathrm{NRT}} - 1)\gamma^{-1/3} < (2\pi)^{1/3}.$$

B. Nearest-neighbor transmission (NNT)

Using similar techniques for the integration of the inverse conditional Laplace transform of the interference as in the infinitely mobile NNT case, we find

$$D_{\min}^{\mathrm{NNT}} \sim \frac{4\gamma}{\pi}\,, \quad \gamma \to \infty\,. \tag{20}$$

As $\gamma \to 0$, $D_{\min}^{\mathrm{NNT}} = 4 + \Theta(\gamma)$. The numerically obtained $D_{\min}^{\mathrm{NNT}}$ and its asymptotic behavior are shown in Fig. 3.

Generally, as $\gamma \to \infty$, there is no difference between the NRT and NNT in terms of interference, but only in the availability of the destination node as a receiver and in the link distance distribution.

C. Nearest-transmitter reception (NTR)

Similarly to the static NRT case, we pre-partition transmitters and receivers. In this case, receivers do not matter (except for the typical receiver considered). We take a fixed point process of transmitters of intensity λp, which implies there is no actual ALOHA involved, or, in terms of the marked point process $\hat{\Phi}$, we take the marks to be fixed also.

Following similar steps as in the previous cases, we obtain

$$D^{\mathrm{NTR}} = \frac{1}{q}\frac{\pi}{\pi - \gamma/q^{1-\delta} + \kappa^{\mathrm{NTR}}}\,, \quad \gamma/q^{1-\delta} - \kappa^{\mathrm{NTR}} < \pi, \tag{21}$$

where $\kappa^{\mathrm{NTR}} = \frac{\pi}{q} H_\delta(-1/(\theta q))$. Since the delay is monotonically decreasing as $p \downarrow 0$ (and thus $q \uparrow 1$),

$$D_{\min}^{\mathrm{NTR}} = \frac{\pi}{\pi - \gamma + \kappa^{\mathrm{NTR}}} = \frac{1}{1 + H_\delta(-1/\theta) - \gamma/\pi} \,. \tag{22}$$

What is interesting about this case is that there is a *hard* phase transition in the sense that a finite local delay cannot be achieved for any p as soon as θ exceeds some critical value θ_{c}, determined by $1 + H_\delta(-1/\theta_{\mathrm{c}}) - \gamma/\pi = 0$. While reducing p reduces the interference, it also increases the link distance in proportion to $p^{-1/2}$, and the net gain is negative if θ is larger than θ_{c}. For $\alpha = 4$, $\theta_{\mathrm{c}} \approx 1.351$. So, the maximum rate that can be supported for finite local delay is $\mathcal{R}_{\max} \approx 1.2333$. As α decreases, θ_{c} decreases also. Since $\alpha < 4.95$ in most environments, the rate supported by NTR cannot exceed 4/3 bits/s/Hz. Therefore the high-θ asymptotics do not exist. For small θ, it follows from (22) that

$$D_{\min}^{\mathrm{NTR}} \sim 1 + \frac{2}{\alpha - 2}\theta = 1 + \frac{\delta}{1-\delta}\theta \,, \quad \theta \to 0 \,. \tag{23}$$

D. Nearest-neighbor reception (NNR)

In this case, it is not difficult to see that the optimum transmit probability tends to $1/2$ as $\theta \to 0$, since the limiting factor is not interference but the availability of transmit-receive pairs. It follows that

$$D_{\min}^{\mathrm{NNR}} \sim 4\left(1 + \frac{\delta/2}{1-\delta}\theta\right) \,, \quad \theta \to 0. \tag{24}$$

The minimum delay is plotted in Fig. 3.

V. Asymptotic Delays

We first summarize the results on the asymptotic delay.

Theorem 2 *As $\theta \to \infty$, the minimum local delay in all four infinitely mobile cases scales as γ/π or*

$$D_{\min} \sim \theta^\delta \frac{\pi\delta}{\sin(\pi\delta)} \,.$$

In the static NRT, NNT, and NNR cases, the scaling behavior is $4\gamma/\pi$ or

$$D_{\min} \sim 4\theta^\delta \frac{\pi\delta}{\sin(\pi\delta)} \,.$$

The exception is the static NTR case, where the delay becomes infinite for all values of p as soon as θ exceeds some critical value θ_{c}.

As $\theta \to 0$, the scaling laws of the minimum local delay are listed in Table I.

Expressed in terms of the transmission rate, the scaling behavior can be summarized as follows.

Corollary 2 *Irrespective of the level of mobility in the network and the choice of the nearest-neighbor transmission scheme, the minimum local delay scales at high rates as*

$$D_{\min} = \Theta(2^{\delta\mathcal{R}}) \,, \quad \mathcal{R} \to \infty \,.$$

	Infinite mobility	Static
NRT	$1 + 2\sqrt{\gamma/\pi}$; $1 + \Theta(\theta^{1/\alpha})$	$1 + O(\gamma^{\max\{1/3, 1/\alpha\}})$
NNT	$4 + c\gamma/\pi$; $4 + \Theta(\theta^\delta)$	$4 + \Theta(\gamma)$; $4 + \Theta(\theta^\delta)$
NTR	$1 + \frac{\delta}{1-\delta}\theta$	$1 + \frac{\delta}{1-\delta}\theta$
NNR	$4 + \frac{2\delta}{1-\delta}\theta$	$4 + \frac{2\delta}{1-\delta}\theta$

TABLE I
Scaling behavior of the minimum local delay as $\theta \to 0$. Without a O or Θ symbol, the asymptotic results are sharp, *i.e.*, "$\sim$". The constant c depends on α and assumes values $c \in (1, 4/3)$.

Again the exception is the static NTR *case. As $\mathcal{R} \to 0$,*

$$D_{\min} = K + O(\mathcal{R}^{1/\alpha}) \quad \text{and} \quad D_{\min} = K + \Omega(\mathcal{R}) \,,$$

where $K = 1$ for NRT and NTR and $K = 4$ for NNT and NNR.

The constant K is the minimum achievable access delay. If one of the nodes is known to be transmitting or listening, $K = 1$.

VI. Conclusions

We have provided a detailed analysis of the asymptotic behavior of the local delay in Poisson networks. The main conclusion is that at high rates, its scaling *is the same for all levels of mobility*, while for low rates, the scaling behavior depends slightly on the communication scheme. In all cases, however, the delay goes to zero faster than $\mathcal{R}$ but at most as fast as $\mathcal{R}^{1/\alpha}$.

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