SIR Asymptotics in General Cellular Network Models

Radha Krishna Ganti
Department of Electrical Engineering
Indian Institute of Technology, Madras
Hyderabad, India 502205
rganti@ee.iitm.ac.in

Martin Haenggi
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, USA
mhaenggi@nd.edu

***Abstract*—It has recently been observed that the SIR distributions of a variety of cellular network models and transmission techniques look very similar in shape. As a result, they are well approximated by a simple horizontal shift of the distribution of the most tractable model, the Poisson point process. This paper makes a first step towards explaining this remarkable property by showing that the asymptotics of the SIR distribution near 0 and near infinity can only differ by a constant.**

***Index Terms*—Cellular networks, stochastic geometry, signal-to-interference ratio, Poisson point processes.**

I. Introduction

A. Motivation

The distribution of the signal-to-interference ratio (SIR) is a key quantity in the analysis and design of interference-limited wireless systems. Here we focus on general single-tier cellular networks where users are connected to the strongest (nearest) base station (BS). Let $\Phi \subset \mathbb{R}^2$ be a point process representing the locations of the BSs and let $x_0 \in \Phi$ be the serving BS of the typical user at the origin, *i.e.*, define $x_0 \triangleq \arg\min\{x \in \Phi \colon \|x\|\}$. Assuming all BSs transmit at the same power level, the downlink SIR is given by

$$\mathsf{SIR} \triangleq \frac{S}{I} = \frac{h_{x_0}\ell(x_0)}{\sum_{x\in\Phi\setminus\{x_0\}} h_x \ell(x)}, \tag{1}$$

where (h_x) are iid random variables representing the fading and ℓ is the path loss law. The complementary cumulative distribution (ccdf) of the SIR is

$$\bar{F}_{\mathsf{SIR}}(\theta) \triangleq \mathbb{P}(\mathsf{SIR} > \theta). \tag{2}$$

Under the SIR threshold model for reception, the ccdf of the SIR can also be interpreted as the success probability of a transmission, *i.e.*, $p_{\mathrm{s}}(\theta) \equiv \bar{F}_{\mathsf{SIR}}(\theta)$.

In the case where Φ forms a homogeneous Poisson point process (PPP), Rayleigh fading, and $\ell(x) = \|x\|^{-\alpha}$, the success probability was determined in [1]. It can be expressed in terms of the Gaussian hypergeometric function ${}_2F_1$ as [2]

$$p_{\mathrm{s,PPP}}(\theta) = \frac{1}{{}_2F_1(1, -\delta; 1-\delta; -\theta)}, \tag{3}$$

where $\delta \triangleq 2/\alpha$. For $\alpha = 4$, remarkably, this simplifies to

$$p_{\mathrm{s,PPP}}(\theta) = \frac{1}{1+\sqrt{\theta}\arctan\sqrt{\theta}}.$$

For all other cases, the success probability is intractable or can at best be expressed using combinations of infinite sums and integrals. Hence there is an important need for techniques that yield good approximations of the SIR distribution for non-Poisson networks.

B. Asymptotic SIR gains and the MISR

It has recently been observed in [3], [4] that the SIR ccdfs for different point processes and transmission techniques (e.g., BS cooperation or silencing) *appear to be merely horizontally shifted versions of each other* (in dB), as long as the resulting diversity gain is the same.

Consequently, the success probability of a network model can be accurately approximated by that of a reference network model by scaling the threshold θ by this SIR gain factor (or shift in dB) G, *i.e.*,

$$p_{\mathrm{s}}(\theta) \approx p_{\mathrm{s,ref}}(\theta/G).$$

Formally, the horizontal gap at target probability p is defined as

$$G_{\mathrm{p}}(p) \triangleq \frac{\bar{F}^{-1}_{\mathsf{SIR}}(p)}{\bar{F}^{-1}_{\mathsf{SIR}_{\mathrm{ref}}}(p)}, \quad p \in (0,1), \tag{4}$$

where $\bar{F}^{-1}_{\mathsf{SIR}}$ is the inverse of the ccdf of the SIR and p is the success probability where the gap is measured. It is often convenient to consider the gap as a function of θ, defined as

$$G(\theta) \triangleq G_{\mathrm{p}}(p_{\mathrm{s,ref}}(\theta)) = \frac{\bar{F}^{-1}_{\mathsf{SIR}}(p_{\mathrm{s,ref}}(\theta))}{\theta}. \tag{5}$$

Due to its tractability, the PPP is a sensible choice as the reference model[1].

So the main focus of this paper are the asymptotic gains relative to the PPP, defined as follows.

Definition 1 (Asymptotic gains relative to PPP)**.** *The asymptotic gains (whenever the limits exist) G_0 and G_∞ are defined as*

$$G_0 \triangleq \lim_{\theta\to 0} G(\theta); \quad G_\infty \triangleq \lim_{\theta\to\infty} G(\theta), \tag{6}$$

where the PPP is used as the reference model.

[1]This is why the method of approximating an SIR distribution by a shifted version of the PPP's SIR distribution is called ASAPPP—"Approximate SIR analysis based on the PPP" [5].

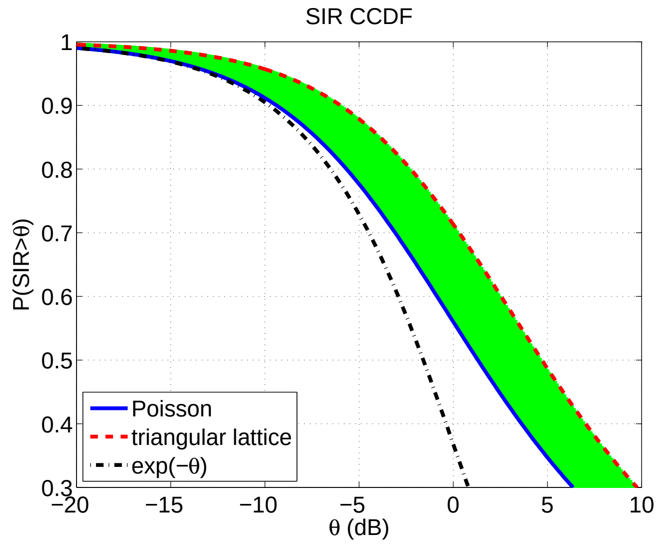

Figure 1. The SIR distributions for the PPP (solid) and the triangular lattice (dashed) for $\alpha = 4$ and the lower bound (which is asymptotically tight) $e^{-\theta}$ for the PPP (dash-dotted). The horizontal gap between the SIR distributions of the PPP and the triangular lattice is quite exactly 3.4 dB for a wide range of θ values. The shaded band indicates the region in which the SIR distributions for all stationary point process fall that are more regular than the PPP.

C. Prior work

Some insights on G_0 are available from prior work. In [4] it is shown that G_0 is closely connected to the mean interference-to-signal ratio (MISR). The MISR is the mean of the interference-to-(average)-signal ratio ISR, defined as

$$\bar{\mathsf{ISR}} \triangleq \frac{I}{\mathbb{E}_h(S)},$$

where $\mathbb{E}_h(S) = \mathbb{E}(S \mid \Phi)$ is the mean received signal power averaged only over the fading. Not unexpectedly, the calculation of the MISR for the PPP is relatively straightforward and yields $\mathsf{MISR}_{\mathrm{PPP}} = 2/(\alpha - 2)$.

In general, the success probability can be expressed as

$$p_{\mathrm{s}}(\theta) = \mathbb{E}\bar{F}_h(\theta\,\bar{\mathsf{ISR}}), \tag{7}$$

where $\bar{F}_h$ is the ccdf of the fading random variables. For Rayleigh fading, $\bar{F}_h(x) = e^{-x}$ and thus $p_{\mathrm{s}}(\theta) \sim 1 - \theta\,\mathsf{MISR}$, $\theta \to 0$, resulting in

$$G_0 = \frac{\mathsf{MISR}_{\mathrm{PPP}}}{\mathsf{MISR}} = \frac{2}{\alpha - 2}\,\frac{1}{\mathsf{MISR}}$$

and

$$p_{\mathrm{s}}(\theta) \sim p_{\mathrm{s,PPP}}(\theta/G_0), \quad \theta \to 0.$$

So asymptotically the shifted ccdf of the PPP is exact.

An example is shown in Fig. 1, where $\alpha = 4$, which results in $\mathsf{MISR}_{\mathrm{PPP}} = 1$, while for the triangular lattice $\mathsf{MISR}_{\mathrm{tri}} = 0.457$. Hence the horizontal shift is $\mathsf{MISR}_{\mathrm{PPP}}/\mathsf{MISR}_{\mathrm{tri}} = 3.4$ dB. For Rayleigh fading, we also have the relationship $p_{\mathrm{s}}(\theta) = \mathcal{L}_{\bar{\mathsf{ISR}}}(\theta) \gtrsim e^{-\theta\,\mathsf{MISR}}$ by Jensen's inequality, also shown in the figure. Here '$\gtrsim$' is a lower bound with asymptotic equality.

D. Contributions

In this paper, we derive similar results for G_∞. In particular, we show that for all stationary point process models and any type of fading, the tail of the SIR distribution always scales as $\theta^{-\delta}$, *i.e.*, we have $p_{\mathrm{s}}(\theta) \sim c\theta^{-\delta}$ where the constant c captures the effects of the network geometry and fading. The asymptotic gain follows as

$$G_\infty = \left(\frac{c}{c_{\mathrm{PPP}}}\right)^{1/\delta}, \tag{8}$$

and we have

$$p_{\mathrm{s}}(\theta) \sim p_{\mathrm{s,PPP}}(\theta/G_\infty), \quad \theta \to \infty.$$

II. System Model

The base station locations are modeled as a stationary point process $\Phi \subset \mathbb{R}^2$. Without loss of generality, we assume that the typical user is located at the origin o. The path loss between the typical user and a BS at $x \in \Phi$ is given by $\ell(x) = \|x\|^{-\alpha}$, $\alpha > 2$. Let $\bar{F}_h$ denote the ccdf of the iid fading random variables. In this paper, we will assume unit mean fading random variables.

We assume nearest-BS association, wherein a user is served by the closest BS. Let x_0 denote the closest BS to the typical user at the origin, and define $R \triangleq \|x_0\|$ and $\Phi^! = \Phi \setminus \{x_0\}$. With the nearest BS association rule, the downlink SIR (1) of the typical user can be expressed as

$$\mathsf{SIR} = \frac{hR^{-\alpha}}{\sum_{x\in\Phi^!} h_x \ell(x)}. \tag{9}$$

Further, let $b(o, r)$ be the open disk of radius r at o.

III. Asymptotics of SIR Distribution

In this section, we analyze the asymptotics of the SIR distribution. The analysis of SIR in (9) is complicated by the fact that the first contact distance R and the interference I are correlated random variables.

A. The expected fading-to-interference ratio (EFIR)

The constants defining the asymptotic gain G_∞ in (8) are closely related to a quantity termed *expected fading-to-interference ratio (EFIR)*, which plays a similar role for $\theta \to \infty$ as the MISR does for $\theta \to 0$.

Definition 2 (Expected fading-to-interference ratio (EFIR))**.** *For a point process Φ, let $I_\infty = \sum_{x\in\Phi} h_x \ell(x)$ and let h be a fading random variable independent of all (h_x). The* expected fading-to-interference ratio *(EFIR) is defined as*

$$\mathsf{EFIR} \triangleq \left(\lambda\pi\mathbb{E}^{!o}\left[\left(\frac{h}{I_\infty}\right)^{\delta}\right]\right)^{1/\delta}, \tag{10}$$

where $\mathbb{E}^{!o}$ is the expectation with respect to the reduced Palm measure of Φ.

Lemma 1 (EFIR for the PPP)**.** *For the PPP, with arbitrary fading,*

$$\mathsf{EFIR}_{\mathrm{PPP}} = (\operatorname{sinc}\delta)^{1/\delta}. \tag{11}$$

Proof: The term $\mathbb{E}^{!o}[I_\infty^{-\delta}]$ in (10) can be calculated by taking the expectation of the following identity which follows from the definition of the gamma function $\Gamma(x)$.

$$I_\infty^{-\delta} \equiv \frac{1}{\Gamma(\delta)} \int_0^\infty e^{-sI_\infty} s^{-1+\delta} \mathrm{d}s.$$

Hence

$$\mathbb{E}^{!o}[I_\infty^{-\delta}] = \frac{1}{\Gamma(\delta)} \int_0^\infty \mathcal{L}_{I_\infty}^{!o}(s) s^{-1+\delta} \mathrm{d}s. \tag{12}$$

From Slivnyak's theorem [6, Thm. 8.10], $\mathbb{E}^{!o} \equiv \mathbb{E}$ for the PPP, so we can replace $\mathcal{L}_{I_\infty}^{!o}(s)$ by the unconditioned Laplace transform $\mathcal{L}_{I_\infty}(s)$, which is well known for the PPP and given by [7]

$$\mathcal{L}_{I_\infty}(s) = \exp(-\lambda\pi\mathbb{E}[h^\delta]\Gamma(1-\delta)s^\delta).$$

From (12), we have

$$\begin{aligned}\mathbb{E}[I_\infty^{-\delta}] &= \frac{1}{\Gamma(\delta)} \int_0^\infty e^{-\lambda\pi\mathbb{E}[h^\delta]\Gamma(1-\delta)s^\delta} s^{-1+\delta} \mathrm{d}s \\ &= \frac{1}{\lambda\pi\mathbb{E}(h^\delta)\Gamma(1-\delta)\Gamma(1+\delta)} = \frac{\operatorname{sinc}\delta}{\lambda\pi\mathbb{E}(h^\delta)}.\end{aligned}$$

So $\lambda\pi\mathbb{E}^{!o}(I_\infty^{-\delta})\mathbb{E}(h^\delta) = \operatorname{sinc}\delta$, and the result follows. ■

Remarkably, $\mathsf{EFIR}_{\mathrm{PPP}}$ only depends on the path loss exponent. It can be closely approximated by $\mathsf{EFIR}_{\mathrm{PPP}} \approx 1-\delta$.

B. Main result

Let $f(R,\Phi^!)$ be a positive function of the distance R and the point process $\Phi^!$. The average $\mathbb{E}[f(R,\Phi^!)]$ can be evaluated using the joint distribution of R and $\Phi^!$, which is known only for a few spatial point processes. Thus we introduce an alternative representation of $f(R,\Phi^!)$ that is easier to analyze.

The indicator variable $\mathbf{1}(\Phi(b(o,\|x\|))=0)$, $x \in \Phi$, equals one only when $x = x_0$ and zero otherwise. Hence it follows that

$$f(R,\Phi^!) \equiv \sum_{x\in\Phi} f(\|x\|, \Phi\setminus\{x\})\mathbf{1}(\Phi(b(o,\|x\|))=0). \tag{13}$$

This representation of $f(R,\Phi^!)$ allows for computing the expectation of $f(R,\Phi^!)$ using the Campbell-Mecke theorem [6, Thm. 8.2]. We will use it to analyze the asymptotics of the ccdf $\bar{F}_{\mathsf{SIR}}$ of the SIR (or, equivalently, the success probability $p_{\mathrm{s}}(\theta)$) as $\theta \to \infty$.

Theorem 1. *For all BS stationary point processes Φ, where the typical user is served by the nearest BS,*

$$p_{\mathrm{s}}(\theta) \sim \left(\frac{\theta}{\mathsf{EFIR}}\right)^{-\delta}, \quad \theta \to \infty.$$

Proof: From (9), we have $p_{\mathrm{s}}(\theta) = \mathbb{E}\bar{F}_h(\theta R^\alpha I)$. Using the representation given in (13), the success probability equals

$$\begin{aligned}&\mathbb{E}\sum_{x\in\Phi} \bar{F}_h\left(\theta\|x\|^\alpha \sum_{y\in\Phi\setminus\{x\}} h_y\|y\|^{-\alpha}\right)\mathbf{1}\big(\Phi(b(o,\|x\|))=0\big) \\ &\overset{(a)}{=} \lambda \int_{\mathbb{R}^2} \mathbb{E}_o^! \bar{F}_h\Big(\theta\|x\|^\alpha \sum_{y\in\Phi_x} h_y\|y\|^{-\alpha}\Big)\mathbf{1}(b(o,\|x\|) \text{ empty})\mathrm{d}x,\end{aligned}$$

where (a) follows from the Campbell-Mecke theorem and $\Phi_x \triangleq \{y\in\Phi\colon y+x\}$ is a translated version of Φ. Substituting $x\theta^{\delta/2} \mapsto x$,

$$\begin{aligned}&= \lambda\theta^{-\delta} \int_{\mathbb{R}^2} \mathbb{E}^{!o}\bar{F}_h\Big(\|x\|^\alpha \sum_{y\in\Phi_{x\theta^{-\delta/2}}} h_y\|y\|^{-\alpha}\Big) \\ &\qquad \cdot \mathbf{1}(b(o,\|x\|\theta^{-\delta/2}) \text{ empty})\mathrm{d}x \\ &\overset{(a)}{\sim} \lambda\theta^{-\delta} \int_{\mathbb{R}^2} \mathbb{E}^{!o}\bar{F}\left(\|x\|^\alpha I_\infty\right)\mathrm{d}x, \quad \theta\to\infty \\ &\overset{(b)}{=} \lambda\theta^{-\delta}\mathbb{E}^{!o}[I_\infty^{-\delta}] \int_{\mathbb{R}^2} \bar{F}_h\left(\|x\|^\alpha\right)\mathrm{d}x, \quad \theta\to\infty,\end{aligned} \tag{14}$$

where (a) follows since $\theta^{-\delta/2} \to 0$ and hence $\mathbf{1}(b(o,\|x\|\theta^{-\delta/2}) \text{ empty}) \to 1$. The equality in (b) follows by using the substitution $xI_\infty^{1/\alpha} \to x$. Here we use I_∞ to denote the interference term since this interference stems from all points in the point process, as opposed to I, which stems from $\Phi^!$. Changing into polar coordinates, the integral can be written as

$$\int_{\mathbb{R}^2} \bar{F}_h\left(\|x\|^\alpha\right)\mathrm{d}x = \pi\delta \int_0^\infty r^{\delta-1}\bar{F}_h(r)\mathrm{d}r \overset{(a)}{=} \pi\mathbb{E}[h^\delta],$$

where (a) follows since h is a positive random variable [8]. ■

Since $h>0$ and $\mathbb{E}(h)<\infty$ and $\delta<1$ we necessarily have $\mathbb{E}(h^\delta)<\infty$.

Corollary 2 (Asymptotic gain at $\theta\to\infty$)**.** *For an arbitrary stationary point process Φ with EFIR given in Def. 2, the asymptotic gain at $\theta\to\infty$ relative to the PPP is*

$$G_\infty = \frac{\mathsf{EFIR}}{\mathsf{EFIR}_{\mathrm{PPP}}} = \left(\frac{\lambda\pi\mathbb{E}^{!o}(I_\infty^{-\delta})\mathbb{E}(h^\delta)}{\operatorname{sinc}\delta}\right)^{1/\delta}.$$

Proof: From Theorem 1, we have that the constant c in (8) is given by $c = \mathsf{EFIR}^\delta$. c_{PPP} follows from Lemma 1 as $c_{\mathrm{PPP}} = \mathsf{EFIR}_{\mathrm{PPP}}^\delta = \operatorname{sinc}\delta$. ■

The Laplace transform of the interference in (12) for general point processes can be evaluated as follows:

$$\begin{aligned}\mathcal{L}_{I_\infty}^{!o}(s) &= \mathbb{E}^{!o}\left(e^{-s\sum_{x\in\Phi}h_x\|x\|^{-\alpha}}\right) \\ &= \mathbb{E}^{!o}\prod_{x\in\Phi}\mathcal{L}_h(s\|x\|^{-\alpha}) = \mathcal{G}^{!o}(\mathcal{L}_h(s\|\cdot\|^{-\alpha})),\end{aligned}$$

where $\mathcal{G}^{!o}(.)$ is the probability generating functional with respect to the reduced Palm measure and $\mathcal{L}_h$ is the Laplace transform of the fading distribution.

Corollary 3 (Rayleigh fading)**.** *With Rayleigh fading, the expected fading-to-interference ratio simplifies to*

$$\mathsf{EFIR} = \left(\lambda\int_{\mathbb{R}^2}\mathcal{G}^{!o}(\Delta(x,\cdot))\mathrm{d}x\right)^{1/\delta},$$

where

$$\Delta(x,y) = \frac{1}{1+\|x\|^\alpha\|y\|^{-\alpha}}.$$

Proof: With Rayleigh fading, the fading power is exponential and $\bar{F}_h(x) = \exp(-x)$. From (14), we have

$$p_{\rm s}(\theta) \sim \lambda\theta^{-\delta} \int_{\mathbb{R}^2} \mathbb{E}^{!o} \bar{F}\left(\|x\|^\alpha I\right) \mathrm{d}x$$
$$= \lambda\theta^{-\delta} \int_{\mathbb{R}^2} \mathbb{E}^{!o} \prod_{y\in\Phi} \frac{1}{1+\|x\|^\alpha\|y\|^{-\alpha}} \mathrm{d}x,$$

and the result follows from the definition of the reduced probability generating functional. ■

C. Relation to the SIR distribution for max-SIR BS association

We now explore the tail of the distribution to the maximum SIR seen by the typical user for exponential h. Assume that the typical user at the origin connects to the BS that provides the *instantaneously strongest* SIR. Also assume that $\theta > 1$. Let $\mathsf{SIR}(x)$ denote the SIR between the BS at x and the user at the origin. Then

$$\mathbb{P}(\max_{x\in\Phi} \mathsf{SIR}(x) > \theta) = \mathbb{E}\sum_{x\in\Phi} \mathbb{P}(\mathsf{SIR}(x) > \theta)$$
$$= \lambda \int_{\mathbb{R}^2} \mathbb{P}^{!o}(\mathsf{SIR}(x) > \theta)\mathrm{d}x$$
$$= \lambda \int_{\mathbb{R}^2} \mathcal{G}^{!o}\left[\frac{1}{1+\theta(\|x\|/\|\cdot\|)^\alpha}\right]\mathrm{d}x$$
$$= \lambda\theta^{-\delta} \int_{\mathbb{R}^2} \mathcal{G}^{!o}(\Delta(x,\cdot))\mathrm{d}x.$$

From the above we observe that (for exponential fading),

$$p_{\rm s}(\theta) \sim \mathbb{P}(\max_{x\in\Phi} \mathsf{SIR}(x) > \theta), \quad \theta\to\infty,$$

which is a well known property for heavy tailed distributions.

IV. Examples

A. Determinantal point processes

Determinantal (fermion) point processes (DPPs) [10] exhibit repulsion and can be used to model minimum separation in a cellular network. The kernel of the DPP Φ is denoted by $K(x,y)$ and due to stationarity is of the form $K(x-y)$. The reduced Palm measure μ^{x_o} pertains to a DPP with kernel K^{x_o} defined by

$$K^{x_o}(x,y) = \frac{1}{K(x_o,x_o)} \det\begin{pmatrix} K(x,y) & K(x,x_o) \\ K(x_o,y) & K(x_o,x_o)\end{pmatrix}, \tag{15}$$

whenever $K(x_o,x_o) > 0$. Let $K^o(x,y)$ denote the kernel associated with the reduced Palm distribution of the DPP process. The reduced Palm probability generating functional is known for a DPP and is given by [10]

$$\mathcal{G}^{!o}(f(\cdot)) \triangleq \mathbb{E}^{!o}\left[\prod_{x\in\Phi} f(x)\right] = \mathrm{detf}(\mathbf{1}-(1-f)K^o), \tag{16}$$

where detf is the Fredholm determinant and $\mathbf{1}$ is the identity operator. The next lemma characterizes the EFIR in a general DPP with Rayleigh fading.

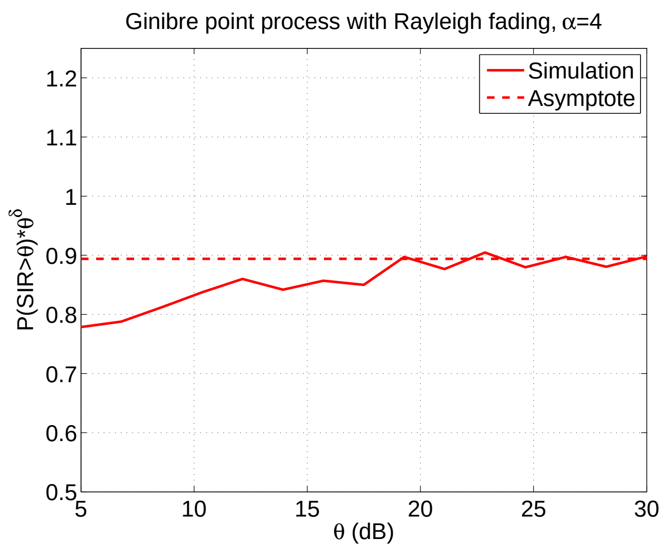

Figure 2. Scaled success probability $p_{\rm s}(\theta)\theta^\delta$ for the GPP of intensity $1/\pi$ for Rayleigh fading with $\alpha = 4$. The asymptote (dashed line) is at $\sqrt{\mathsf{EFIR}} \approx 0.89$.

Lemma 2. *When the BSs are distributed as a stationary DPP, the EFIR with Rayleigh fading is*

$$\mathsf{EFIR} = \left(\lambda\int_{\mathbb{R}^2} \mathrm{detf}(\mathbf{1}-(1-\Delta(x,\cdot))K^o)\mathrm{d}x\right)^{1/\delta}. \tag{17}$$

Proof: Follows from Corollary 3 and (16). ■

Ginibre point processes: Ginibre point processes (GPPs) are determinantal point processes with density $\lambda = c/\pi$ and kernel given by $K(x,y) = \frac{c}{\pi}e^{-\frac{c}{2}(|x|^2+|y|^2)}e^{cx\bar{y}}$. Using the properties of GPPs [11], it can be shown that

$$\mathbb{E}^{!o}[e^{-sI_\infty}] = \prod_{k=1}^{\infty}\int_0^\infty \mathcal{L}_h(sr^{-\alpha/2})\frac{r^{k-1}e^{-cr}}{c^{-k}\Gamma(k)}\mathrm{d}r,$$

from which $\mathbb{E}^{!o}[I^{-\delta}]$ can be evaluated using (12). In Figure 2, the scaled success probability $p_{\rm s}(\theta)\theta^\delta$ and its asymptote $\mathsf{EFIR}^\delta = \sqrt{\mathsf{EFIR}}$ are plotted for a GPP with $\alpha = 4$ as a function of θ. The computed value of the EFIR is $\mathsf{EFIR}_{\rm GPP} \approx 0.80$. We observe a close match as soon as $\theta > 15$ dB.

Fig. 3 shows the simulated values of the gains G_0 and G_∞ for the GPP. $G_0 \approx 1.5$ for all values of α, while G_∞ increases slightly with α.

B. Square lattice point processes

Let u_1, u_2 be iid uniform random variables in $[0,1]$. The unit intensity (square) lattice point process Φ is defined as $\Phi \triangleq \mathbb{Z}^2 + (u_1,u_2)$. For this lattice, with Rayleigh fading, the Laplace transform of the interference is bounded as [12]

$$e^{-sZ(2/\delta)} \le \mathcal{L}^{!o}_{I_\infty}(s) \le \frac{1}{1+sZ(2/\delta)}, \tag{18}$$

where $Z(x) = 4\zeta(x/2)\beta(x/2)$ is the Epstein zeta function, $\zeta(x)$ is the Riemann zeta function, and $\beta(x)$ is the Dirichlet beta function. From (12) we have

$$Z(2/\delta)^{-\delta} \le \mathbb{E}^{!o}[I_\infty^{-\delta}] \le \frac{\pi\csc(\pi\delta)}{\Gamma(\delta)Z(2/\delta)^\delta}.$$

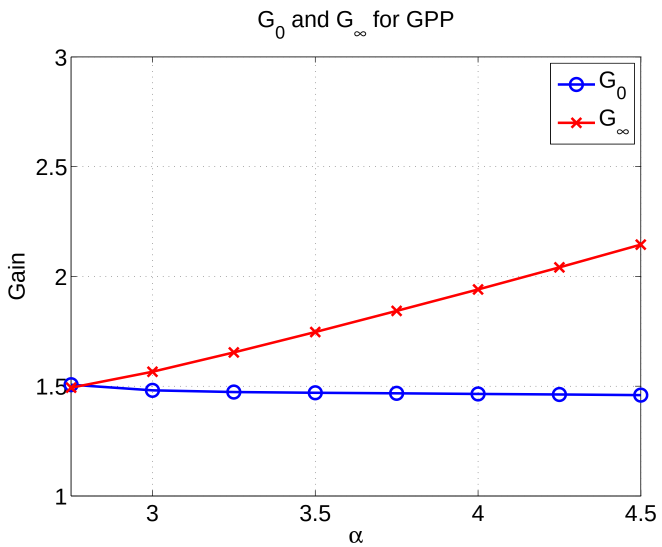

Figure 3. Simulated gains G_0 and G_∞ for the GPP of intensity $1/\pi$ with Rayleigh fading as a function of α.

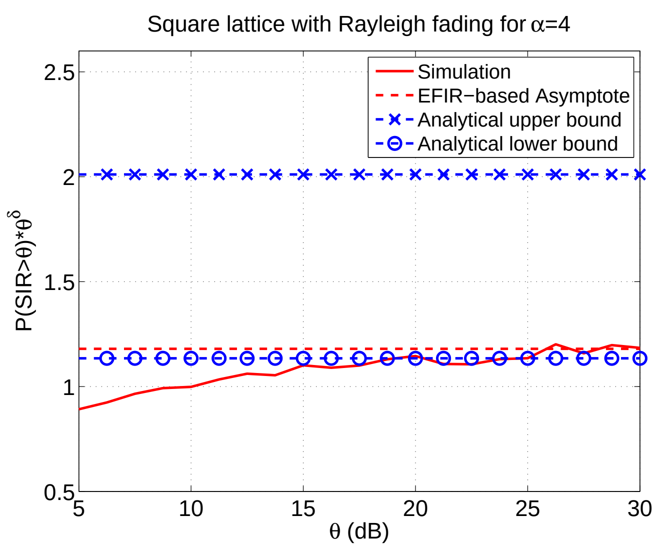

Figure 4. Scaled success probability $p_s(\theta)\theta^\delta$ for the square lattice point process with Rayleigh fading and $\alpha = 4$. The asymptote (dashed line) is $\sqrt{\mathsf{EFIR}} \approx 1.19$.

The upper bound equals $(Z(2/\delta)^\delta \Gamma(1+\delta) \operatorname{sinc} \delta)^{-1}$, and it follows that for Rayleigh fading,

$$\frac{(\pi\Gamma(1+\delta))^{1/\delta}}{Z(2/\delta)} \leq \mathsf{EFIR}_{\mathrm{lat}} \leq \left(\frac{\pi}{\operatorname{sinc}\delta}\right)^{1/\delta} \frac{1}{Z(2/\delta)}. \qquad (19)$$

As α increases ($\delta \to 0$), the upper and lower bound approach each other and thus both bounds get tight.

The success probability multiplied by θ^δ, the EFIR asymptote and its bounds (19) for a square lattice process are plotted in Figure 4 for $\alpha = 4$. We observe that the lower bound, which is 1.29, is indeed a good approximation to the numerically obtained value EFIR ≈ 1.40, and that for $\theta > 15$ dB, the ccdf is already quite close to the asymptote.

V. Conclusions

For all stationary point process, the asymptotics of the SIR ccdf (or success probability) are of the form

$$p_s(\theta) \sim 1 - c_0\theta^m, \quad \theta \to 0; \qquad p_s(\theta) \sim c_\infty \theta^{-\delta}, \quad \theta \to \infty$$

for a fading cdf $F_h(x) = \Theta(x^m)$, $x \to 0$. Both constants c_0 and c_∞ depend on the path loss exponent and the point process model, and c_0 also depends on the fading statistics. Fading *may* also affect c_∞.

The constant c_0 is related to the mean interference-to-signal-ratio (MISR). For $m = 1$, $c_0 =$ MISR, while c_∞ is related to the expected fading-to-interference ratio (EFIR) through $c_\infty = \mathsf{EFIR}^\delta$.

This result partially explains the empirical observation that the success probabilities all appear to be shifted versions of each other. They show that interference affects the SIR ccdf merely through a horizontal shift (in dB). Consequently, the success probabilities for arbitrary stationary point process can be well approximated by shifting the one for the Poisson point process, which is known in analytical form.

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