Stability Analysis of Static Poisson Networks

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***Abstract*—To be considered for an 2015 IEEE Jack Keil Wolf ISIT Student Paper Award. The stable packet arrival rate region of the discrete-time slotted ALOHA network with the sources distributed as a static Poisson point process is investigated here. The problem is a generalization and extension of interacting queues problem, in which the physical layer is abstracted. Employing tools from queueing theory as well as point process theory, we obtain sufficient conditions and necessary conditions for stability by the concept of dominance. Numerical results show that the gap between sufficient conditions and necessary conditions is small, and the results also reveal how these conditions vary with system parameters.**

***Index Terms*—dominant system, interacting queues, Poisson bipolar process, static network, stability.**

I. Introduction

The protocol of slotted ALOHA is studied and analyzed extensively in the literature. However, most of these works either concentrate on capacity analysis or assume that terminals are backlogged, i.e., the terminals always have packets to transmit and no queueing of packets at the terminals occurs. If each terminal provides a buffer for queueing, the problem becomes more practically relevant and more challenging. It is complicated because it involves interacting queues, i.e., the serving rate of each queue depends on the sizes of queues, the analysis of which requires the combination of queueing theory and multi-access information theory, which is notoriously difficult to cope with.

Previous analyses of interacting queues are mostly based on a physical layer that is abstracted. Most works consider a discrete-time slotted ALOHA system with N terminals. In each time slot, each terminal attempts to transmit the head-of-line packet with a certain probability if its buffer is not empty. A collision occurs if two or more terminals transmit in the same time slot. Even for this simplified ALOHA system, the exact stability region has been found only when the number of terminals is $N = 2$ [1], [2] or $N = 3$ [3]. For $N > 3$, only sufficient conditions and necessary conditions for stability could be derived.

In practical networks, concurrent transmissions lead to interference between transmissions, which cannot be accurately modeled as collisions. Moreover, the randomness in the deployment of transmitters makes accurate modeling and analysis of interference complicated. Therefore, the interaction between the queues at the transmitters in practical networks is much more intricate than the aforementioned simplified ALOHA system.

In this work, we model a large-scale network by using tools from point process theory, which is widely adopted to analyze the performance of wireless networks [4]–[6]. A common and meaningful model is the Poisson point process (PPP), in which each transmitter in the network is modeled as one point of the PPP. We combine queueing theory and stochastic geometry to analyze the stability region for the arrival rate at each transmitter in a static network, i.e., the transmitters and the receivers are generated at first and remain static during all the time slots. If each transmitter maintains a buffer of infinite capacity to store the packets generated, the analysis becomes complicated since the serving rate of each queue depends on the status of other queues as well as the channel status and the ALOHA protocol. By applying the concept of dominance, we derive sufficient conditions and necessary conditions for stability, and by slightly relaxing the results, we obtain the results in closed form.

Previous analyses have yielded only bounds to the regions of arrival rate for which the system is stable [1]–[3], [7], [8]. The stability of multi-access systems with an infinite number of transmitters is studied in [9]. The stability region of two-user interference channel is obtained in [10]. The stability and delay of high-mobility networks are analyzed in [11] using a combination of queueing theory and stochastic geometry. In high-mobility networks, the sizes of queues and the serving rates are decoupled; however, this does not hold in static networks.

II. System Model

We consider a discrete-time slotted ALOHA system with transmitters and receivers distributed as a Poisson bipolar network [5, Def. 5.8], i.e., we model the locations of the transmitters as a PPP $\Phi = \{x_i\} \subset \mathbb{R}^2$ of intensity λ. Each transmitter is associated with a receiver at a fixed distance r_0 and a random orientation. In the analysis, we condition on $x_0 \in \Phi$ which is the typical transmitter under consideration, where $r_0 = |x_0|$ is the distance of x_0 to the origin where the corresponding receiver is located (see Fig. 1). Time is divided into discrete slots with equal duration, and each transmission attempt occupies one time slot. We assume the network is static, i.e., the locations of transmitters and receivers are generated once and then kept unchanged during all time slots.

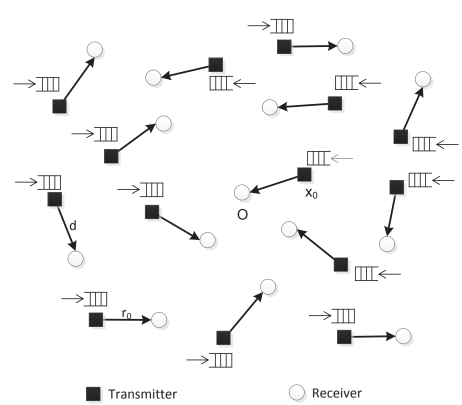

Fig. 1. A snapshot of the bipolar model with ALOHA.

We use the Rayleigh block fading model in which the power fading coefficients remain constant over each time slot and are spatially and temporally independent with exponential distribution of mean 1. Let α be the path loss exponent and $h_{k,x}$ be the fading coefficient between transmitter x and the considered receiver at origin o in time slot k. All transmitters are assumed to transmit at unit power. The power spectral density of the thermal noise is N_0 and the bandwidth is W. We assume that the SINR threshold model is applied, i.e., as long as the SINR is above a threshold θ, a link can be successfully used for information transmission at spectral efficiency $\log_2(1+\theta)$ bits/s/Hz.

Each transmitter has a buffer of infinite capacity to store the packets generated. Letting ΔT be the duration of each time slot, the amount of information of a packet is $W\Delta T\log_2(1+\theta)$ bits. Each transmitter generates packets according to a Bernoulli process with arrival rate λ_a $(0 \leq \lambda_a \leq 1)$ packets per time slot. With these notations, the arrival rate of the amount of information is $\lambda_a W \log_2(1+\theta)$ bits/s. To simplify the notation, we normalize the arrival rate of the amount of information by the bandwidth, which yields $\lambda_s = \lambda_a \log_2(1+\theta)$ bits/s/Hz. The arrival processes of different transmitters are independent. In each time slot, each transmitter attempts to send its head-of-line packet with probability p if its buffer is not empty. If an attempt of transmission is failed, the transmitter attempts to retransmit the packet at the next time slot with probability p; if an attempt of transmission is successful, the transmitter deletes the packet from the buffer. The SINR of the typical receiver in time slot k is

$$\mathrm{SINR}_k = \frac{h_{k,x_0} r_0^{-\alpha}}{WN_0 + \sum_{x\in\Phi\setminus\{x_0\}} h_{k,x}|x|^{-\alpha}\mathbf{1}(x\in\Phi_k)}. \tag{1}$$

In order to clearly define the stability of the large scale network, we use the following notation. We condition on Φ having a point at x_0, thus the relevant probability measure of the point process is the Palm probability $\mathbb{P}^{x_0}$. Correspondingly, the expectation, denoted by $\mathbb{E}^{x_0}$, is taken with respect to the measure $\mathbb{P}^{x_0}$. Let $\mathcal{C}_\Phi^k$ be the event that the transmission of the typical transmitter x_0 succeeds in time slot k conditioned on the PPP Φ, i.e., $\mathcal{C}_\Phi^k$ consists of two events: that the transmission is scheduled by ALOHA in time slot k and that the scheduled transmission is successful. Even if the realization of the PPP Φ and the time slot index k are given, it is still uncertain whether the typical transmission is successful because of the effect of fading and ALOHA. Let $\mathbb{P}^{x_0}(\mathcal{C}_\Phi^k)$ be the success probability conditioned upon Φ and k. We define the stability region for the network as follows.

Definition 1. *The stability region $\mathcal{S}$ is defined as the set of traffic arrival rates $\lambda_s = \lambda_a\log_2(1+\theta)$ bits/s/Hz given by*

$$\mathcal{S} \triangleq \left\{\lambda_s\in\mathbb{R}^+ : \mathbb{P}^{x_0}\left\{\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\mathbb{P}^{x_0}(\mathcal{C}_\Phi^k) < \lambda_a\right\} < \varepsilon\right\}. \tag{2}$$

Remark 1. $\mathbb{P}^{x_0}\big\{\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\mathbb{P}^{x_0}(\mathcal{C}_\Phi^k) < \lambda_a\big\}$ *is the probability that the queue at the typical transmitter is unstable, due to the conclusion from Loynes [12], which states that the condition for stability of an isolated queue is that the service rate is larger than the arrival rate. We declare that the network is stable when the probability that the queue at the typical transmitter to be unstable is less than a certain threshold ε. Strictly speaking, the stability in the definition should be named ϵ-stability since there always exist some transmitters whose queues are unstable in the static Poisson network.*

In the following, we derive sufficient conditions and necessary conditions for the stability of the static Poisson network.

III. Sufficient Conditions

In order to derive sufficient conditions for stability, we consider a dominant system. In the dominant system the typical transmission behaves exactly the same as in the original system; however, for other transmissions, we assume that when the queues become empty, the transmitters continue to transmit "dummy" packets with probability p, thus continuing their interference to other transmissions no matter whether their queues are empty or not. Therefore, the stability region we obtain under these assumptions, denoted by $\overline{\mathcal{S}}$, will be a subset of $\mathcal{S}$. Under these assumptions, the success probability for the typical transmission given Φ, denoted by $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$, is the same for each time slot since the fading and the ALOHA are i.i.d. between different time slots. Thus $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$ is a random variable uniquely determined by the realization of the PPP Φ. The mathematical description of $\overline{\mathcal{S}}$ is

$$\overline{\mathcal{S}} \triangleq \left\{\lambda_s\in\mathbb{R}^+ : \mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) < \lambda_a\right\} \leq \varepsilon\right\}. \tag{3}$$

Theorem 1. *Given a slotted ALOHA system with transmitters distributed as a PPP and with Bernoulli packet arrivals, a sufficient condition for the system to be stable is*

$$\lambda_s \in \Bigg\{\lambda_s\in\mathbb{R}^+ : \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\mathrm{Im}\Bigg\{p^{j\omega}\exp\Bigg(- j\omega\theta r_0^\alpha WN_0 \\ -2\pi\lambda\int_0^\infty \Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{j\omega}\Big) r\mathrm{d}r \\ -j\omega\ln\Bigg(\frac{\lambda_s}{\log_2(1+\theta)}\Bigg)\Bigg)\Bigg\}\mathrm{d}\omega < \varepsilon\Bigg\}. \tag{4}$$

Proof: The proof is in Appendix A. ■

The proof of Theorem 1 relies on deriving the cumulative distribution function (cdf) of $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$, which is implemented by deriving the moments of $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$ and by applying the Gil-Pelaez Theorem [13]. We omit the proof due to the space limitations.

The sufficient condition given by Theorem 1 is difficult to evaluate. Using the Chernoff bound, we obtain sufficient conditions that are easier to evaluate, as stated in the following corollary.

Corollary 1. *Given a slotted ALOHA system with the transmitters distributed as a PPP and with Bernoulli packet arrivals, a sufficient condition for the system to be stable is*

$$\lambda_s \in \Bigg\{\lambda_s\in\mathbb{R}^+ : \lambda_s \leq p\varepsilon^{\frac{1}{t}}\log_2(1+\theta)\exp\Bigg(- \theta r_0^\alpha WN_0 \\ +\frac{2\pi\lambda}{t}\int_0^\infty \Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{-t}\Big) r\mathrm{d}r\Bigg)\Bigg\}, \tag{5}$$

for all $t > 0$. In particular, if t is chosen as a positive integer, we obtain a sufficient condition in closed form as

$$\lambda_s \in \left\{\lambda_s\in\mathbb{R}^+ : \lambda_s \leq \lambda_{\mathrm{suff}}(n)\right\}, \tag{6}$$

where

$$\lambda_{\mathrm{suff}}(n) = p\varepsilon^{\frac{1}{n}}\log_2(1+\theta)\exp\Bigg(- \theta r_0^\alpha WN_0 \\ -\pi\lambda\delta(1-p)^\delta\theta^\delta r_0^2\sum_{i=1}^{n}((1-p)^{-i} - 1)\frac{\Gamma(i-\delta)\Gamma(n-i+\delta)}{\Gamma(i+1)\Gamma(n-i+1)}\Bigg) \tag{7}$$

and $\delta = 2/\alpha$ for all $n \in \mathbb{N}^+$. Let $n_{\max} = \mathrm{argmax}_{n\in\mathbb{N}^+}\lambda_{\mathrm{suff}}(n)$. An improved closed-form sufficient condition for the system to be stable is then given by

$$\lambda_s \in \Big\{\lambda_s \in \mathbb{R}^+ : \lambda_s \le \lambda_{\mathrm{suff}}(n_{\max})\Big\}. \tag{8}$$

Proof: The proof is in Appendix B. ■

IV. Necessary Conditions

To derive necessary conditions for stability, we consider some simplified systems. In the following, we propose two approaches to derive two different types of necessary conditions for stability, namely *type I necessary conditions* and *type II necessary conditions*. In the derivation of type I necessary conditions, we consider a simplified system in which only the effect of the nearest interferer is considered. In the derivation of type II necessary conditions, we consider a modified system that drops the packets in the interfering transmitters that are not scheduled by ALOHA or whose transmissions are failed.

A. Type I necessary conditions

We consider a simplified version of the original system, in which only two pairs of transmitters and receivers are considered. One pair is the typical pair, and the other pair is the pair containing the nearest interferer. Let $T_1 = (r_m \cos\varphi, r_m \sin\varphi)$ be the location of the nearest transmitter, where r_m is the distance from the origin and φ is the angle. Let $R_1 = (r_m \cos\varphi + r_0 \cos\psi, r_m \sin\varphi + r_0 \sin\psi)$ be the location of the associated receiver, where ψ is the angle between T_1 and R_1 (see Fig. 2). Thus φ and ψ are uniformly distributed random variables in $[0, 2\pi]$. The probability density function of r_m is given by

$$f_{r_m}(r) = 2\pi\lambda r \exp\left(-\pi\lambda r^2\right). \tag{9}$$

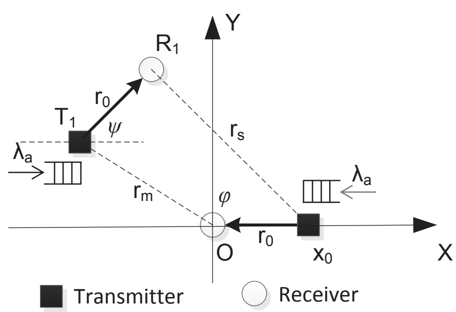

Fig. 2. The simplified system which consists of two pairs of transmitters and receivers, i.e., the typical transmission and the nearest transmission.

The following lemma gives the sufficient and necessary condition for the typical transmission to be stable with given φ, ψ, r_m.

Lemma 1. *For the simplified system with given φ, ψ, r_m (see Fig. 2), the sufficient and necessary condition for the typical transmission to be stable is*

$$\lambda_s \in \Big\{\lambda_s \in \mathbb{R}^+ : \lambda_s < \lambda_{u,1}\Big\}, \text{ for } r_s > r_m \tag{10}$$

$$\lambda_s \in \Big\{\lambda_s \in \mathbb{R}^+ : \lambda_s < \lambda_{u,2}\Big\}, \text{ for } r_s \le r_m \tag{11}$$

where

$$\lambda_{u,1} = \frac{\left(\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} + 1 - p\right) p \log_2(1+\theta) e^{-WN_0\theta r_0^\alpha}}{\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} - \frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1},$$

$$\lambda_{u,2} = \left(\frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1 - p\right) p \log_2(1+\theta) e^{-WN_0\theta r_0^\alpha},$$

$$r_s = \sqrt{(r_m \cos\varphi + r_0 \cos\psi - r_0)^2 + (r_m \sin\varphi + r_0 \sin\psi)^2}.$$

Proof: The proof is in Appendix C. ■

The proof of Lemma 1 is similar to the derivation of the sufficient and necessary condition for the stability of two interacting queues [10]. Consider a dominant system of the simplified system, in which the typical transmitter still transmits "dummy" packets when its queue is empty; however, the nearest interfering transmitter in the dominant system behaves the same as the original simplified system. If the queue at the typical transmitter in the dominant system is unstable, it will be also unstable in the original simplified system. Because when the queue at the typical transmitter is unstable, it tends to grow to infinity, and the probability of ever sending a dummy packet by the typical transmitter in the dominant system is zero, thus the dominant system is indistinguishable from the original system under saturation. Therefore, we only need to derive the sufficient and necessary condition for stability of the dominant system, which is exactly the sufficient and necessary condition for stability of the original simplified system.

Since φ, ψ, r_m are random variables, by applying Lemma 1, we get the following theorem which gives a type I necessary condition for stability of the original system.

Theorem 2. *Given a slotted ALOHA system with the transmitters distributed as a PPP and with packet arrival being Bernoulli processes, a type I necessary condition for the system to be stable is*

$$\lambda_s \in \big\{\lambda_s \in \mathbb{R}^+ : \lambda_s < \lambda_{\mathrm{nec}}\big\}, \tag{12}$$

where

$$\lambda_{\mathrm{nec}} = \left(1 - \frac{\theta p}{\theta + \left(F_Z^{-1}(\varepsilon)\right)^\alpha}\right) p \log_2(1+\theta) e^{-WN_0\theta r_0^\alpha}, \tag{13}$$

$Z = \frac{1}{r_0}\max\{r_m, r_s\}$ with $F_Z(z)$ being the cdf of Z, and r_s is defined in Lemma 1.

Proof: The proof is in Appendix D. ■

The necessary condition given by Theorem 2 is not in closed form. In the following, we derive a closed-form necessary condition by considering the simplified system in the special case $\varphi = \psi = -\pi$. With given r_m, if the queue at the typical transmitter in the simplified system is unstable for $\varphi = \psi = -\pi$, it will also be unstable for other φ and ψ, due to the fact that when $\varphi = \psi = -\pi$, the interaction between the two pairs of transceivers is the smallest among all φ and ψ.

Corollary 2. *Given a slotted ALOHA system with the transmitters distributed as a PPP and with packet arrival being Bernoulli processes, a closed-form type I necessary condition for the system to be stable is given by*

$$\lambda_s \in \big\{\lambda_s \in \mathbb{R}^+ : \lambda_s < \widehat{\lambda}_{\mathrm{nec}}\big\}, \tag{14}$$

where

$$\widehat{\lambda}_{\mathrm{nec}} = \left(1 + \left(\frac{p\theta r_0^\alpha}{\left(\sqrt{-\frac{\ln(1-\varepsilon)}{\pi\lambda}} + 2r_0\right)^\alpha + \theta r_0^\alpha}\right)^2\right)^{-1} p \log_2(1+\theta) \exp(-WN_0\theta r_0^\alpha). \tag{15}$$

Proof: The proof is in Appendix E. ∎

B. Type II necessary condition

Here we consider a modified system, where packets at the interfering transmitters are dropped if the transmitters are silenced due to ALOHA or if the transmission fails due to the SINR condition, thus an interfering transmitter is active with probability $\lambda_a p$. Since the packets will not accumulate at interfering transmitters in the modified system, a necessary condition for the queue at the typical transmitter in the original system to be stable is that the queue at the typical transmitter in the modified system is stable. The following theorem gives a type II necessary condition for the original system to be stable.

Theorem 3. *Given a slotted ALOHA system with the transmitters distributed as a PPP and with packet arrival being Bernoulli processes, a type II necessary condition for the system to be stable is*

$$\lambda_s \in \Bigg\{\lambda_s \in \mathbb{R}^+ : \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\mathrm{Im}\Big\{p^{j\omega}\exp\Big(-j\omega\theta r_0^\alpha W N_0 -2\pi\lambda\int_0^\infty \Big(1-\Big(\frac{\lambda_s p/\log_2(1+\theta)}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - \frac{\lambda_s p}{\log_2(1+\theta)}\Big)^{j\omega}\Big) r\mathrm{d}r -j\omega\ln\Big(\frac{\lambda_s}{\log_2(1+\theta)}\Big)\Big)\Big\}\mathrm{d}\omega < \varepsilon\Bigg\}. \tag{16}$$

Proof: The proof is in Appendix F. ∎

The proof of Theorem 3 is similar to that of Theorem 1. By applying the Markov inequality, we obtain a type II necessary condition as follows that is easier to evaluate.

Corollary 3. *Given a slotted ALOHA system with the transmitters distributed as a PPP and with packet arrival being Bernoulli processes, for all $t > 0$, a type II necessary condition for the system to be stable is*

$$\lambda_s \in \Bigg\{\lambda_s \in \mathbb{R}^+ : \lambda_s \le p(1-\varepsilon)^{-\frac{1}{t}}\log_2(1+\theta) \exp\Big(-\theta r_0^\alpha W N_0 - \frac{2\pi\lambda}{t}\int_0^\infty \Big(1-\Big(\frac{\lambda_s p/\log_2(1+\theta)}{1+\theta r_0^\alpha r^{-\alpha}} +1 - \frac{\lambda_s p}{\log_2(1+\theta)}\Big)^t\Big) r\mathrm{d}r\Big)\Bigg\}. \tag{17}$$

When $t=1$, we obtain a closed-form result as

$$\lambda_s \in \Big\{\lambda_s \in \mathbb{R}^+ : \lambda_s \le \widetilde{\lambda}_{\mathrm{nec}}\Big\}. \tag{18}$$

where

$$\widetilde{\lambda}_{\mathrm{nec}} = \frac{\log_2(1+\theta)}{p\lambda\pi r_0^2\theta^\delta\frac{\pi\delta}{\sin(\pi\delta)}}\mathcal{W}\left(\frac{p^2}{1-\varepsilon}\lambda\pi r_0^2\theta^\delta\frac{\pi\delta}{\sin(\pi\delta)}e^{-\theta r_0^\alpha W N_0}\right) \tag{19}$$

and $\mathcal{W}(z)$ is the Lambert W function.

Proof: The proof is in Appendix G. ∎

V. Asymptotic Behavior

In this section, we obtain some asymptotic results based on the previous analysis.

When p approaches 0, we obtain

$$\lambda_{\mathrm{suff}}(\infty) \sim p\log_2(1+\theta)\exp\left(-\theta r_0^\alpha W N_0\right); \tag{20}$$

$$\widehat{\lambda}_{\mathrm{nec}} \sim p\log_2(1+\theta)\exp\left(-\theta r_0^\alpha W N_0\right); \tag{21}$$

$$\widetilde{\lambda}_{\mathrm{nec}} \sim \frac{1}{1-\varepsilon}p\log_2(1+\theta)\exp(-\theta r_0^\alpha W N_0). \tag{22}$$

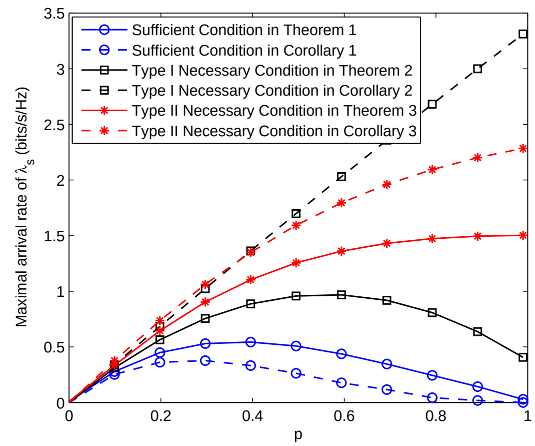

Fig. 3. Comparison of sufficient conditions and necessary conditions as functions of p. The parameters are set as $\varepsilon = 0.1$, $\theta = 10$dB, $r_0 = 1$, $N_0 = -173$dBm, $W = 20$MHz, $\alpha = 4$ and $\lambda = 0.05$.

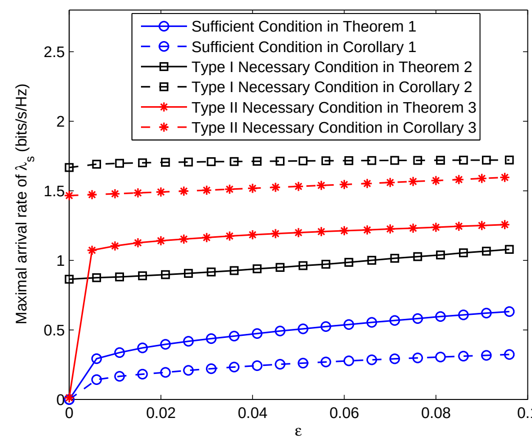

Fig. 4. Comparison of sufficient conditions and necessary conditions as functions of ε. The parameters are set as $p = 0.5$, $\theta = 10$dB, $r_0 = 1$, $N_0 = -173$dBm, $W = 20$MHz, $\alpha = 4$ and $\lambda = 0.05$.

When θ approaches 0, we obtain

$$\lambda_{\mathrm{suff}}(\infty) \sim \frac{p\theta}{\ln 2}; \quad \widehat{\lambda}_{\mathrm{nec}} \sim \frac{p\theta}{\ln 2}; \quad \widetilde{\lambda}_{\mathrm{nec}} \sim \frac{p\theta}{(1-\varepsilon)\ln 2}. \tag{23}$$

More asymptotic results can be obtained when ε or λ approaches 0; however, we omit them due to space limitations.

VI. Comparison of Sufficient Conditions and Necessary Conditions

In this section, we numerically compare the sufficient conditions and necessary conditions derived in the previous sections.

Fig. 3 shows the maximal arrival rates in the sufficient and the necessary conditions as functions of p. It is observed that when $p \to 0$, all curves converge to 0. As p increases, the curves for the sufficient condition in Theorem 1 (blue solid line) and the type I necessary condition in Theorem 2 (black solid line) first increase then decrease. It can be inferred that the actual maximal arrival rate also first increases then decreases. This can be explained as follows: for small p, the capacity is limited by the small ALOHA probability, and for large p, it is limited by the large interference.

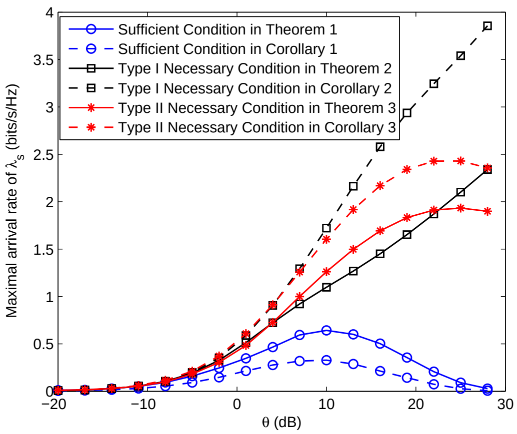

Fig. 5. Comparison of sufficient conditions and necessary conditions as functions of θ. The parameters are set as $p = 0.5$, $\varepsilon = 0.1$, $r_0 = 1$, $N_0 = -173$dBm, $W = 20$MHz, $\alpha = 4$ and $\lambda = 0.05$.

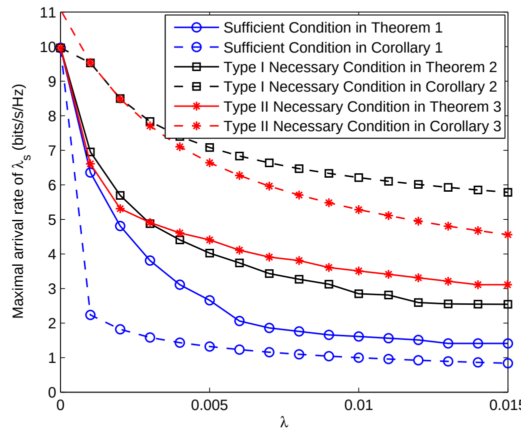

Fig. 6. Comparison of sufficient conditions and necessary conditions as functions of λ with optimal pair of (p, θ). The parameters are set as $\varepsilon = 0.1$, $r_0 = 1$, $N_0 = -173$dBm, $W = 20$MHz and $\alpha = 4$.

Fig. 4 plots the maximal arrival rates in the sufficient and the necessary conditions as functions of ε. The curves for sufficient conditions and necessary conditions do not depend strongly on ε. Since the gap between the curve for sufficient conditions and that for necessary conditions is not large, it can be inferred that the actual maximal arrival rate does not change much either as the increasing of ε, indicating that small changes in the arrival rate λ_s will greatly affect the stability of the network.

Fig. 5 plots the maximal arrival rates in the sufficient and the necessary conditions as functions of θ. It is observed that when $\theta \to 0$, all curves converge to 0. By comparing the curve for type I necessary condition in Theorem 2 (black solid line) and that for type II necessary condition in Theorem 3 (red solid line), we observe that when θ is small, the type II necessary condition is better than the type I necessary condition because for small θ the success probability is large and the probability of dropping a packet is small, thus the modified system is close to the original system. When θ starts to grow, the type I necessary condition becomes tighter since the success probability becomes smaller and the probability of dropping a packet increases. However, when θ continues to grow, the type II necessary condition becomes better again, which is due to the fact that very large values of θ make it almost impossible for a transmission to be successful in the presence of interference. Since the derivation of the type I necessary condition only considers the effect of the nearest interferer, the accuracy is worse than the type II necessary condition.

Lastly, consider the case where the transmit probability p and the SINR threshold θ are designable parameters to maximize the maximal arrival rate. To obtain realistic values, we choose p from $[0, 1]$ and choose θ from $[-20, 30]$ dB. Fig. 6 plots the maximal arrival rates in terms of the sufficient and the necessary conditions as functions of λ when the optimal p and θ are chosen. It is observed that all curves except the curve for the type II necessary condition in Corollary 3 converge to the same value, because for small λ, the effect of interference is negligible, thus the dominant system and the simplified system tend to be the same.

VII. Conclusions

In this paper, we investigated the stable packet arrival rate region of the discrete-time slotted ALOHA network where the transmitters and receivers distributed as a static Poisson bipolar network. We employed tools from queueing theory as well as point process theory and proposed several novel approaches to study stability of this system by the concept of dominance. We obtained sufficient conditions and necessary conditions for stability in closed form. The numerical results show that the gap between sufficient conditions and necessary conditions is small and reveal how the conditions vary with the system parameters.

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Appendix A
Proof of Theorem 1

The success probability for the typical transmission conditioned on Φ in the dominant system is denoted as $\mathbb{P}^{x_0}(\mathcal{C}_\Phi) = p\mathbb{P}^{x_0}(\mathrm{SINR} > \theta \mid \Phi)$, which can be evaluated as

$$\begin{aligned}
&\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\\
\overset{(a)}{=}& p\mathbb{P}^{x_0}(\mathrm{SINR} > \theta \mid \Phi)\\
=& p\mathbb{P}^{x_0}\left(h_{k,x_0} r_0^{-\alpha} > \theta\left(WN_0 + I_k\right) \mid \Phi\right)\\
\overset{(b)}{=}& p\mathbb{E}^{x_0}\left(\exp\left(-\theta r_0^\alpha\left(WN_0 + I_k\right)\right) \mid \Phi\right)\\
=& p\exp\left(-\theta r_0^\alpha WN_0\right)\\
&\prod_{x\in\Phi\setminus\{x_0\}} \mathbb{E}^{x_0}\left(\exp\left(-\theta r_0^\alpha h_{k,x}|x|^{-\alpha}\mathbf{1}(x\in\Phi_k)\right) \mid \Phi\right)\\
=& p\exp\left(-\theta r_0^\alpha WN_0\right)\\
&\prod_{x\in\Phi\setminus\{x_0\}}\Big(p\mathbb{E}^{x_0}\Big(\exp\big(-\theta r_0^\alpha h_{k,x}|x|^{-\alpha}\big) \mid \Phi\Big) + 1 - p\Big)\\
\overset{(c)}{=}& p\exp\big(-\theta r_0^\alpha WN_0\big)\prod_{x\in\Phi\setminus\{x_0\}}\Big(\frac{p}{1+\theta r_0^\alpha|x|^{-\alpha}} + 1 - p\Big),
\end{aligned} \tag{24}$$

where (a) is because a transmission occurs with probability p, and (b) and (c) follows because the fading coefficients $h_{k,x}$ are i.i.d. random variables with exponential distribution of unit mean.

Letting $Y \overset{\Delta}{=} \ln\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right)$, the moment generating function of Y is

$$\begin{aligned}
M_Y(s) =& \mathbb{E}\left(e^{s\ln(\mathbb{P}^{x_0}(\mathcal{C}_\Phi))}\right)\\
=& \mathbb{E}\left(\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right)^s\right)\\
=& p^s\exp\left(-s\theta r_0^\alpha WN_0\right)\\
&\mathbb{E}\Big(\prod_{x\in\Phi\setminus\{x_0\}}\Big(\frac{p}{1+\theta r_0^\alpha|x|^{-\alpha}} + 1 - p\Big)^s\Big)\\
=& p^s\exp\Big(-s\theta r_0^\alpha WN_0\\
&-\lambda\int_{R^2}\Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha|x|^{-\alpha}} + 1 - p\Big)^s\Big)\mathrm{d}x\Big)\\
=& p^s\exp\Big(-s\theta r_0^\alpha WN_0\\
&-2\pi\lambda\int_0^\infty\Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^s\Big)r\mathrm{d}r\Big).
\end{aligned} \tag{25}$$

The pdf of Y can be derived by applying the inverse transform of the characterized function $M_Y(j\omega)$. The cdf of Y, denoted by $F_Y(y) = \mathbb{P}(Y \le y)$, can be derived by applying the Gil-Pelaez Theorem.

$$F_Y(y) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\{e^{-j\omega y}M_Y(j\omega)\}}{\omega}\mathrm{d}\omega. \tag{26}$$

The probability that the queue at the typical transmitter in the dominant system is unstable is given by the cdf of $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$, which is

$$\begin{aligned}
&\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \le \lambda_a\right\}\\
=& \mathbb{P}^{x_0}\left\{\ln\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right) \le \ln(\lambda_a)\right\}\\
=& F_Y(\ln(\lambda_a))\\
=& \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\{e^{-j\omega\ln(\lambda_a)}M_Y(j\omega)\}}{\omega}\mathrm{d}\omega.
\end{aligned} \tag{27}$$

The condition for the queue at the typical transmitter in the dominant system to be stable is $\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \le \lambda_a\right\} < \varepsilon$. By combining (25) and (27), and plugging in the equality $\lambda_s = \lambda_a\log_2(1+\theta)$, we get the condition for the queue at the typical transmitter to be stable as follows

$$\begin{aligned}
&\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\mathrm{Im}\Big\{p^{j\omega}\exp\Big(-j\omega\theta r_0^\alpha WN_0\\
&-2\pi\lambda\int_0^\infty\Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{j\omega}\Big)r\mathrm{d}r\\
&-j\omega\ln\Big(\frac{\lambda_s}{\log_2(1+\theta)}\Big)\Big)\Big\}\mathrm{d}\omega < \varepsilon.
\end{aligned} \tag{28}$$

Solving the above inequality, we get the results.

Appendix B
Proof of Corollary 1

For all $t > 0$, by applying the Markov inequality, we obtain the following inequality.

$$\begin{aligned}
&\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) < \lambda_a\right\}\\
=& \mathbb{P}^{x_0}\left\{e^{-t\ln(\mathbb{P}^{x_0}(\mathcal{C}_\Phi))} > e^{-t\ln(\lambda_a)}\right\}\\
<& \frac{1}{e^{-t\ln(\lambda_a)}}\mathbb{E}\left(e^{-t\ln(\mathbb{P}^{x_0}(\mathcal{C}_\Phi))}\right)\\
=& e^{t\ln(\lambda_a)}\mathbb{E}\left(\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right)^{-t}\right)\\
=& p^{-t}\exp\left(t\ln(\lambda_a) + t\theta r_0^\alpha WN_0\right)\\
&\mathbb{E}\Big(\prod_{x\in\Phi\setminus\{x_0\}}\Big(\frac{p}{1+\theta r_0^\alpha|x|^{-\alpha}} + 1 - p\Big)^{-t}\Big)\\
=& p^{-t}\exp\Big(t\ln(\lambda_a) + t\theta r_0^\alpha WN_0\\
&-2\pi\lambda\int_0^\infty\Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{-t}\Big)r\mathrm{d}r\Big).
\end{aligned} \tag{29}$$

Solving the following inequality and plugging in $\lambda_s = \lambda_a\log_2(1+\theta)$, we have

$$\begin{aligned}
&p^{-t}\exp\Big(t\ln(\lambda_a) + t\theta r_0^\alpha WN_0\\
&-2\pi\lambda\int_0^\infty\Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{-t}\Big)r\mathrm{d}r\Big) \le \varepsilon,
\end{aligned} \tag{30}$$

we get an upper bound for the arrival rate λ_s in (5) within which the queues at the typical transmitter in the dominant system and in the original system will both be stable.

By setting $t = n \in \mathbb{N}^+$, we get

$$\begin{aligned}
&\lambda_s \le p\varepsilon^{\frac{1}{n}}\log_2(1+\theta)\exp\Big(-\theta r_0^\alpha WN_0\\
&+\frac{2\pi\lambda}{n}\int_0^\infty\Big(1 - \Big(\frac{p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - p\Big)^{-n}\Big)r\mathrm{d}r\Big)\\
\overset{(a)}{=}& p\varepsilon^{\frac{1}{n}}\log_2(1+\theta)\exp\Big(-\theta r_0^\alpha WN_0\\
&-\frac{2\pi\lambda}{n}\sum_{i=0}^n C_n^i(1-(1-p)^i)\int_0^\infty\frac{(\theta r_0^\alpha r^{-\alpha})^i r}{(1+(1-p)\theta r_0^\alpha r^{-\alpha})^n}\mathrm{d}r\Big)\\
\overset{(b)}{=}& p\varepsilon^{\frac{1}{n}}\log_2(1+\theta)\exp\Big(-\theta r_0^\alpha WN_0\\
&-\pi\lambda\delta(1-p)^\delta\theta^\delta r_0^2\sum_{i=1}^n((1-p)^{-i}-1)\frac{\Gamma(i-\delta)\Gamma(n-i+\delta)}{\Gamma(i+1)\Gamma(n-i+1)}\Big),
\end{aligned} \tag{31}$$

where $C_n^i = n!/(i!(n-i)!) = \Gamma(n+1)/(\Gamma(i+1)\Gamma(n-i+1))$ is the binomial coefficient and $\delta = 2/\alpha$. The equation (a) holds from the binomial expansion and from the exchange of summation and integral. The equation (b) follows from the relationship between beta function and gamma function and from the fact that the term for $i = 0$ equals to zero.

Appendix C
Proof of Lemma 1

Consider a dominant system of the simplified system shown in Fig. 2, i.e., the typical transmission still transmits "dummy" packets when its queue is empty, thus it keeps causing interference to the nearest transmission. Unlike the typical transmission, the nearest interfering transmission in the dominant system behaves the same as the original simplified system.

In the dominant system, the typical transmission is active with probability p. The probability that the nearest interfering transmission is scheduled by ALOHA and also successfully transmitted is

$$\begin{aligned} p_2 &\overset{\Delta}{=} p^2 \mathbb{P}\left\{\frac{h_1 r_0^{-\alpha}}{h_2 r_s^{-\alpha} + WN_0} > \theta\right\} + p(1-p)\mathbb{P}\left\{\frac{h_1 r_0^{-\alpha}}{WN_0} > \theta\right\} \\ &\overset{(a)}{=} \left(\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} + 1 - p\right) p \exp\left(-WN_0\theta r_0^\alpha\right), \end{aligned} \tag{32}$$

where h_1 is the fading coefficient between the transmitter and the receiver of the nearest transmission, and h_2 is the fading coefficient between the typical transmitter and the receiver of the nearest transmission. And (a) follows because h_1 and h_2 are both exponentially distributed. In the following, we divide the proof into two cases, i.e., $\lambda_a \geq p_2$ and $\lambda_a < p_2$.

1) The case when $\lambda_a \geq p_2$: When $\lambda_a \geq p_2$, the queue at the nearest interfering transmitter is unstable and will never be empty, thus the nearest interfering transmitter will cause interference to the typical transmission with probability p. Therefore, the probability that the typical transmission is scheduled by ALOHA and also successfully transmitted is

$$\begin{aligned} p_1 &\overset{\Delta}{=} p^2 \mathbb{P}\left\{\frac{h_3 r_0^{-\alpha}}{h_4 r_m^{-\alpha} + WN_0} > \theta\right\} + p(1-p)\mathbb{P}\left\{\frac{h_3 r_0^{-\alpha}}{WN_0} > \theta\right\} \\ &= \left(\frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1 - p\right) p \exp\left(-WN_0\theta r_0^\alpha\right), \end{aligned} \tag{33}$$

where h_3 is the fading coefficient between the transmitter and the receiver of the typical transmission, and h_4 is the fading coefficient between the nearest interfering transmitter and the receiver of the typical transmission.

If $r_s > r_m$, by comparing (32) with (33), we have $\lambda_a > p_2 > p_1$, which implies that the queue at the typical transmitter is unstable for the case $\lambda_a \geq p_2$. This conclusion can be explained intuitively by Fig. 2 since the interference from the typical transmitter to the nearest interfering transmission is less than that from the nearest interfering transmitter to the typical transmission. Thus, when the queue of the nearest interfering transmitter is unstable, the queue at the typical transmitter will also be unstable.

If $r_s \leq r_m$, by comparing (32) with (33), we have $p_1 \geq p_2$, which implies that the queue at the typical transmitter is stable for $p_1 \geq \lambda_a \geq p_2$ and unstable for $\lambda_a > p_1$ for the case $\lambda_a \geq p_2$.

2) The case when $\lambda_a < p_2$: When $\lambda_a < p_2$, the queue of the nearest interfering transmitter is empty with probability $1 - \lambda_a/p_2$ and is nonempty with probability λ_a/p_2. Therefore, the probability that the typical transmission is scheduled by ALOHA and also successfully transmitted is as

$$\begin{aligned} p_1' &= p^2 \frac{\lambda_a}{p_2} \mathbb{P}\left\{\frac{h_3 r_0^{-\alpha}}{h_4 r_m^{-\alpha} + WN_0} > \theta\right\} \\ &\quad + \left(p(1-p)\frac{\lambda_a}{p_2} + p\left(1 - \frac{\lambda_a}{p_2}\right)\right) \mathbb{P}\left\{\frac{h_3 r_0^{-\alpha}}{WN_0} > \theta\right\} \\ &= \left(\frac{p\lambda_a}{p_2}\frac{1}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1 - \frac{p\lambda_a}{p_2}\right) p \exp\left(-WN_0\theta r_0^\alpha\right). \end{aligned} \tag{34}$$

In order to make the typical transmission stable, the arrival rate should satisfy $\lambda_a < p_1'$ which can be evaluated into

$$\begin{aligned} \lambda_a &< \frac{pp_2}{p_2 \exp\left(WN_0\theta r_0^\alpha\right) + p^2 - p^2 \frac{1}{1+\theta r_0^\alpha r_m^{-\alpha}}} \\ &= \frac{p_2}{\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} - \frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1} \\ &= \frac{\left(\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} + 1 - p\right) p \exp\left(-WN_0\theta r_0^\alpha\right)}{\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} - \frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1}. \end{aligned} \tag{35}$$

If $r_s > r_m$, it can be verified that the right side of the above inequality is less than p_2. Therefore, for the case $\lambda_a < p_2$, the queue at the typical transmitter in the dominant system will be stable only when the inequality (35) is fulfilled.

If $r_s \leq r_m$, the right side of the above inequality is larger than p_2. Therefore, for the case $\lambda_a < p_2$, the queue at the typical transmitter in the dominant system will be stable.

Combining the cases of $\lambda_a \geq p_2$ and $\lambda_a < p_2$, we get the stability region of the dominant system as follows.

$$\lambda_a < \begin{cases} \frac{\left(\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} + 1 - p\right) p \exp\left(-WN_0\theta r_0^\alpha\right)}{\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} - \frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1} & \text{if } r_s > r_m \\ \left(\frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1 - p\right) p \exp\left(-WN_0\theta r_0^\alpha\right) & \text{if } r_s \leq r_m \end{cases} \tag{36}$$

Plugging in $\lambda_s = \lambda_a \log_2(1+\theta)$, we get

$$\lambda_s < \begin{cases} \frac{\left(\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} + 1 - p\right) p \log_2(1+\theta) \exp\left(-WN_0\theta r_0^\alpha\right)}{\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} - \frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1} \\ \qquad \text{if } r_s > r_m \\ \left(\frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1 - p\right) p \log_2(1+\theta) \exp\left(-WN_0\theta r_0^\alpha\right) \\ \qquad \text{if } r_s \leq r_m \end{cases} \tag{37}$$

If the queue at the typical transmitter in the dominant system is unstable, then it is also unstable in the original system, because in this case, the queue at the typical transmitter tends to grow to infinity, so the probability of ever sending a dummy packet by the typical transmitter in the dominant system is zero, thus the dominant system is indistinguishable from the original system under saturation. Therefore, we get the sufficient and necessary condition for the original simplified system.

Appendix D
Proof of Theorem 2

According to Lemma 1, if $r_s > r_m$, from (36) we have

$$\begin{aligned} \lambda_a &< \frac{\left(\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} + 1 - p\right) p \exp\left(-WN_0\theta r_0^\alpha\right)}{\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} - \frac{p}{1+\theta r_0^\alpha r_m^{-\alpha}} + 1} \\ &< \left(\frac{p}{1+\theta r_0^\alpha r_s^{-\alpha}} + 1 - p\right) p \exp\left(-WN_0\theta r_0^\alpha\right). \end{aligned} \tag{38}$$

Since Lemma 1 gives the sufficient and necessary condition for stability of the typical transmitter in the simplified system when φ, ψ, r_m are given, by comparing (36) and (38), we obtain a necessary condition as follows.

$$\lambda_a < \left(\frac{p}{1+\theta r_0^{\alpha}\left(\max\{r_m, r_s\}\right)^{-\alpha}} + 1 - p\right) p \exp\left(-WN_0\theta r_0^{\alpha}\right).$$

According to the equation (2) and Lemma 1, when φ, ψ, r_m are random variables, letting $Z = \frac{1}{r_0}\max\{r_m, r_s\}$, Z is a random variable determined by r_m and r_s A necessary condition for the queue at the typical transmitter in the simplified system to be stable is

$$\begin{aligned}\varepsilon \quad &\geq \quad \mathbb{P}\left\{\lambda_a \geq \left(\frac{p}{1+\theta Z^{-\alpha}} + 1 - p\right) p e^{-WN_0\theta r_0^{\alpha}}\right\} \\ &= \quad \mathbb{P}\left\{Z \leq \left(\theta \frac{\lambda_a \exp\left(WN_0\theta r_0^{\alpha}\right) + p^2 - p}{p - \lambda_a \exp\left(WN_0\theta r_0^{\alpha}\right)}\right)^{\frac{1}{\alpha}}\right\}. \quad (39)\end{aligned}$$

Denote the cdf of Z as $F_Z(z)$, whose closed-form expression is hard to derive. Then, the equation (39) can be written as

$$\varepsilon \quad \geq \quad F_Z\left(\left(\theta \frac{\lambda_a \exp\left(WN_0\theta r_0^{\alpha}\right) + p^2 - p}{p - \lambda_a \exp\left(WN_0\theta r_0^{\alpha}\right)}\right)^{1/\alpha}\right), \quad (40)$$

which evaluates to

$$\lambda_a \leq \left(1 - \frac{\theta p}{\theta + \left(F_Z^{-1}(\varepsilon)\right)^{\alpha}}\right) p \exp\left(-WN_0\theta r_0^{\alpha}\right). \quad (41)$$

Plugging in the equality $\lambda_s = \lambda_a \log_2(1+\theta)$, we obtain the result in Theorem 2.

Appendix E
Proof of Corollary 2

According to (2) and Lemma 1, when r_m is a random variable, the sufficient and necessary condition for the queue at the typical transmitter in the simplified system to be stable when $\varphi = \psi = -\pi$ is

$$\begin{aligned}\varepsilon \geq &\mathbb{P}\left\{\lambda_s \geq \frac{p_2 \log_2(1+\theta)}{\frac{p}{1+\theta r_0^{\alpha}(r_m+2r_0)^{-\alpha}} - \frac{p}{1+\theta r_0^{\alpha} r_m^{-\alpha}} + 1}\right\} \\ = &\mathbb{P}\Bigg\{\lambda_s \geq \frac{(r_m^{\alpha} + \theta r_0^{\alpha})\left((r_m+2r_0)^{\alpha} + (1-p)\theta r_0^{\alpha}\right) p \exp(-WN_0\theta r_0^{\alpha})}{(r_m^{\alpha} + (1+p)\theta r_0^{\alpha})\left((r_m+2r_0)^{\alpha} + (1-p)\theta r_0^{\alpha}\right) + p^2\theta^2 r_0^{2\alpha}} \\ &\log_2(1+\theta)\Bigg\}. \quad (42)\end{aligned}$$

Since $f(x) = x/(1+x)$ is an increasing function of x, we obtain a necessary condition for stability as follows

$$\begin{aligned}\varepsilon > &\mathbb{P}\left\{\lambda_s \geq \frac{\left((r_m+2r_0)^{\alpha} + \theta r_0^{\alpha}\right)^2 p \exp(-WN_0\theta r_0^{\alpha})}{\left((r_m+2r_0)^{\alpha} + \theta r_0^{\alpha}\right)^2 + p^2\theta^2 r_0^{2\alpha}} \log_2(1+\theta)\right\} \\ = &\mathbb{P}\Bigg\{\left(p\log_2(1+\theta)\exp(-WN_0\theta r_0^{\alpha}) - \lambda_s\right) \\ &\left((r_m+2r_0)^{\alpha} + \theta r_0^{\alpha}\right)^2 \leq \lambda_s p^2\theta^2 r_0^{2\alpha}\Bigg\}. \quad (43)\end{aligned}$$

Since the inequality $p\exp(-WN_0\theta r_0^{\alpha}) - \lambda_a > 0$ is satisfied from Lemma 1, we have

$$\begin{aligned}&\varepsilon > \mathbb{P}\Bigg\{r_m \leq \\ &\underbrace{\left(\sqrt{\frac{\lambda_s p^2\theta^2 r_0^{2\alpha}}{p\log_2(1+\theta)\exp(-WN_0\theta r_0^{\alpha}) - \lambda_s}} - \theta r_0^{\alpha}\right)^{1/\alpha} - 2r_0}_{A}\Bigg\}. \\ &\quad (44)\end{aligned}$$

When $A \leq 0$, the probability at the right side of the inequality is zero; thus the above inequality (44) always holds. When $A > 0$, by applying the probability distribution of r_m given by (9), we have

$$\varepsilon > 1 - \exp\left(-\pi\lambda A^2\right). \quad (45)$$

Then, we have

$$0 < A < \sqrt{-\frac{\ln(1-\varepsilon)}{\pi\lambda}}. \quad (46)$$

Combining the cases of $A \leq 0$ and $A > 0$, we have

$$\begin{aligned}&\left(\sqrt{\frac{\lambda_s p^2\theta^2 r_0^{2\alpha}}{p\log_2(1+\theta)\exp(-WN_0\theta r_0^{\alpha}) - \lambda_s}} - \theta r_0^{\alpha}\right)^{1/\alpha} \\ &-2r_0 < \sqrt{-\frac{\ln(1-\varepsilon)}{\pi\lambda}}. \quad (47)\end{aligned}$$

Solving the above inequality, we get

$$\begin{aligned}&\lambda_s < \left(1 + \left(\frac{p\theta r_0^{\alpha}}{\left(\sqrt{-\frac{\ln(1-\varepsilon)}{\pi\lambda}} + 2r_0\right)^{\alpha} + \theta r_0^{\alpha}}\right)^2\right)^{-1} \\ &p\log_2(1+\theta)\exp(-WN_0\theta r_0^{\alpha}). \quad (48)\end{aligned}$$

Appendix F
Proof of Theorem 3

By introducing the modified system, the packets in the interfering transmitters will be abandoned if they are not scheduled or failed for transmitting, thus an interfering transmitter is active with probability $\lambda_a p$. Similar to the derivations of (24), we get the success probability for the typical transmission conditioned on Φ in the modified system as follows

$$\begin{aligned}&\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \\ = \quad &p\exp\left(-\theta r_0^{\alpha} WN_0\right) \\ &\prod_{x\in\Phi\setminus\{x_0\}}\Big(\lambda_a p \mathbb{E}^{x_0}\Big(\exp\big(-\theta r_0^{\alpha} h_{k,x}|x|^{-\alpha}\big) \mid \Phi\Big) + 1 - \lambda_a p\Big) \\ = \quad &p\exp\big(-\theta r_0^{\alpha} WN_0\big) \\ &\prod_{x\in\Phi\setminus\{x_0\}}\Big(\frac{\lambda_a p}{1+\theta r_0^{\alpha}|x|^{-\alpha}} + 1 - \lambda_a p\Big). \quad (49)\end{aligned}$$

Let $Y \triangleq \ln\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right)$, then the moment generating function of Y is given by

$$\begin{aligned}&M_Y(s) = p^s \exp\Big(-s\theta r_0^{\alpha} WN_0 \\ &-2\pi\lambda\int_0^{\infty}\Big(1 - \Big(\frac{\lambda_a p}{1+\theta r_0^{\alpha} r^{-\alpha}} + 1 - \lambda_a p\Big)^s\Big) r\mathrm{d}r\Big). \quad (50)\end{aligned}$$

The cdf of Y can be derived as follows by applying the Gil-Pelaez Theorem given by (26).

$$F_Y(y) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\{e^{-j\omega y} M_Y(j\omega)\}}{\omega} \mathrm{d}\omega. \quad (51)$$

The probability that the queue at the typical transmitter in the modified system being unstable is

$$\begin{aligned} & \mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \le \lambda_a\right\} \\ = \; & F_Y(\ln(\lambda_a)) \\ = \; & \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\{e^{-j\omega \ln(\lambda_a)} M_Y(j\omega)\}}{\omega} \mathrm{d}\omega. \end{aligned} \quad (52)$$

The condition for the queue at the typical transmitter in the modified system to be stable is $\mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) \le \lambda_a\right\} < \varepsilon$. By combining (50) and (52), and plugging in the equality $\lambda_s = \lambda_a \log_2(1+\theta)$, we get the condition for the queue at the typical transmitter in the modified system to be stable as follows

$$\begin{aligned} & \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{1}{\omega}\mathrm{Im}\Big\{ p^{j\omega} \exp\Big(- j\omega\theta r_0^\alpha W N_0 \\ & -2\pi\lambda \int_0^\infty \Big(1 - \Big(\frac{\lambda_s p/\log_2(1+\theta)}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - \frac{\lambda_s p}{\log_2(1+\theta)}\Big)^{j\omega}\Big) r\mathrm{d}r \\ & -j\omega \ln\Big(\frac{\lambda_s}{\log_2(1+\theta)}\Big)\Big)\Big\}\mathrm{d}\omega < \varepsilon. \end{aligned} \quad (53)$$

Solving the above inequality, we get an upper bound for the arrival rate λ_s which gives a necessary condition for the original system to be stable.

Appendix G
Proof of Corollary 3

For any $t > 0$, by applying the Markov's inequality, we obtain the following inequality

$$\begin{aligned} & \mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) < \lambda_a\right\} \\ = \; & \mathbb{P}^{x_0}\left\{(\mathbb{P}^{x_0}(\mathcal{C}_\Phi))^t < \lambda_a^t\right\} \\ > \; & 1 - \lambda_a^{-t}\mathbb{E}\left((\mathbb{P}^{x_0}(\mathcal{C}_\Phi))^t\right) \\ = \; & 1 - p^t \exp\big(- t\ln(\lambda_a) - t\theta r_0^\alpha W N_0\big) \\ & \mathbb{E}\Big(\prod_{x\in\Phi\setminus\{x_0\}} \Big(\frac{\lambda_a p}{1+\theta r_0^\alpha |x|^{-\alpha}} + 1 - \lambda_a p\Big)^t\Big) \\ = \; & 1 - p^t \exp\Big(- t\ln(\lambda_a) - t\theta r_0^\alpha W N_0 \\ & -\lambda \int_{R^2} \Big(1 - \Big(\frac{\lambda_a p}{1+\theta r_0^\alpha |x|^{-\alpha}} + 1 - \lambda_a p\Big)^t\Big)\mathrm{d}x\Big) \\ = \; & 1 - p^t \exp\Big(- t\ln(\lambda_a) - t\theta r_0^\alpha W N_0 \\ & -2\pi\lambda \int_0^\infty \Big(1 - \Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - \lambda_a p\Big)^t\Big) r\mathrm{d}r\Big). \end{aligned} \quad (54)$$

Solving the following inequality and plugging in the equality $\lambda_s = \lambda_a \log_2(1+\theta)$,

$$\begin{aligned} & 1 - p^t \exp\Big(- t\ln(\lambda_a) - t\theta r_0^\alpha W N_0 \\ & -2\pi\lambda \int_0^\infty \Big(1 - \Big(\frac{\lambda_a p}{1+\theta r_0^\alpha r^{-\alpha}} + 1 - \lambda_a p\Big)^t\Big) r\mathrm{d}r\Big) \le \varepsilon, \end{aligned} \quad (55)$$

we get an upper bound for the arrival rate λ_s in (17) which gives a type II necessary condition for the original system to be stable.

By applying the Markov inequality, we obtain the following inequality

$$\begin{aligned} & \mathbb{P}^{x_0}\left\{\mathbb{P}^{x_0}(\mathcal{C}_\Phi) < \lambda_a\right\} \\ > \; & 1 - \lambda_a^{-1}\mathbb{E}\left(\mathbb{P}^{x_0}(\mathcal{C}_\Phi)\right) \\ = \; & 1 - p\lambda_a^{-1}\exp\Big(- \theta r_0^\alpha W N_0 - \lambda_a p \lambda \pi r_0^2 \theta^\delta \frac{\pi\delta}{\sin(\pi\delta)}\Big). \end{aligned} \quad (56)$$

Letting $\mathcal{W}(z)$ be the Lambert W function, which is defined by $z = \mathcal{W}(z)e^{\mathcal{W}(z)}$ for any complex number z. Solving the following inequality and plugging in the equality $\lambda_s = \lambda_a \log_2(1+\theta)$,

$$1 - p\lambda_a^{-1}\exp\Big(- \theta r_0^\alpha W N_0 - \lambda_a p \lambda \pi r_0^2 \theta^\delta \frac{\pi\delta}{\sin(\pi\delta)}\Big) \le \varepsilon, \quad (57)$$

we get an upper bound for the arrival rate λ_s in Corollary 3 which gives a type II necessary condition for the original system to be stable.