ALOHA Performs Optimal Power Control in Poisson Networks

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***Abstract*—This paper studies power control strategies in interference-limited wireless networks with Poisson distributed nodes. We focus on the case where each transmitter knows its distance to its desired receiver but does not know the topology of the rest of the network. We study three sets of strategies: the single-node optimal power control (SNOPC) strategy, the Nash equilibrium power control (NEPC) strategy, and the globally optimal power control (GOPC) strategy. SNOPC strategies maximize the expected throughput of the power controllable link given that all the other transmitters do not use power control. Under NEPC strategies, no individual node of the network can achieve a higher expected throughput by unilaterally deviating from these strategies. The GOPC strategy maximizes the throughput of a typical node in the network.**

This paper shows that under mean and peak power constraints at each transmitter, all of the three strategies are ALOHA-type random on-off power control policies in bipolar networks. For links of iid random distances, we show both SNOPC and NEPC strategies are ALOHA-type random on-off policies. These results suggest that ALOHA can be viewed not only as a MAC scheme but also as an efficient and stable power control scheme.

I. Introduction

In wireless networks, power control provides interference management and trade-offs between energy and throughput [1]. Relatively recently, benefits of random power control have been observed in different contexts, *e.g.*, [2], [3]. In the case where channel state information is not completely known, randomly varying the transmit power can boost the performance of wireless communication. In particular, [4] shows that an ALOHA-type random on-off power control policy maximizes the expected throughput in a noise-limited wireless network. This paper extends this result to interference-limited networks, where concurrent transmissions limit the network throughput.

This paper concentrates on three types of strategies: 1) The single-node optimal power control (SNOPC) strategy, where only one node in the network uses power control; 2) Nash equilibrium power control (NEPC) strategy and 3) Globally optimal power control (GOPC) strategy when all the nodes in the network use power control; The SNOPC strategy maximize the expected throughput of the power controllable link. The NEPC strategy ensures that no individual node of the network can achieve a higher expected throughput by unilaterally deviating from these strategies. The GOPC strategy maximizes the throughput of a typical link in the network.

This paper shows that ALOHA-type random on-off power control policies are single-node optimal and constitute Nash equilibria. In Poisson bipolar network, all of the three strategies are ALOHA-type random on-off power control policies, and the transmit power and transmit probability can be expressed in closed-form.

II. System Model

A. Network Model

The network topology is represented as a marked Poisson point process (PPP) $\hat{\Phi} = \{(x_i, y_{x_i})\} \subset \mathbb{R}^2 \times \mathbb{R}^2$, where $\Phi = \{x_i\}$ is a homogeneous PPP with intensity λ and denotes the location of the transmitters, and the marks y_x denote the location of a dedicated receiver of transmitter x. The link distance $R_x \triangleq \|x - y_x\|$ is iid with distribution f_R.

We consider the following SIR model, where a transmission attempt from z to y_z is considered successful iff

$$\mathrm{SIR}_z \triangleq \frac{S_z}{I_z} > \theta,$$

where $S_z = P_z h_z \|z - y_z\|^{-\alpha}$, $I_z = \sum_{x \in \Phi \setminus \{z\}} P_x h_{xz} \|x - y_z\|^{-\alpha}$, P_x is the transmit power at node $x \in \Phi$, $\alpha > 2$ is the path-loss exponent, θ is the SIR threshold, h_z and h_{xz} are (power) fading coefficients from the desired transmitter and the interferer x to z respectively. We focus on the iid Rayleigh fading case, thus both h_z and (h_{xz}) are iid exponentially distributed with unit mean. In the following, we use I for I_z for simplicity.

B. Game-Theoretic Formulation

The players in the game are all the transmitters in the network $x \in \Phi$. Each of the players can select a strategy s_x from a common set of stationary strategies $\mathcal{S}$. Here, $\mathcal{S}$ is the set of distributions with (at most) unit mean and with support (at most) $[0, P_{\max}]$, where $P_{\max} > 1$ (otherwise, the mean power constraint would always be loose).

The strategy each node chooses is based on its knowledge about the network. In particular, we consider the case where the transmitters knows the network density, the distance to its desired receiver, and the distribution of the link distances in the network. In other words, if $\mathcal{I}_x$ is the information available at node x, we have $\mathcal{I}_x = \{\lambda, R_x, f_R\}$.

The pay-off of node $x \in \Phi$ is its own expected throughput (success probability) averaged over all the randomness in the rest of the network, *i.e.*, $\pi_x(s_x) = p_{s|\mathcal{I}_x}(s_x) = \mathbb{P}(\frac{S_x}{I_x} > \theta \mid \mathcal{I}_x, s_x(\mathcal{I}_x))$. The single-node optimal power control (SNOPC) strategy of node x maximizes $\pi_x(\cdot)$ if all the other transmitters in the network transmit with unit power (no power control). If all the transmitters in the network use power control, we say that a strategy set $\{s_x(\mathcal{I}_x), x \in \Phi\}$ is a *Nash equilibrium* and $s_x(\mathcal{I}_x)$ is the Nash equilibrium power control (NEPC) strategy if none of the transmitters is willing to unilaterally deviate from its current strategy as that cannot increase its pay-off (expected throughput).

In addition to the game-theoretic framework above, we study the global impact of SNOPC and NEPC by evaluating the spatially averaged throughput, (or, simply spatial throughput), defined as the throughput (success probability) of a typical node in the network, which can be expressed as

$$p_s = \mathbb{E}^{!x}[\pi_x(s_x)],$$

where $\mathbb{E}^{!x}$ is the expectation with respect to the reduced Palm measure. In the case of a PPP, by Slivnyak's theorem, $\mathbb{E}^{!x} = \mathbb{E}$, *i.e.*, having a node at location x does not change the distribution of the point process [5].

A strategy set $\{s_x(\mathcal{I}_x), x \in \Phi\}$ is said to be the globally optimal power control (GOPC) strategy if it maximizes the spatial throughput of the network.

III. SNOPC and NEPC Strategies for General Link Distances

This section derives the SNOPC and NEPC strategies for general f_R. We start with the SNOPC strategy and then study the Nash equilibrium. First, we introduce two lemmas:

Lemma 1. *If the interferers are distributed as a homogeneous Poisson point process Φ with intensity λ and the transmit power at each transmitter is drawn iid from the same distribution f_P, the interference observed at any receiver y_z with $z \in \Phi$ has the Laplace transform*

$$\mathcal{L}_I(s) = \exp(-\lambda c_d \mathbb{E}[P^\delta]\mathbb{E}[h^\delta]\Gamma(1-\delta)s^\delta),$$

where $\delta = 2/\alpha$.

Proof: First, by Slivnyak's theorem, $\mathcal{L}_I(s) = \mathbb{E}[\prod_{x\in\Phi} e^{-sP_x h_x \|x\|^{-\alpha}}]$, where P_x is the transmit power at x. Then, since P_x, $\forall x \in \Phi$, is iid, $P_x h_x$ can be considered as a new fading coefficient $\tilde{h}_y$. The proof is then completed by the Laplace transform of the interference distribution for arbitrary iid fading with finite δ-th moment [6, Sec. 3.2]. ∎

Lemma 2. *Given a link of length $R = r$, if there exists $x_0 > 0$ such that $x\mathcal{L}_I(\theta r^\alpha x)$ is monotonically increasing for $x < x_0$ and monotonically decreasing for $x > x_0$, the power control strategy that maximizes the throughput at node x is random on-off power control with transmit power γ and transmit probability γ^{-1} where $\gamma = \max\{1, \min\{P_{\max}, x_0^{-1}\}\}$.*

Proof: For interference-limited Rayleigh fading networks, the success probability of a transmission at power P is $\mathcal{L}_I(s)\,|_{s=\frac{\theta r^\alpha}{P}}$. Thus, the success probability of any power control strategy characterized by the pdf f_P of the random variable P is

$$p_s = \mathbb{E}_P\left[\mathcal{L}_I(s)\,|_{s=\frac{\theta r^\alpha}{P}}\right] = \int_0^\infty \mathcal{L}_I\left(\frac{\theta r^\alpha}{x}\right) f_P(x)\mathrm{d}x. \quad (1)$$

It is easy to show that $\mathcal{L}_I(x)$ is a valid ccdf, *i.e.*, $\mathcal{L}_I(0) = 1$, $\lim_{x\to\infty}\mathcal{L}_I(x) = 0$, and $\mathcal{L}_I(x)$ is monotonically decreasing on $[0,\infty)$. So, instead, we can consider an interferenceless link of distance r with another fading random variable $\tilde{h}$ whose ccdf is $\bar{F}_{\tilde{h}}(x) = \mathcal{L}_I(x)$. The success probability is

$$\begin{aligned}\tilde{p}_s &= \mathbb{P}(P\tilde{h}r^{-\alpha} > \theta) = \mathbb{E}_P\left[\bar{F}_{\tilde{h}}\left(\frac{\theta r^\alpha}{P}\right)\right] \\ &= \int_0^\infty \mathcal{L}_I\left(\frac{\theta r^\alpha}{x}\right) f_P(x)\mathrm{d}x.\end{aligned} \quad (2)$$

Comparing (1) and (2), we see that finding the SNOPC strategy that maximizes p_s and finding the one for $\tilde{p}_s$ are two identical problems. The latter problem has already been solved in [4]. In particular, Theorem 2 in [4] shows that if there exists such a x_0 as in the statement of the lemma, subject to the constraints $\mathbb{E}[P] \le 1$ and $P \le P_{\max}$, $\tilde{p}_s$ is maximized when $f_P(x) = (1-\gamma^{-1})\delta(x) + \gamma^{-1}\delta(x-\gamma)$, where $\gamma = \max\{1, \min\{P_{\max}, x_0^{-1}\}\}$. ∎

Corollary 1. *If the Laplace transform of the interference I has the form $\mathcal{L}_I(s) = \exp(-as^\delta)$, where $\delta = 2/\alpha$ and $a > 0$, the throughput-maximizing power control strategy at any transmitter $z \in \Phi$ with $R_z = r$ is a random on-off power control strategy with transmit power γ and transmit probability γ^{-1}, where $\gamma = \max\{1, \min\{P_{\max}, (a\delta)^{1/\delta}\theta r^\alpha\}\}$.*

Corollary 1 is proved by simply verifying that the Laplace transform of the interference distribution satisfies the conditions in Lemma 2.

Proposition 1. *If only one node $z \in \Phi$ with $R_z = r$ uses power control and all other nodes $\Phi\backslash\{z\}$ transmit at unit power, the SNOPC strategy of z is an ALOHA-type random on-off power control strategy with transmit power γ and transmit probability γ^{-1}, where $\gamma = \max\{1, \min\{P_{\max}, \left(\lambda\frac{\pi^2\delta^2}{\sin(\pi\delta)}\right)^{1/\delta}\theta r^\alpha\}\}$.*

Proof: The proposition follows directly from Lemma 1 ($P \equiv 1$) and Corollary 1. ∎

Moreover, since the transmit power at each node $x \in \Phi$ is a (stochastic) function of the link distances $R_x = r$, where the R_x are spatially iid, Lemma 1 shows that the interference always has a Laplace transform of the form $\exp(-as^\delta)$, regardless of what kind of power control strategy is applied at each node. Then, the proposition below follows.

Proposition 2. *ALOHA-type random on-off power control is the unique NEPC strategy in a wireless network where the transmitters are distributed as a homogeneous Poisson point process Φ and $\mathcal{I}_x = \{\lambda, R_x, f_R\}$, for all $x \in \Phi$.*

Proof: The fact that ALOHA-type random on-off power control at each node is a Nash equilibrium can be deduced directly from Lemma 1 and Corollary 1. In particular, we can write $\mathbb{E}[P^\delta]$ in terms of the throughput-maximizing random on-off strategy at each link, which yields

$$\mathbb{E}[P^\delta] = \mathbb{E}_R[P_R^\delta] = \mathbb{E}_R[\gamma_R^{-1}\gamma_R^\delta]$$
$$= \mathbb{E}\left[\min\left\{1, \max\{P_{\max}^{\delta-1}, \left(\lambda\mathbb{E}[P^\delta]C(\delta)\right)^{1-\frac{1}{\delta}} (\theta R^\alpha)^{\delta-1}\}\right\}\right], \quad (3)$$

where $C(\delta) = \frac{\pi^2\delta^2}{\sin(\pi\delta)}$. Note that the RHS of (3) is a monotonically decreasing function of $\mathbb{E}[P^\delta]$ (since $1 - 1/\delta < 0$), and when $\mathbb{E}[P^\delta] = 0$, its value is $P_{\max}^{\delta-1} > 0$. Thus, there is a unique $\mathbb{E}[P^\delta] > 0$ satisfying (3). Once this value is found, the optimal power control strategy at x is simply an ALOHA policy with transmit power γ and transmit probability γ^{-1}, where $\gamma = \max\{1, \min\{P_{\max}, \left(\lambda\mathbb{E}[P^\delta]\frac{\pi^2\delta^2}{\sin(\pi\delta)}\right)^{1/\delta} \theta r_x^\alpha\}\}$.

Moreover, Lemma 1 also says that no matter what kind of power control policy is applied in the rest of the network, the interference distribution observed at an arbitrary receiver has a Laplace transform of the form $\mathcal{L}_I(s) = \exp(-as^\delta)$. Thus, Corollary 1 also indicates the uniqueness. ■

IV. Power Control in Bipolar Networks

As a special case of the networks discussed in previous sections, in (standard) *bipolar networks*, the link distances are a known and constant r, *i.e.*, $f_R(x) = \delta(x-r)$ [7]. This section focuses on this type of network and shows that, in bipolar networks[1], the NEPC strategy derived in Section III can be further expressed in closed-form, and the GOPC strategy can be derived.

A. The NEPC Strategy

For general f_R, finding the NEPC strategy involves solving $\mathbb{E}[P^\delta]$ from (3), which has to be done numerically. However, in Poisson bipolar networks [7], closed-form expressions for the NEPC strategy can be obtained as follows.

Corollary 2. *If all the link distances are r, the NEPC strategy is an ALOHA-type random on-off policy with transmit power γ and transmit probability γ^{-1} where $\gamma = \max\{1, \min\{P_{\max}, \lambda\frac{\pi^2\delta^2}{\sin(\pi\delta)}\theta^\delta r^2\}\}$.*

Proof: For Rayleigh fading, h is exponentially distributed with mean 1, and thus $\mathbb{E}[h^\delta] = \Gamma(1+\delta)$. Then, when the link distances are the same, (3) becomes

$$\mathbb{E}[P^\delta] = \left(\max\left\{1, \min\left\{P_{\max}, \left(\lambda C(\delta)\mathbb{E}[P^\delta]\right)^{1/\delta}\theta r^\alpha\right\}\right\}\right)^{\delta-1},$$

where $C(\delta) = \frac{\pi^2\delta^2}{\sin(\pi\delta)}$. Solving this equation for $\mathbb{E}[P^\delta]$ and applying to $\gamma = \max\{1, \min\{P_{\max}, (\lambda\mathbb{E}[P^\delta]\frac{\pi^2\delta^2}{\sin(\pi\delta)})^{1/\delta}\theta r^\alpha\}\}$ yields the desired result. ■

Corollary 2 says that in any case, an ALOHA-type random on-off policy is the NEPC policy in a Poisson bipolar network. For ALOHA-type random on-off strategies with transmit power γ and transmit probability γ^{-1}, we define the following regimes to facilitate our illustration.

Definition 1. *A random on-off power control strategy is said to be in its peak-power-limited regime if the transmit power is $P_{\max}$.*

Definition 2. *A random on-off power control strategy is said to be in its bandwidth-limited regime if the transmit power is 1.*

B. The GOPC Strategy

The NEPC strategy characterized by Corollary 2 is a stable in the sense that no selfish user is motivated to deviate from this strategy. However, in general, the NEPC strategy is suboptimal in terms of the spatial throughput of the network. In this subsection, we show that a GOPC strategy which maximizes the spatial throughput can also be derived based on the same framework, and this GOPC strategy is also a ALOHA type random on-off power control policy.

Definition 3. *For a link of length r, a power control policy $P(r)$ is* δ-optimal *under interference I iff it maximizes the success probability under the constraint $\mathbb{E}[P^\delta] \le 1$ and $P \le P_{\max}$.*

Similarly, we say a power control policy is δ-NEPC iff $\{s_x, x \in \Phi\}$ constitutes a Nash equilibrium, and a power control policy is δ-GOPC iff the throughput of a typical link in the network is maximized, both under the constraint $\mathbb{E}[P_x^\delta] \le 1$ and $P_x \le P_{\max}$ for all $x \in \Phi$. Then, we have the following lemma:

Lemma 3. *Given a link of length $R = r$, if there exists $x_0 > 0$ such that $x\mathcal{L}_I(\theta r^\alpha x^{1/\delta})$ is monotonically increasing for $x < x_0$ and monotonically decreasing for $x > x_0$, the δ-optimal power control strategy is random on-off policy with transmit power $\gamma^{1/\delta}$ and transmit probability γ^{-1} where $\gamma = \max\{1, \min\{P_{\max}^\delta, x_0^{-1}\}\}$.*

Proof: The success probability of the link of length r can be written as

$$p_s = \mathbb{E}_P\left[\mathcal{L}_I(s)|_{s=\frac{\theta r^\alpha}{P}}\right] = \mathbb{E}_P\left[\mathcal{L}_{\tilde{I}}(s)|_{s=\frac{\theta l^\alpha}{P^\delta}}\right], \quad (4)$$

where $l^\alpha = \theta^{\delta-1}r^2$ and $\mathcal{L}_{\tilde{I}}(s) \triangleq \mathcal{L}_I(s^{1/\delta})$ for all $s \ge 0$. Thus, p_s can be interpreted as the success probability of a transmission over a link of length l when the transmit power P^δ is applied and the interference is $\tilde{I}$. Moreover, the peak power constraint is equivalent to $P^\delta \le P_{\max}^\delta$. Combining Lemma 2 with the fact that $x\mathcal{L}_{\tilde{I}}(\theta l^\alpha x) = x\mathcal{L}_I(\theta r^\alpha x^{1/\delta})$ completes the proof. ■

Since Lemma 1 shows that the interference in Poisson networks always has the Laplace transform $\exp(-ax^\delta)$ for some constant a, Lemma 3 naturally leads to the following corollary.

Corollary 3. *An ALOHA-type random on-off power control policy is the unique δ-NEPC strategy in Poisson networks.*

[1]Following the most common use of this term in the literature, we refer *bipolar networks* always to the networks where the links are all of the same deterministic length.

The idea of the proof is analogous to that of Proposition 2, and thus is omitted in this paper. It is easy to verify that at the equilibrium, the transmit power at each link of length r is $\gamma^{1/\delta}$ and the transmit probability is γ^{-1}, where $\gamma = \max\{1, \min\{\lambda\frac{\pi^2\delta}{\sin(\pi\delta)}\theta^\delta r^2, P_{\max}^\delta\}\}$.

Lemma 4. *Without the peak power constraint,* i.e., $P_{\max} = \infty$*, the δ-NEPC strategy is the δ-GOPC strategy in Poisson bipolar networks.*

Proof: First, since all the link distances are the same, the information available at each transmitter $\mathcal{I}_x,\ x \in \Phi$ are the same, which results in the fact that δ-GOPC strategy must have all the nodes in the network use the same power control strategy. We denote such a power control strategy by random variable P and the spatial throughput achieved by such strategy by $p_s(P)$.

Second, if we fix $\mathbb{E}[P^\delta] = 1$, the spatial throughput is maximized by the δ-NEPC strategy. Because $\mathbb{E}[P^\delta]$ is fixed to be 1, the interference distribution is fixed with Laplace transform $\exp(-\lambda\frac{\pi^2\delta}{\sin(\pi\delta)}s^\delta)$. By definition, the δ-NEPC strategy maximizes the expected throughput at each link under this interference distribution and thus maximizes the spatial throughput.

Finally, if we use P_γ to denote the transmit power under the δ-NEPC strategy, we must have $p_s(P) \leq p_s(P_\gamma)$ for all P with $\mathbb{E}[P^\delta] \in \mathbb{R}^+$. This can be proved by contradiction: Assume there exists $\tilde{P}$ such that $p_s(\tilde{P}) > p_s(P_\gamma)$ and $\mathbb{E}[\tilde{P}^\delta] = W \in \mathbb{R}^+$. We can construct another power control policy with transmit power $\hat{P} = \tilde{P}/W^{1/\delta}$. Obviously, $\mathbb{E}[\hat{P}] = 1$ but $p_s(\hat{P}) = p_s(\tilde{P}) > p_s(P_\gamma)$, which contradicts the fact that the δ-NEPC strategy maximizes the spatial throughput when $\mathbb{E}[P^\delta] = 1$. ∎

A direct consequence of Lemma 4 is the GOPC strategy in the following proposition.

Proposition 3. *In the bipolar networks where $R \equiv r$, the GOPC strategy is ALOHA-type random on-off policy with transmit power $k\gamma^{1/\delta}$ and transmit probability γ^{-1}, where $k \leq \min\{P_{\max}\gamma^{-1/\delta}, \gamma^{1-1/\delta}\}$ and $\gamma = \max\{1, \lambda\frac{\pi^2\delta}{\sin(\pi\delta)}\theta^\delta r^2\}$.*

Proof: As is shown in the proof if Lemma 4, it is straightforward to show that no power control strategy can achieve a higher spatial throughput than the spatial throughput achieved by the δ-NEPC strategy. Then, the proof can be completed by verify that the strategy stated above achieves this maximum spatial throughput while satisfying the mean and peak power constraints. ∎

Not surprisingly, the GOPC strategy characterized by Proposition 3 matches the ALOHA scheme derived in [7] which maximizes the spatial throughput in Poisson bipolar networks. However, there are two major differences between Proposition 3 and the results in [7]. First, the ALOHA scheme in [7] only specifies the transmit probability while the GOPC strategy also specifies the range of the transmit power since mean and peak power constraints are considered. Second, [7] finds its ALOHA scheme by optimizing the transmit probability, *i.e.*, its optimality is among all ALOHA policies. However, Proposition 3 shows such an ALOHA scheme is optimal among all random power control strategies.

V. Comparison of Power Control Strategies

A. Bipolar Networks

In Rayleigh fading network, the throughput (success probability) of a transmission can be expressed in terms of the Laplace transform of the interference distribution. In particular, if no power control is applied and all the transmitters transmit with unit power, the throughput of a typical link is given by [6]

$$p_s(r) = \exp\left(-\lambda\pi s^\delta \frac{\pi\delta}{\sin(\pi\delta)}\right)\bigg|_{s=\theta r^\alpha}.$$

Similarly, when the SNOPC strategy described in Proposition 1 is applied at a single link of length r, the throughput can be expressed as $p_s(r) =$

$$\begin{cases} \exp\left(-\lambda\pi(\theta r^\alpha)^\delta \frac{\pi\delta}{\sin(\pi\delta)}\right), & r \leq R_1 \\ \frac{\exp(-1/\delta)}{\theta r^\alpha}\left(\frac{\sin(\pi\delta)}{\lambda\pi^2\delta^2}\right)^{1/\delta}, & R_1 < r \leq R_2 \\ P_{\max}^{-1}\exp\left(-\lambda\pi\left(\frac{\theta r^\alpha}{P_{\max}}\right)^\delta \frac{\pi\delta}{\sin(\pi\delta)}\right), & r > R_2, \end{cases} \tag{5}$$

where $R_1 = \theta^{-1/\alpha}\sqrt{\frac{\sin(\pi\delta)}{\lambda\pi^2\delta^2}}$ and $R_2 = \left(\frac{P_{\max}}{\theta}\right)^{1/\alpha}\sqrt{\frac{\sin(\pi\delta)}{\lambda\pi^2\delta^2}}$.

In bipolar networks, the NEPC strategy is described in Corollary 2, and the expected throughput at each link can be calculated analogously as

$$p_s(r) = \gamma^{-1}\exp\left(-\lambda\gamma^{-1}\frac{\pi^2\delta}{\sin(\pi\delta)}\theta^\delta r^2\right),$$

where $\gamma = \max\{1, \min\{P_{\max}, \lambda\pi r^2\frac{\pi\delta^2}{\sin(\pi\delta)}\theta^\delta\}\}$.

As indicated by Proposition 3, the GOPC strategy is not unique. In fact, the optimality of GOPC strategies only depends on the properly chosen transmit probability, and the absolute transmit power does not matter as long as the mean and peak power constraints are satisfied. However, in order to make a fair comparison with the NEPC strategy, we always choose the maximum transmit power for the GOPC strategy, *i.e.*, $k = \min\{P_{\max}\gamma^{-1/\delta}, \gamma^{1-1/\delta}\}$. Let P_{NEPC} and p_{NEPC} be the transmit power and transmit probability of the NEPC strategy respectively, and P_{GOPC} and p_{GOPC} be the transmit power and transmit probability of the GOPC strategy respectively. It can be shown that, for the same parameters, we always have $P_{\text{GOPC}} \geq P_{\text{NEPC}}$ and $p_{\text{GOPC}} \leq p_{\text{NEPC}}$. In other words, the GOPC strategy achieves higher spatial throughput by forcing each transmitter to back off on their transmit probability.

However, GOPC is unstable in the sense that any selfish link can apply another power control strategy and thus obtain a performance far better than anyone else. It is not difficult to see (by slight variation to Proposition 1) that the best response of any individual link in a bipolar network applying this GOPC strategy is an ALOHA policy with transmit power γ_{BR} and transmit probability γ_{BR}^{-1}, where the subscript BR stands for

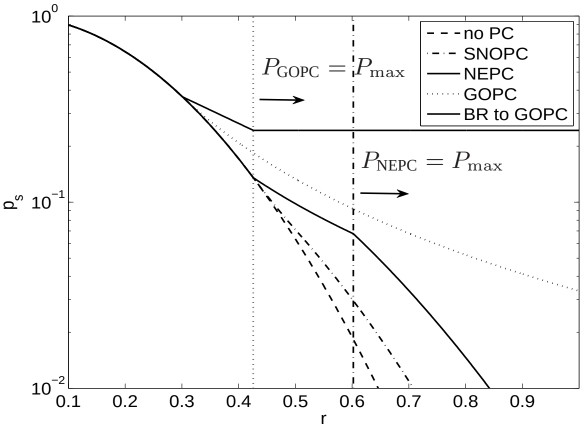

Fig. 1: Comparison of throughput in bipolar networks using 1) constant power transmission (no power control) 2) the SNOPC strategy (at a single link) 3) the NEPC strategy 4) the GOPC strategy 5) the best response to the GOPC strategy. Here, $\lambda = 1$, $P_{\max} = 2$, $\alpha = 4$, $\theta = 5$. At the RHS of the two vertical lines, the transmit power of the GOPC/NEPC strategy hits the peak power limit.

best response and $\gamma_{\text{BR}} = \max\{1, P_{\text{GOPC}}\delta^{\delta}\}$

$$= \max\left\{1, \min\left\{P_{\max}, \left(\lambda p_{\text{GOPC}} P_{\text{GOPC}}^{\delta} \frac{\pi^2 \delta^2}{\sin(\pi\delta)}\right)^{1/\delta} \theta r^{\alpha}\right\}\right\}.$$

Here, $\gamma_{\text{BR}}^{-1} \geq p_{\text{GOPC}}$, and the equality holds only when $\gamma_{\text{BR}}^{-1} = p_{\text{GOPC}} = 1$, *i.e.*, both strategies operate in the bandwidth-limited regime.

Fig. 1 compares the throughput/spatial throughput of 5 strategies: constant-power transmission (no power control), the SNOPC strategy when the rest of the network does not use power control, the NEPC strategy, the GOPC strategy, and the best response to the GOPC strategy in Poisson bipolar networks. We can see from the figure that the NEPC policy has a better performance than constant power transmission. As expected, outside the bandwidth-limited regime of both GOPC and NEPC, NEPC has a spatial throughput strictly smaller than GOPC. However, the performance gain of GOPC over NEPC mostly comes from forcing each transmitter in the network to reduce its mean transmit power and thus manage the interference, *i.e.*, for large r, $p_{\text{GOPC}} P_{\text{GOPC}} < 1$. Fig. 1 shows that in such cases, if any node cheats by using another power control strategy, in particular, the best response to GOPC, its expected throughput gain is significant. Such gain can be a strong incentive for individual links to cheat.

Another interesting observation of Fig. 1 is that by allowing all the transmitters in the network selfishly use power control, the spatial throughput of the network can be improved. In particular, the comparison of SNOPC and NEPC shows that the throughput gain of a smart user is larger when all the other users are also smart. This result is somewhat surprising, since it is natural to conjecture that a smart user should be able to take more advantage of others if they are all dumb. The root of this counter-intuitive phenomenon lies in the special form of the Nash-equilibrium, *i.e.*, each node transmits with (the same) power $\gamma \geq 1$ and probability γ^{-1}. At this equilibrium, the interference I_γ observed at any receiver has the Laplace transform $\mathcal{L}_{I_\gamma}(s) = \exp(-\lambda\pi\gamma^{\delta-1}\frac{\pi\delta}{\sin(\pi\delta)}s^{\delta})$, which is larger than the Laplace transform of the interference without power control $\mathcal{L}_I(s) = \exp(-\lambda\pi\frac{\pi\delta}{\sin(\pi\delta)}s^{\delta})$ for all $s > 0$. Due to the relation between success probability and Laplace transform, this implies that any power control strategy achieves a higher expected throughput when the network operates at a certain the Nash equilibrium than when all other nodes transmit with constant power. Moreover, the NEPC strategy, by definition, maximizes the (individual) throughput at the Nash equilibrium, and thus the spatial throughput of NEPC is always higher than what SNOPC can achieve in a network without power control.

The fact that $\mathcal{L}_{I_\gamma}(s) > \mathcal{L}_I(s),\ \forall s > 0$ suggests that by selfishly choosing its power control strategy, each node is essentially *reducing* its interference to other nodes. Therefore, the spatial throughput of NEPC is always larger than the throughput of SNOPC when no power control is applied in the rest of the network.

B. Variable Link Distances

When the link distances are not a known constant but iid subject to some distribution f_R, the NEPC strategy hinges on solving for $\mathbb{E}[P^{\delta}]$ in (3). A closed-form solution is not available, but a numerical solution is easy to obtain. Given $\mathbb{E}[P^{\delta}]$, the spatial throughput can be calculated by taking the expectation over the distribution of R and can be expressed in terms of the incomplete gamma function.

Unlike the bipolar case where the GOPC strategy can be derived by a similar approach to the one we used to find SNOPC and NEPC strategies, the GOPC strategy in variable link distance case is difficult to find and depends on the f_R. Moreover, since GOPC can easily circumvent the peak power constraint by uniformly reducing the transmit power at every node, in some cases, a GOPC strategy may not exist, *i.e.*, the spatial throughput of the network can always be increased by uniformly driving the transmit power at each node to zero.

However, based on the intuition we get from the bipolar case, we define a globally suboptimal power control (GSOPC) strategy as follows:

Definition 4. *δ-GSOPC(W) is a ALOHA-type random on-off power control policy. At each link of length r, the transmit power is $\min\{W^{1/\delta}\gamma^{1/\delta}, \gamma, P_{\max}\}$ and transmit probability γ^{-1}, where $W \in \mathbb{R}^+$ and $\gamma = \max\{1, \lambda\frac{\pi^2\delta}{\sin(\pi\delta)}\theta^{\delta}r^2\}$.*

The δ-GSOPC(W) strategy is based on the δ-NEPC strategy which has been shown closely related to the GOPC strategy in bipolar networks (Proposition 3). If the δ-th moment constraint is modified to $\mathbb{E}[P^{\delta}] \leq W$, it is not difficult to see that a Nash equilibrium can be achieved by applying a random on-off policy at each node with transmit power $W^{1/\delta}\gamma^{1/\delta}$ and transmit probability γ^{-1}, where $\gamma = \max\{1, \min\{\lambda\frac{\pi^2\delta}{\sin(\pi\delta)}\theta^{\delta}r^2, P_{\max}^{\delta}/W\}\}$. But, such a power control policy violates the constraint $\mathbb{E}[P] \leq 1$, and thus is not a valid power control strategy. Therefore, we put a hard limit

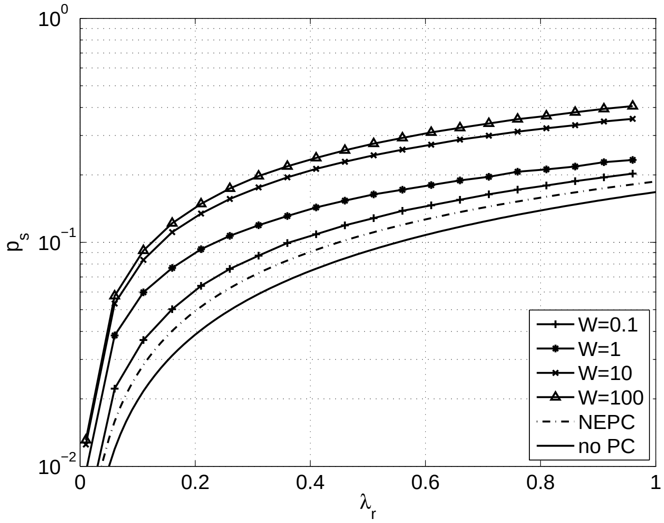

Fig. 2: Spatial throughput comparison of δ-GSOPC(W) and NEPC power control strategies when the link distances are Rayleigh distributed with mean $1/2\sqrt{\lambda_r}$. Here, $\lambda = 1$, $P_{\max} = 2$, $\alpha = 4$, $\theta = 10$.

on the transmit power and obtain the δ-GSOPC(W) strategy as defined.

Note that when $W \to 0$, the δ-GSOPC(W) is equivalent to the δ-NEPC strategy with scaled transmit power. When $W \to \infty$, the δ-th moment constraint becomes dominated by the mean power constraint. Since the δ-th moment of the transmit power is closely related to the interference distribution, smaller W means more fairness in terms of interference contribution among links of different length, while larger W implies larger throughput penalty to long links.

Fig. 2 compares the spatial throughput between δ-GSOPC(W) and NEPC strategies when the link distance R is Rayleigh distributed with mean $1/2\sqrt{\lambda_r}$, *i.e.*, $f_R(x) = 2\lambda_r \pi x \exp(-\lambda_r \pi x^2)$. This distribution is interesting because f_R is the distribution of the link distances when each node of Φ tries to connect to its nearest neighbor in an independent homogeneous PPP of intensity λ_r [8].

The spatial throughput of different δ-GSOPC(W) shows that a less fair strategy (larger W) results in larger spatial throughput. Here, the fairness is measured by the interference contribution, *i.e.*, $\mathbb{E}[P_x^\delta],\ x \in \Phi$.

Fig. 2 also shows that the NEPC strategy achieves a higher spatial throughput than constant power transmission. However, it does not maximize the spatial throughput. In fact, its performance can be easily beaten by δ-GSOPC(W) strategies. The reason of this disadvantage lies in the selfishness of the NEPC strategy which always tries to maximizes each node's own throughput regardless of the interference it causes. However, δ-GSOPC(W) tries to maximize each node's own throughput under the constraint of not causing additional interference to the network (by fixing $\mathbb{E}[P^\delta]$).

VI. Conclusions

This paper studies (random) power control strategies in random wireless networks where the node distribution is governed by a Poisson point process. We show that, in terms of throughput, a set of ALOHA-type random on-off power control policies is single-node optimal and constitutes a Nash equilibrium.

This framework also enables us to show that, in Poisson bipolar networks, ALOHA-type random on-off power control policy is globally optimal in terms of spatial throughput. While the study of ALOHA schemes in Poisson bipolar networks have been carried out in many contexts (*e.g.*, [7]), to the best of our knowledge, this paper is the first to show ALOHA-type random on-off is the optimal power control strategy under mean and peak power constraints.

Based on the intuition obtained in Poisson bipolar networks, we presented a globally suboptimal ALOHA-type power control strategy, δ-GSOPC(W), which achieves higher throughput than the NEPC strategy in networks with random link distances, and provides a trade-off between fairness and spatial throughput.

Since in many cases, the random on-off power control scheme is SNOPC/NEPC/GOPC, this paper provides a new view of ALOHA as a versatile power control scheme as opposed to as a simple but inefficient MAC scheme.

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