SHOT NOISE MODELS FOR OUTAGE AND THROUGHPUT ANALYSES IN WIRELESS AD HOC NETWORKS

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ABSTRACT

Shot noise processes with a decaying power law impulse response function are ideally suited for modeling the network self-interference in an ad hoc network where the nodes are distributed in a plane according to a Poisson point process. A stochastic impulse response function can be used to model different random parameters like channel fluctuation, variable transmission power of the nodes etc. However, for a decaying power law, all the moments of the interference diverge. In this paper, we extend the 2D shot noise model to represent a modified power law that is not only more realistic but also allows for finite moments for the interference. For this model, we present performance curves for the AWGN channel with and without variable transmission power at the nodes as well as for the fading channel. Then, we impose a guard zone around each receiver node in the network, modify the shot noise model to this channel access scheme and demonstrate the improvement in per-node throughput that a guard zone is capable of providing.

I. INTRODUCTION

Ad hoc networks are characterized by a distributed set of nodes that compete for common network resources potentially interfering with every other transmitter in the network. Due to the lack of infrastructure, implementing power control algorithms remains a challenge and an active area of research and motivates the need to come up with exact models for the network self-interference in order to determine the capacity of the system. In order to do this, the most commonly adopted path loss model in literature is that the signal strength falls off as a decaying power law of the distance of transmission. For this model, when the nodes are distributed in $\mathbb{R}^2$ according to a Poisson point process (PPP), the exact distribution of the interference is analytically tractable only for a path loss exponent of 4 for the AWGN [1] and Rayleigh fading channels [2]. Other authors like [3] and the references therein simply use the characteristic function of the interference to model the system performance without attempting to determine the exact distribution for path loss exponents other than 4.

For practical channels, therefore, it is of great interest to determine under what conditions the interference power converges to a Gaussian in distribution. The moments of the interference are excellent indicators in this regard [4]. However, when there is no power control, interferers close to a receiver contribute a lot more interference power than those further away, thus, violating the conditions for the central limit theorem [5]. Further, interferers arbitrarily close to a receiver can cause infinite interference in the decaying power law model and this causes all the moments of the interference to diverge.

In this paper, we eliminate diverging moments by using a modified power law to model the path loss so that it becomes physically meaningful for arbitrarily small distances as well. We derive the moment generating function of the interference for this modified power law by modeling the interference to be a 2D Poisson shot noise process with a stochastic impulse response function. Our work is similar in spirit to [3], and we derive the outage and throughput performances for the additive white gaussian noise (AWGN) as well as the Rayleigh fading channel for a wide range of path loss exponents. We also present performance results when the nodes employ variable transmission powers with nearest neighbor transmission. The results show that while the modified path loss model does not differ by much from the original decaying power law model in terms of outage performance, it does result in finite interference moments.

Even for the modified power law, interferers close to the receiver can easily swamp the desired signal, thus, resulting in low per-node throughputs. A solution to this problem is to modify the channel access scheme such that there is a guard zone around every receiver in which no interferers can be present. This results in significant improvements in the system performance even for small values of the guard zone [6][7]. We extend our shot noise model to such a channel access scheme and present performance curves for this model as well.

II. SYSTEM MODEL

The system model is as follows

- Nodes are distributed in $\mathbb{R}^2$ according to a homogeneous Poisson point process (PPP) Π of density λ.
- Nodes make independent decisions on whether to transmit or listen, and each node is equipped with an omni-directional antenna. In any given time slot, a node transmits with probability α so that the set of all transmitting nodes forms a PPP Π_m of intensity $\lambda\alpha$.
- Transmitters can all use either a constant transmission power or choose to draw their powers from a known distribution.
- Node mobility is not considered in this paper. Rather, the performance results are obtained for an "average" network configuration, i.e., by averaging over all possible network configurations.
- Each node generates information packets of fixed length and all transmissions are assumed to be synchronized slot-wise (slotted ALOHA).
- The interference model assumes that each transmitting node could potentially interfere with any receiving node. For transmission over a distance r, the power law decay is given by $r^{-\eta}$, where η is the path loss exponent.
- The total interference seen at a typical receiver node is $\mathcal{I} = \sum_{i=1}^{n} \mathcal{I}_i$, where the summation is over all transmitting nodes and $n \to \infty$ for infinite networks. In order to keep $\mathbb{E}[\mathcal{I}]$ finite, it is necessary (but not sufficient) that the path loss exponent $\eta > 2$ (Maclaurin and Cauchy criterion) [8].
- An outage occurs when the signal-to-interference ratio (SIR) γ is less than a certain threshold Θ, i.e., $\mathcal{O} = \mathbb{P}(\gamma < \Theta)$. The background noise power, σ_n^2, is assumed to be much smaller than the network self-interference and is ignored in the outage analysis. In the Rayleigh fading case, noise and interference can be treated independently [9], so the noise analysis simply yields an additional factor in the reception probability.
- The per-node throughput is defined to be the overall probability with which a transmitting node a successfully transfers a packet to a receiving node b. For a network with uniform traffic across all links, the per-node throughput will be the same for all the nodes (boundary effects do not exist if the nodes are assumed to be distributed over the surface of a sphere). The per-node throughput is defined as $\zeta = \alpha(1-\alpha)(1-\mathcal{O})$, where α is the probability that a transmits to b and $(1-\alpha)$ is the probability that b does not choose to transmit in the same time slot.

III. SHOT NOISE BACKGROUND

Shot noise results when a memoryless linear filter is excited by a train of impulses derived from a homogeneous PPP with arrival rate μ [10]. The impulse response of the filter, $f(t)$, can assume different shapes like a triangle, rectangle, decaying exponential, decaying power law etc. More generally, the impulse shapes can be stochastic and may be randomly chosen from a family of shapes, $f(k,t)$, with a random variable k. In this paper, we consider the stochastic impulse response model since specialization to the deterministic case is trivial. The shot noise amplitude is given by

$$I(t) = \sum_j f(k_j, t - t_j). \tag{1}$$

The arrival times $\{t_j\}$ are Poisson with rate μ and $\{k_j\}$ are iid random variables drawn from a common distribution and independent of $\{t_j\}$. All impulse functions $f(k,t)$ are assumed to be integrable over $-\infty < t < \infty$ so that the series in (1) converges. As the driving rate μ increases, under some weak conditions, the amplitude distribution of shot noise approaches a Gaussian distribution. This is true of most impulse response functions. However, when the impulse response is a decaying power law, the amplitude distribution does not tend to a Gaussian for any value of μ [5]. In this paper, we are interested only in the decaying power law shot noise process.

The moment generating function of $I(t)$, $\Phi(s) = \mathbb{E}\left[e^{-sI(t)}\right]$, can be obtained as follows. Let the k_j's be drawn from a discrete set $K_1, K_2, \cdots$ with probabilities $p_1, p_2, \cdots$. The shot noise process can then be written as the sum of independent shot noise processes, i.e., $I(t) = I_1(t) + I_2(t) + \cdots$, where $I_i(t)$ is the sum of deterministic impulse responses with a Poisson arrival and a constant parameter K_i, i.e.,

$$I_i(t) = \sum_j f(K_i, t - t_j) \tag{2}$$

Since the I_i are independent,

$$\begin{aligned} \Phi(s) &= \mathbb{E}\left[e^{-s(I_1(t)+I_2(t)+\cdots)}\right] \\ &= \Phi_1(s)\Phi_2(s)\cdots. \end{aligned} \tag{3}$$

For a deterministic impulse response, it is a well-known result [11] that

$$\Phi_i(s) = \exp\left\{-\mu p_i \int_{-\infty}^{\infty} \left(1 - \exp\left[-sf(K_i,t)\right]\right) dt\right\} \tag{4}$$

After evaluating every $\Phi_i(s)$ using (4), $\Phi(s)$ is given by

$$\Phi(s) = \exp\left\{-\mu \int_{-\infty}^{\infty} \mathbb{E}_k \left(1 - \exp\left[-sf(k,t)\right]\right) dt\right\}, \tag{5}$$

where $\mathbb{E}_k[\cdot]$ is expectation w.r.t k. Though k is assumed to be drawn from a discrete distribution, the above expression can be extended to continuous distributions using limiting arguments so that (5) is true in general.

A decaying power law impulse response function is given by $f(k,t) = kt^{-\eta}$. For t varying between A and B, the moment generating function, after simplification using integration by parts, is given by [5]

$$\begin{aligned}\Phi(s) &= \exp\left\{-\mu\int_A^B \mathbb{E}_k\left[1-\exp\left(-skt^{-\eta}\right)\right]dt\right\}\\ &= \exp\left\{\mu A\mathbb{E}_k\left[1-\exp\left(-skA^{-\eta}\right)\right]\right.\\ &\quad -\mu B\mathbb{E}_k\left[1-\exp\left(-skB^{-\eta}\right)\right]\\ &\quad +\mu\mathbb{E}_k\left[(sk)^{1/\eta}\Gamma\left(1-1/\eta, skA^{-\eta}\right)\right]\\ &\quad \left.-\mu\mathbb{E}_k\left[(sk)^{1/\eta}\Gamma\left(1-1/\eta, skB^{-\eta}\right)\right]\right\}, \quad (6)\end{aligned}$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is the incomplete Gamma function. For the special case when $A=0$ and $B=\infty$, (6) reduces to

$$\Phi(s) = \exp\left(-\mu\mathbb{E}_k\left[k^{1/\eta}\right]\Gamma\left(1-1/\eta\right)s^{1/\eta}\right). \quad (7)$$

This completes the description of the 1D shot noise process. Extending the moment generating function in (6) to a 2D PPP is straightforward and the derivation is given in Appendix A. Intuitively, this can be understood as projecting a 2D homogeneous PPP on to a 1D process in r, the distance of a transmitting node from the origin. This new process, however, is not homogeneous and the intensity of the transmitting nodes increases linearly as $\lambda\alpha r$. Upon using this projection in (6) and assuming uniform angle distribution for the node location, we get back the exponent derived in the appendix to within a constant.

The following section adapts this 2D shot noise process to model the interference in a random ad hoc network, which is then used to derive outage and throughput bounds.

IV. INTERFERENCE MODELING

The distribution of the point process in $\mathbb{R}^2$ is unaffected by the addition of a transmitter node at the origin (by Slivnyak's Theorem [4]). Given this transmitter node, we consider a receiver at unit distance from this transmitter[1], shift the origin to this receiver node, and develop the interference model around this "typical" receiver node. This conditional distribution is sometimes referred to as the Palm distribution and since the network is homogeneous, the interference measure at the origin is representative of the interference seen by all other receiver nodes in the network.

The magnitude of interference seen by the receiver at the origin can be likened to the amplitude of the shot noise process described in Section III. Let r_i be the distance of the i^{th} interferer to the origin. The path loss model is the decaying power law impulse response so that $f(k_i, r_i) = k_i/r_i^\eta$. The driving rate of the arrival process is modeled by the intensity of the transmitting nodes, i.e., $\mu = \lambda\alpha$. The total interference seen at the origin is given by

$$\mathcal{I} = \sum_{i\in\Pi}\mathcal{B}_i f(k_i, r_i) = \sum_{i\in\Pi_m} k_i/r_i^\eta, \quad (8)$$

where $\{\mathcal{B}_i\}$ constitute a set of iid Bernoulli random variables such that $\mathbb{P}(\mathcal{B}_i = 1) = 1 - \mathbb{P}(\mathcal{B}_i = 0) = \alpha$. For a AWGN channel, k_i is a constant. For a block Rayleigh fading channel, k_i is drawn from an exponential distribution with unit mean and remains constant over one transmission slot. There is also the possibility of the transmitters employing variable transmission powers, in which case k_i is drawn from a known distribution for the transmit power. For all these different transmission models, we present the exact distribution of $\mathcal{I}$ for the special case of $\eta = 4$ in the following subsection and demonstrate the idea of equivalent shot noise processes.

A. *Decaying power law,* $A = 0, B = \infty$

The moment generating function for the decaying power law model $f(k,r) = k/r^\eta,\ 0 \le r < \infty$ is obtained by evaluating (28) in the limit $A = 0$ and $B = \infty$ to be

$$\Phi(s) = \exp\left(-\pi\lambda\alpha\mathbb{E}_k\left[k^{2/\eta}\right]s^{2/\eta}\Gamma\left(1-2/\eta\right)\right). \quad (9)$$

Owing to the singularity at $r = 0$, however, the mean and variance of the interference obtained from this moment generating function diverge. Nevertheless, this form for $\Phi(s)$ allows for some interesting observations. Notice that the moment generating function is of the form $\Phi(s) = \exp\left[-(cs)^{2/\eta}\right]$, where c is a constant so that for all vales of $\lambda\alpha$, the interference is a one-sided Lévy-stable random variable with asymmetry of dimension $D = 2/\eta$ [12]. Similar to a Gaussian distribution, a Lévy-stable distribution has the property that the sum of two Lévy-stable random variables is another Lévy-stable random variable whose distribution is of the same form as the individual random variables. Therefore, even when the intensity of the interferers is infinite, i.e., $\lambda\alpha \to \infty$, the form of the interference distribution remains the same. However, the conditions for the central limit theorem are violated as long as $0 < D < 1$ and the interference never converges to a Gaussian distribution for practical path loss models with $\eta \ge 2$. Ideally, for the

[1]Even if the transmitter-receiver distance is not unity, all distances in the network can be normalized by this distance so that the desired link always has unit distance.

interference to be strictly Gaussian in the limit $\lambda\alpha \to \infty$, a necessary condition is that $\eta = 0$, i.e., there is no path loss. In such a network, all interferers would contribute evenly and the interference converges to a Gaussian distribution in the limit. The moments of the interference are an indication of how quickly the distribution tends a Gaussian and this motivates the modified path loss model presented in the next subsection. Before that, we present the exact distributions of the interference for the special case of $\eta = 4$ for 3 different transmission models.

1) Deterministic channel: Here, k is a constant and without loss of generality can be taken to be 1. The moment generating function becomes $\Phi(s) = \exp\left(-\pi\lambda\alpha s^{2/\eta}\Gamma\left(1-2/\eta\right)\right)$ and for the special case of $\eta = 4$, the distribution of the interference obtained through the inverse Fourier transform, attains the well-known form [1]

$$f_{\mathcal{I}}(x) = \frac{\pi}{2}\lambda\alpha x^{-3/2}\exp\left(-\pi^3\frac{\lambda^2\alpha^2}{4x}\right) \tag{10}$$

2) Rayleigh block fading channel: k is an exponential random variable with unit mean so that $\mathbb{E}_k\left[k^{2/\eta}\right] = \Gamma\left(1+2/\eta\right)$. Substituting this back into (9) and evaluating the distribution for $\eta = 4$ yields

$$f_{\mathcal{I}}(x) = \frac{\lambda\alpha}{4}\left(\frac{\pi}{x}\right)^{3/2}\exp\left(-\pi^4\frac{\lambda^2\alpha^2}{16x}\right) \tag{11}$$

Once again, this is the exact same distribution derived in [2].

3) Deterministic channel, variable transmission powers: So far, we have assumed that all transmitters transmit at the same power level which is to say that they all have receivers at the same link distance. A more practical model is one where each transmitter adopts nearest neighbor transmission. Here, k_i represents the variable transmission power of the i^{th} interferer.

When the transmitting nodes are distributed according to a PPP of intensity $\lambda\alpha$, the distance to the nearest neighbor, d, is Rayleigh distributed with mean $\frac{1}{2\sqrt{\lambda(1-\alpha)}}$ [13], i.e.,

$$f_D(x) = 2\pi\lambda\left(1-\alpha\right)xe^{-\pi\lambda(1-\alpha)x^2}. \tag{12}$$

The path loss from the transmitter to its receiver is d^η and, hence, the transmitter needs to adjust its power proportional to d^η in order to reach its receiver with the same expected power during all time slots[2]. For $\eta = 4$, the transmission power for every transmitting node is a random variable with distribution corresponding to $k = d^4$ so that

$$f_K(x) = \frac{\pi\lambda\left(1-\alpha\right)}{2}x^{-1/2}e^{-\pi\lambda(1-\alpha)x^{1/2}} \tag{13}$$

For this distribution of k, we have $\mathbb{E}_k\left[k^{1/2}\right] = \frac{1}{\pi\lambda(1-\alpha)}$. Substituting this back in (9), we determine the distribution of the interference to be

$$f_{\mathcal{I}}(x) = \frac{\alpha}{2}\left(\frac{1}{\left(1-\alpha\right)x}\right)^{3/2}\exp\left(-\pi\frac{\alpha^2}{4\left(1-\alpha\right)^2 x}\right) \tag{14}$$

The interference distributions given by (10), (11) and (14) correspond to random variables that are related to each other through just a scaling factor. This demonstrates that as long as the distribution of k is known, the stochastic transmission model can be replaced by an equivalent deterministic model k_0/r^η such that $k_0^{2/\eta} = \mathbb{E}_k\left[k^{2/\eta}\right]$, where k_0 is constant. The interference values obtained for the different transmission models would then differ only by the ratio of their corresponding k_0 values. This result is a specialization of a more general equivalence result presented in Appendix B and holds only for $A = 0$ and $B = \infty$.

B. Modified power law

The decaying power law model is a good way to model path loss when the transmitter or interferer is far away from the receiver, but the model becomes physically meaningless for very small distances since the receiver can never receive more than the transmitted power. To avoid this scenario, some authors adopt a path loss model of the form $(1+r)^{-\eta}$. Equivalently, we use the following modified power law decay

$$f(k,r) = \begin{cases} k, & r < 1 \\ kr^{-\eta}, & r \geq 1. \end{cases} \tag{15}$$

The advantage of using this model is that it eliminates the singularity at $r = 0$ present in the original power law and provides a finite mean and variance for the total interference. This can be seen by modeling the total interference caused by all the transmitters at the origin to be the sum of 2 terms $\mathcal{I}_1$ and $\mathcal{I}_2$, where $\mathcal{I}_1$ is the total interference caused by all transmitters within a distance of 1 from the origin and $\mathcal{I}_2$ is is the total interference power due to all transmitters at distances greater than 1. Since the nodes are distributed according to a PPP, $\mathcal{I}_1$ and $\mathcal{I}_2$ are independent. The moment generating function for $\mathcal{I}_1$ can easily be obtained as $\Phi_1(s) = \exp\left(-\pi\lambda\alpha\mathbb{E}_k\left[1-e^{-sk}\right]\right)$. The corresponding function for $\mathcal{I}_2$ is obtained by substituting $A = 1$ and $B = \infty$ in (28),

$$\Phi_2(s) \quad = \quad \exp\left\{\pi\lambda\alpha\left[\mathbb{E}_k\left(1-e^{-sk}\right) - \right.\right.$$

[2]If it is assumed that every transmitter knows the distance to its receiver exactly so that it can adjust its power accordingly, the point process model is no longer valid since we are conditioning on all receiver locations. Instead we assume that every transmitter randomly draws its power according to $f_K(k)$ with no knowledge about the location of its receiver

$$s^{2/\eta}\mathbb{E}_k\left(k^{2/\eta}\right)\Gamma\left(1-2/\eta\right)+$$
$$s^{2/\eta}\mathbb{E}_k\left(k^{2/\eta}\Gamma\left(1-2/\eta,sk\right)\right)\Big]\Big\} \quad (16)$$

Let k be a unit mean exponential random variable. For a PPP, the mean and variance of $\mathcal{I}_1$ are given by $\mu_1 = \pi\lambda\alpha$ and $\sigma_1^2 = 2\pi\lambda\alpha$. The corresponding values for $\mathcal{I}_2$ are obtained as $\mu_2 = -\frac{d}{ds}\ln\Phi_2(s)|_{s=0} = \frac{2\pi\lambda\alpha}{\eta-2}$ and $\sigma_2^2 = \frac{d^2}{ds^2}\ln\Phi_2(s)|_{s=0} = \frac{2\pi\lambda\alpha}{\eta-1}$. Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are independent,

$$\mu_{\mathcal{I}} = \mu_1+\mu_2 = \frac{\pi\lambda\alpha\eta}{\eta-2}$$
$$\sigma_{\mathcal{I}}^2 = \sigma_1^2+\sigma_2^2 = \frac{2\pi\lambda\alpha\eta}{\eta-1}. \quad (17)$$

Thus, the modified path loss model results in finite first and second order moments which, together with other higher order moments, can be used to analyze convergence to a Gaussian in distribution [4]. Further, as Section VI shows, the outage performance for the modified path loss model is given in terms of $\Phi(s) = \Phi_1(s)\Phi_2(s)$.

In the following subsection, we will adapt this finite interference model to include a guard zone around every receiver in the network.

C. Guard zone in ad hoc networks

Often, the interferers that are really close to the origin are strong enough to completely swamp the desired signal. This leads to undesirably high outage probabilities. One way to ovecome this problem is to modify the channel access scheme by imposing a guard zone of radius d_0 around every receiver node. In other words, every receiver has an exclusion zone of radius d_0 around it within which no interferers are allowed to transmit. As we will see in Section VI, this results in an improved outage and throughput performance. In this paper, we will only consider $d_0 \geq 1$ so that the path loss model is always given by the power law $kr^{-\eta}$. $\Phi(s)$ is obtained by substituting $A = d_0$ and $B = \infty$ in (28),

$$\Phi(s) = \exp\left\{\pi\lambda\alpha\left[d_0^2\mathbb{E}_k\left(1-e^{-skd_0^{-\eta}}\right) - s^{2/\eta}\mathbb{E}_k\left(k^{2/\eta}\right)\Gamma\left(1-2/\eta\right)+ s^{2/\eta}\mathbb{E}_k\left(k^{2/\eta}\Gamma\left(1-2/\eta,skd_0^{-\eta}\right)\right)\right]\right\} \quad (18)$$

Since there are no interferers within d_0, the mean and variance are directly obtained from $\Phi(s)$ to be $\mu_{\mathcal{I}} = \frac{2\pi\lambda\alpha d_0^{2-\eta}}{\eta-2}$ and $\sigma_{\mathcal{I}}^2 = \frac{2\pi\lambda\alpha d_0^{2-2\eta}}{\eta-1}$ for the Rayleigh fading channel. Note that fading doubles the variance compared to a AWGN channel while the mean interference remains unaltered.

V. PERFORMANCE ANALYSIS

In this section, we will use the interference model developed earlier to determine the outage and throughput performances of the network. The link distance between the transmitter and the target receiver at the origin is taken to be 1 so that irrespective of η, the received signal power is given by k. The signal-to-interference ratio (SIR) at the origin is then given by $\gamma = k/\mathcal{I}$. For a given SIR threshold Θ, outage is defined as $\mathcal{O} = \mathbb{P}\left[\gamma < \Theta\right]$. The per-node throughput follows from the definition in the system model.

A. Rayleigh block fading channel

When k is an exponential random variable with unit mean the probability of packet success, $p_s = 1-\mathcal{O}$, is given by

$$\begin{aligned} p_s &= \mathbb{E}_{\mathcal{I}}\left[\mathbb{P}\left(k>\Theta\mathcal{I}|\mathcal{I}\right)\right] \\ &= \mathbb{E}_{\mathcal{I}}\left[\exp\left(-\Theta\mathcal{I}\right)\right] \\ &= \Phi(\Theta). \end{aligned} \quad (19)$$

For $A = 0, B = \infty$, i.e., when there is no guard zone in place, the probability of packet success is directly obtained from (9) to be

$$p_s = \exp\left(-\pi\lambda\alpha\Theta^{2/\eta}\Gamma\left(1-2/\eta\right)\Gamma\left(1+2/\eta\right)\right). \quad (20)$$

For the modified path loss model, we use $\Phi(s)$ given in Section IV-B to obtain

$$p_s = \exp\left\{\pi\lambda\alpha\left[-\Theta^{2/\eta}\Gamma\left(1-2/\eta\right)\Gamma\left(1+2/\eta\right)+ \Theta^{2/\eta}\mathbb{E}_k\left[k^{2/\eta}\Gamma\left(1-2/\eta,\Theta k\right)\right]\right]\right\}. \quad (21)$$

Finally, when there is a guard zone $d_0 > 0$ in place, outage can only occur due to interferers beyond d_0 from the receiver and the outage probability is obtained from (18) to be

$$p_s = \exp\left\{\pi\lambda\alpha\left[d_0^2\left(1-\frac{1}{1+\Theta d_0^{-\eta}}\right) - \Theta^{2/\eta}\Gamma\left(1-2/\eta\right)\Gamma\left(1+2/\eta\right)+ \Theta^{2/\eta}\mathbb{E}_k\left[k^{2/\eta}\Gamma\left(1-2/\eta,\Theta kd_0^{-\eta}\right)\right]\right]\right\} \quad (22)$$

The final terms in the exponential in (21) and (22) involve expectation over an incomplete Gamma function and since the distribution of k is known, we resort to numerical integration to obtain the throughput curves that are presented later in this paper. However, it is possible to bound this term using the incomplete Gamma inequality [14]

$$\begin{aligned} \int_x^\infty t^{-2/\eta}e^{-t}dt &\lessapprox \Gamma\left(1-2/\eta\right)\left[1-\left(1-e^{-x}\right)^{1-2/\eta}\right] \\ &< \Gamma\left(1-2/\eta\right)e^{-x}, \quad \eta>2. \end{aligned} \quad (23)$$

Alternatively, we can also use the Cauchy-Schwartz inequality as follows

$$\int_x^{\infty} t^{-2/\eta} e^{-t} dt \quad < \quad \sqrt{\int_x^{\infty} t^{-4/\eta} dt \int_x^{\infty} e^{-2t} dt}, \eta < 4 \tag{24}$$

For higher path loss exponents, we can apply the inequality again to obtain

$$\int_x^{\infty} t^{-4/\eta} dt < \sqrt{\int_x^{\infty} t^{-8/\eta} dt}, \quad \eta < 8. \tag{25}$$

B. Deterministic channel, variable transmit powers, $\eta = 2$

For free space propagation with nearest neighbor transmission, k is an exponential random variable with mean $1/(\pi\lambda)$. The outage results derived in the previous subsection are directly applicable for this transmission model by simply replacing Θ everywhere with $\Theta' = \Theta/(\pi\lambda)$.

The following section presents outage and throughput curves for the various path loss models that we have considered so far.

VI. RESULTS

In this section, we present 3 sets of throughput curves. The first set compares the per-node throughputs for 3 different transmission models when there is no guard zone in place. The second set of curves illustrates the usefulness of the guard zone in considerably improving the per-node throughput in an ad hoc network. The third set explores the impact that the modified power law has on the per-node throughput.

We choose the system parameters to be $\lambda = 1$, $\Theta = 8$ dB and $\eta = 4$ for all these curves. Fig. 1(a) shows that when all nodes transmit at the same power level, fading degrades the system performance and the maximum value of the per-node throughput, $\zeta_{\max}$, decreases by about 20%. However, as our follow-up paper will show, even a 1-bit noiseless feedback about the channel state is good enough to exploit fading and improve the per-node throughput in an interference limited system.

Fig. 1(a) also presents ζ values when nodes draw their powers randomly according to (13)[3]. Similar to the fading channel, this model also degrades $\zeta_{\max}$ by about 30% but the throughput curve has a very heavy tail. This means that the channel access scheme can be designed for a higher value of α without affecting the per-node throughput by much. Therefore, the intensity of nodes that can transmit during the same time slot, $\lambda\alpha$, is higher with random transmission powers, thereby, resulting in a higher sum throughput in the network.

[3]For ease of analysis, we assume there is no power constraint

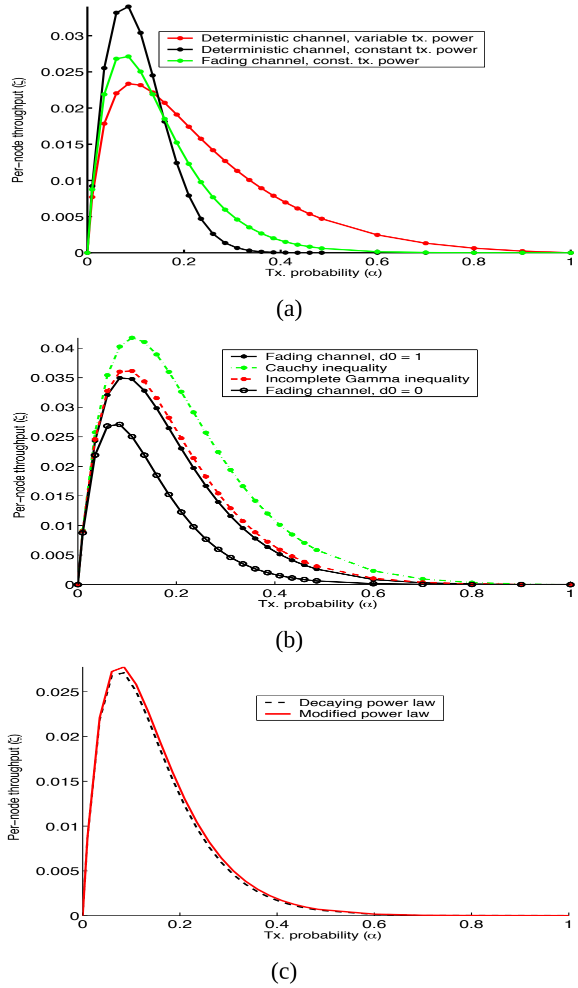

Fig. 1. (a) Per-node throughput for $d_0 = 0$ (b) Per-node throughputs for the fading channel for $d_0 = 0$ and $d_0 = 1$ together with 2 upper bounds for the $d_0 = 1$ case (c) Comparing the per-node throughputs for the decaying power law and the modified power law for the same set of system parameters.

Fig. 1(b) presents ζ values for the fading channel with and without a guard zone. All nodes transmit at the same power level and we choose a guard zone of $d_0 = 1$. The guard zone improves $\zeta_{\max}$ by almost 35% while simultaneously increasing the transmit probability that attains this throughput. However, the effect this has on increasing the sum throughput in the network is not immediately obvious, since the guard zone also reduces the effective number of nodes from which we can choose the set of transmitters. Fig. 1(b) also illustrates the 2 upper bounds that were developed in Section V. Finally, Fig. 1(c) shows that the throughput curves for the decaying and modified power laws are essentially the same and that we can use the compact outage expressions of the former with reasonable accuracy for the latter model also rather than use numerical integration or bounds for the incomplete Gamma function. Note that $\eta = 4$ is chosen only for the sake of illustration and that performance curves can be obtained for other non-integer and irrational path loss exponents as well.

VII. CONCLUSION

In this paper, we have demonstrated the utility of the 2D shot noise process to model the self-interference and perform outage and throughput analyses in a large wireless ad hoc network, where the nodes are distributed in $\mathbb{R}^2$ according to a PPP. We have analyzed both the decaying power law path loss model as well as a modified version that has advantage of having finite interference moments. Further, we have used the theory of equivalent shot noise processes to capture the stochastic nature of the transmission channel, variable transmission powers etc. We have also adapted this shot noise model to include a guard zone around every receiver and presented exact values as well as bounds on the throughput and outage probabilities that demonstrate the utility of the guard zone in improving the performance of an ad hoc network.

APPENDIX A

For a 2-dimensional decaying power law shot noise process, the characteristic function is given by

$$\Phi(s) = \exp\left[-\mu \mathbb{E}_k\left(\psi(s)\right)\right], \tag{26}$$

where

$$\psi(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1 - \exp\left[-sk\left(x^2+y^2\right)^{-\eta/2}\right] dxdy \tag{27}$$

$\psi(s)$ is derived as follows,

$$\begin{aligned}
\psi(s) &\overset{(a)}{=} \int_A^B 1 - \exp\left[-skr^{-\eta}\right] 2\pi r dr \\
&\overset{(b)}{=} \pi\left(sk\right)^{2/\eta}\int_{skA^{-\eta}}^{skB^{-\eta}}\left(1-e^{-t}\right) d\left(t^{-2/\eta}\right) \\
&= \pi B^2\left[1-e^{-skB^{-\eta}}\right] - \pi A^2\left[1-e^{-skA^{-\eta}}\right] \\
&\quad + \pi(sk)^{2/\eta}\int_{skB^{-\eta}}^{skA^{-\eta}} t^{-2/\eta}e^{-t}dt \\
&= \pi B^2\left[1-e^{-skB^{-\eta}}\right] - \pi A^2\left[1-e^{-skA^{-\eta}}\right] \\
&\quad + \pi\left(sk\right)^{2/\eta}\Gamma(1-2/\eta, skB^{-\eta}) \\
&\quad - \pi\left(sk\right)^{2/\eta}\Gamma(1-2/\eta, skA^{-\eta}).
\end{aligned} \tag{28}$$

(a) is obtained by switching to polar coordinates and generalizing the limits of integration while (b) is obtained through a change of variables and integration by parts.

APPENDIX B

Gilbert and Pollack [10] showed that the integral in (5) remains unchanged if $f(k,t)$, defined over a family of values for k, is replaced an equivalent $f(k_0,t)$ as long as

$$\mathbb{E}_k\left[\mathcal{L}\{t : f(k,t) > x\}\right] = \mathcal{L}\{t : f(k_0,t) > x\}, \tag{29}$$

where $\mathcal{L}\{f(\cdot) > x\}$ represents the Lebesgue measure of the set with values greater than x. When $f(k,t) = kt^{-\eta}, 0 \le t < \infty$, $\mathbb{E}_k\left[\mathcal{L}\{t : f(k,t) > x\}\right] = \mathbb{E}_k\left[k^{1/\eta}\right]x^{-1/\eta}$. Thus, the ensemble of stochastic impulse responses $kt^{-\eta}$ is equivalent to the deterministic impulse response $k_0t^{-\eta}$ for all first order statistics such that $k_0 = \mathbb{E}_k\left[k^{1/\eta}\right]^{\eta}$. However, the first-order statistics agree only as long as $A = 0$ and $B = \infty$ and the theory of equivalence is not true otherwise.

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