Delay-Based Connectivity of Wireless Networks

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***Abstract*— Interference in wireless networks causes intricate dependencies between the formation of links. In current graph models of wireless networks, where vertices represent transceivers and edges represent links, such dependencies are not included. In this paper we propose a random geometric graph that explicitly captures the effect of interference. The graph connects nodes which can communicate with a certain maximum expected delay. We analyze some basic properties of the graph where nodes form a Poisson point process and use ALOHA as the channel access scheme.**

I. Introduction

Although interference is known to be the main performance-limiting factor of wireless networks, it is usually completely ignored when studying the connectivity of these networks. The most prominent model is Gilbert's disk graph [1], where the nodes form a stationary Poisson point process (PPP) and are assumed to be connected if they are within a distance r. Such models not only disregard interference, but they are also completely static and thus cannot account for half-duplex constraints (nodes cannot transmit and receive at the same time). Due to the half-duplex constraint, a network graph representing successful transmissions in a given time slot is necessarily disconnected at all times. More precisely, it is a directed forest consisting of many trees of depth one rooted at the transmitters[1].

To overcome these shortcomings, we take a radically different approach by defining connectivity on the basis of the mean delay that it takes to form a connection between two nodes.

II. Network Model and Definitions

Let Φ be a motion-invariant point process on $\mathbb{R}^2$, partitioned at each time $k \in \mathbb{N}$ into a transmitter process $\Phi_t(k)$ and a receiver process $\Phi_r(k)$ by a channel access (MAC) mechanism.

Let $\mathbf{1}_k(x \to y) = 1$ if $x \in \Phi_t(k)$ and $y \in \Phi_r(k)$ and the following condition on the signal-to-interference ratio (SIR) holds:

$$\mathrm{SIR}_{xy} \triangleq \frac{h_{xy}(k)g(x-y)}{\sum_{z\in\Phi_t(k)\setminus\{x\}} h_{zy}(k)g(z-y)} > \theta\,.$$

Otherwise $\mathbf{1}_k(x \to y) = 0$. The fading random variables $h_{xy}(k)$ are assumed iid in time and space with $\mathbb{E}(h) = 1$, and the path loss function g is monotonically decreasing with $g(x) = o(\|x\|^{-2})$ as $\|x\| \to \infty$.

We assume a motion- and time-invariant MAC scheme, in the sense that $\mathbb{P}(x \in \Phi_t(k)) =: p$ does not depend on $x \in \Phi$ nor k and the joint probability $\mathbb{P}(x \in \Phi_t(k) \cap \Phi_t(i))$ only depends on $|k-i|$.

The most complete and comprehensive graph model is the *spatio-temporal* SIR graph, as introduced in [2]. It is a weighted directed multigraph, where directed edges $\overrightarrow{xy}$ exist with weights $\{k \in \mathbb{N} \colon \mathbf{1}_k(x \to y) = 1\}$, *i.e.*, an edge with weight k is present whenever a successful transmission from x to y has occurred. This graph obviously captures all the relevant information about the network, but it is not easy to work with. In this paper, we extract a simpler graph that can be analyzed in more detail. The graph is based on the *single-hop delay*, defined as follows:

Definition 1 (Single-hop delay) *The single-hop delay $D \colon \Phi^2 \to \mathbb{R}$ is defined as*

$$D(x,y) \triangleq \mathbb{E}\left[\min_{k\in\mathbb{N}} \mathbf{1}_k(x \to y)\right] \tag{1}$$

where the expectation is taken with respect to the MAC scheme and the fading.

Since the MAC scheme is time-invariant, we can also define the delay as the inverse of the long-term average throughput, *i.e.*,

$$D(x,y) = \left(\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} \mathbf{1}_k(x \to y)\right)^{-1}.$$

If $0 < p < 1$, then it is ensured that $D(x,y)$ is finite for all $x, y \in \Phi$, since the SIR condition is satisfied when a large enough ball around y is free from interfering transmitters and the fading is not too unfavorable, which happens with positive probability. In other words, the SIR graph described above has an infinite number of edges between all pairs of nodes a.s. Generally, the situation is not symmetric, *i.e.*, $D(x,y) \neq D(y,x)$, and $D(x_1,y_1)$ may be rather different than $D(x_2,y_2)$ even if $\|x_1 - y_1\| = \|x_2 - y_2\|$ since one receiver may have more potential interferers in its vicinity.

We are ready to define the delay graph:

Definition 2 (Delay graph G_τ) *The delay graph is the random geometric digraph $G_\tau = (\Phi, \vec{E}_\tau)$, where $(x,y) \in \vec{E}_\tau$ if $D(x,y) \leq \tau$.*

So a source and a destination node are connected by a directed edge if the source can be expected to reach the destination (in a single hop) in at most τ time slots. The

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[1] If nodes can receive packets from several transmitters simultaneously, common leave nodes are possible.

delay graph is related to the SIR graph as follows: In the delay graph, the edge $\overrightarrow{xy}$ is present if the expected smallest edge weight in the SIR graph is at most τ. In other words, in the delay graph, the randomness due to the MAC and fading in the SIR graph is averaged out, and a single edge indicates whether the two nodes can be expected to connect in τ or less time slots.

We would like to explore the connectivity properties of G_τ. A necessary condition for successful transmission is that the desired transmitters transmits and the intended receiver listens. If these two events are perfectly correlated, a transmission succeeds with probability at most $\max\{p, 1-p\} = 1/2$. If they are independent, the lower bound is $\max_p p(1-p) = 1/4$. This gives lower bounds on the delay of 2 and 4, respectively. Focusing on the independent case, we have that for $\tau < 4$, all nodes are isolated, while for $\tau \to \infty$, the graph is fully connected. Thus it can be expected that the connectivity exhibits a *phase transition* with respect to τ, in the sense that there exists a finite critical value τ_c, such that G_τ a.s. has an infinite out-component for $\tau > \tau_c$ (*i.e.*, there is a node from which an infinite number of nodes can be reached, or, equivalently, there is a positive probability that an infinite number of nodes can be reached from the typical node), while it a.s. consists of finite out-components only if $\tau < \tau_c$. The critical value

$$\tau_c = \inf\{\tau \in \mathbb{R} \mid G_\tau \text{ has an infinite out-component a.s.}\}$$

is called the *percolation threshold*.

The delay graph is also a *throughput graph* for throughput τ^{-1}, in the sense that if there is an edge $\overrightarrow{xy}$, this means that the *direct* throughput from x to y is at least τ^{-1} (packets transferred per unit time). The *indirect* throughput $x \to z \to y$, defined as the minimum throughput of $x \to z$ and $z \to y$, may be larger.

We focus on G_τ for the case where Φ is a homogeneous PPP of intensity 1, and transmitters are chosen at each time with probability p independently, such that $\Phi_t(k)$ is a PPP of intensity p for each k. The path loss law is the power law $\ell(x) = \|x\|^{-\alpha}$, and the fading random variables are exponential (Rayleigh fading).

III. Delay Graphs in Static Poisson Networks

A. The mean delay in the average network

Consider the typical node in Φ, assumed at the origin o, and add a reference receiver at distance R that listens with probability $q \triangleq 1-p$. Conditioning on Φ having a point at o implies that the expectations that involve the point process are taken with respect to the Palm distribution $\mathbb{P}^o$ of Φ and denoted by $\mathbb{E}^o$ [3].

We define the *mean delay at distance* R to be the number of time slots it takes for the typical node to connect to the receiver at distance R, *averaged over the fading, MAC scheme, and* Φ. We have the following result:

Lemma 1 *In a PPP with Rayleigh fading and ALOHA, the mean delay at distance R is*

$$D(R) = \frac{1}{pq} \exp\left(\frac{p\lambda\gamma R^2}{q^{1-2/\alpha}}\right) , \qquad (2)$$

where $\gamma = \pi\theta^{2/\alpha}\Gamma(1-2/\alpha)\Gamma(1+2/\alpha)$ *is the* spatial contention parameter *for the PPP [4].*

Proof: Let $\mathcal{C}$ be the event that the node at the origin successfully connects to its receiver in a single transmission (one time slot) *conditioned on* Φ. Conditioned on Φ, the transmission success events are temporally independent with probability $\mathbb{P}^o(\mathcal{C})$, so the conditional delay is geometric with mean $\mathbb{P}^o(\mathcal{C})^{-1}$. The mean delay is then obtained by integration with respect to (w.r.t.) Φ:

$$D(R) = \mathbb{E}^o\left(\frac{1}{\mathbb{P}^o(\mathcal{C})}\right) , \qquad (3)$$

where the inner probability is a conditional probability given Φ, and the expectation is taken over the point process.

To calculate $\mathbb{P}^o(\mathcal{C})$, consider an arbitrary time slot, and let the location of the receiver be z. The interference at z is

$$I = \sum_{x\in\Phi\setminus\{o\}} h_{xz} e_x \|x-z\|^{-\alpha} ,$$

where h_{xz} is the fading coefficient and $e_x \in \{0,1\}$ is 1 if node x transmits in this time slot. Both sets of random variables are iid, $h_x \sim \exp(1)$ and $e_x \sim \text{Bernoulli}(p)$. The desired signal power is $S = h_{oz}R^{-\alpha}$, since $\|z\| = R$, and $\mathbb{P}^o(\mathcal{C}) = \mathbb{P}^o(\text{SIR} > \theta \mid \Phi) = \mathbb{P}(h_o R^{-\alpha} > \theta I) = \mathbb{E}\exp(-\theta R^\alpha I \mid \Phi)$. This last expression is the conditional Laplace transform of the interference $\mathcal{L}_I(s \mid \Phi) = \mathbb{E}\exp(-sI \mid \Phi)$, where $s = \theta R^\alpha$. From (3), it follows that the local delay given R is then given by $\mathbb{E}^o\left(\frac{1}{\mathcal{L}_I(s|\Phi)}\right)$, which follows from [5, Lemma 17.30, vol. II, p. 90]:

$$\begin{aligned}\mathbb{E}^o\left(\frac{1}{\mathcal{L}_I(s\mid\Phi)}\right) &= \frac{1}{pq}\exp\left(\lambda\int_{\mathbb{R}^2}\frac{ps}{sq+\|x\|^\alpha}\mathrm{d}x\right) ,\\ &= \frac{1}{pq}\exp\left(\frac{p\lambda C(\alpha)s^{2/\alpha}}{q^{1-2/\alpha}}\right) ,\end{aligned}$$

with $C(\alpha) = \pi\Gamma(1+2/\alpha)\Gamma(1-2/\alpha)$. The local delay conditioned on a link distance R is obtained by replacing s by θR^α. The result follows since $C(\alpha)(\theta R^\alpha)^{2/\alpha} = \gamma R^2$. □

As expected, $D(R)$ is finite for all R for $0 < p < 1$. The delay-minimizing choice of p cannot be expressed in closed-form. We can give good bounds, though. Let $c = \lambda\gamma R^2$. Then $D(R) = \exp(pc/q^{1-2/\alpha})/(pq)$. Since $0 < 1-2/\alpha < 1$, we have

$$\frac{\exp(pc)}{pq} < D(R) < \frac{\exp(pc/q)}{pq} . \qquad (4)$$

The two bounds can be minimized, yielding

$$\begin{aligned}\frac{1}{4}\left(3+c-\sqrt{1+6c+c^2}\right) &< p_{\text{opt}}\\ &< \frac{1}{2c}\left(1+c-\sqrt{4+c^2}\right) . \qquad (5)\end{aligned}$$

Both bounds tend to $1/c$ for large c, so they are asymptotically tight, and $p_{\rm opt} < 1/c$ is a trivial but reasonably tight upper bound. It follows that, as $R \to \infty$,

$$p_{\rm opt}(R) = \Theta(R^{-2})$$

and

$$\min_p D(R) \sim \lambda\gamma R^2 \,,$$

which shows that when the optimum p is chosen, the delay is increasing quadratically in the distance, rather than exponentially, as is the case for fixed p per (2).

It is interesting to note that the lower bound in (4) is the *high-mobility bound*, since the delay $\exp(pc)/(pq)$ would be the exact delay achieved if a new realization of the point process were drawn in each time slot. The deviation of $D(R)$ from this lower bound is due to the dependence induced by the static point process, and the upper bound in (4) gives the maximum "penalty" caused by this dependence. This upper bound gets tighter as α grows, which is intuitive since for large α, only a few nearby nodes contribute significantly to the interference.

Lemma 1 provides insight into whether splitting a long hop into two (or more) shorter hops is beneficial from a delay perspective. For a delay comparison, pre-constants do not matter, so we may consider $\tilde{D}(R) = \exp(bR^2)$ for some $b > 0$. It pays off to split a hop of length $2R$ into two hops of length R (if possible) if $\tilde{D}(R) > 2\tilde{D}(R/2)$ or

$$bR^2 > \frac{4\log 2}{3} \,. \tag{6}$$

Generally, there is an optimum number of hops for each distance R. Let

$$h(n) \triangleq \frac{n(n+1)\sqrt{\log(1+1/n)}}{\sqrt{b(2n+1)}} \,, \quad n > 0 \,,$$

and $h(0) \triangleq 0$. This function yields the distance $R_n = h(n)$ for which $n\tilde{D}(R_n/n) = (n+1)\tilde{D}(R_n/(n+1))$. So at distance R_n, the delay for n hops is the same as the delay for $n+1$ hops, hence for smaller distances, n hops is better than $n+1$. It follows that

$$n \text{ hops is optimum} \iff h(n-1) < R \le h(n) \,.$$

As $n \to \infty$, $h(n) \sim n/\sqrt{2b}$, so to cover a distance R, the optimum number of hops $n_{\rm opt} \approx \lceil R\sqrt{2b} \rceil$. A more detailed analysis reveals that $h(n)$ is tightly lower bounded as $h(n) > (n + 2^{-3/2})/\sqrt{2b}$, so $\hat{n}_{\rm opt} = \lceil R\sqrt{2b} - 2^{-3/2} \rceil$ is the true optimum for almost all $R > 1/(4\sqrt{b})$, and too large by 1 hop for the other values of R. If n was a real number, the optimum would be

$$\tilde{n}_{\rm opt} = \frac{\sqrt{2p\lambda\gamma}}{q^{1/2-1/\alpha}} R \,,$$

in agreement with the asymptotic results for $n_{\rm opt} \in \mathbb{N}$.

The optimum hop length $l_{\rm opt}$ is thus about $1/\sqrt{2b}$. Within this distance the typical transmitter finds an average of $\lambda\pi l_{\rm opt}^2 = \pi q^{1-2/\alpha}/(2p\gamma)$ nodes in a PPP of intensity λ. This value can give an indication whether it makes sense to split a hop into two shorter hops if possible, although, to determine the optimum hop length, many other factors need to be considered [6].

We observe that with multi-hopping, the delay scales linearly in R, while the single-hop delay increases exponentially.

Since the optimum p is not available in closed-form, it is not possible to jointly optimize the number of hops and the channel access probability analytically.

B. Connectivity

Lemma 1 relates the delay and the maximum admissible link distance. The condition $D(R) \le \tau$ implies an upper bound on R:

$$R^2 \le \frac{q^{1-2/\alpha}}{p\lambda\gamma} \log(\tau p q) \,. \tag{7}$$

Clearly, if $p < 1/\tau$ or $p > 1 - 1/\tau$, the right hand side (RHS) is negative, so there is no distance $R \ge 0$ such that the delay is smaller than τ. This is obvious already from (2), since in these cases, $1/(pq) > \tau$. The exact condition for the existence of a positive range of R is $p \in (p_{\min}, 1 - p_{\min})$, where

$$p_{\min} = \frac{1}{2}\left(1 - \frac{1}{\tau}\sqrt{\tau^2 - 4\tau}\right) \,.$$

A natural question is what is the optimum choice of p that maximizes the RHS of (7). There is no closed-form solution, but for τ not too small (not too close to 4), the optimum p is essentially given by maximizing $\log(\tau p)/p$ (since the terms in $1-p$ matter less if τ is not too small), which yields the approximation and upper bound $p_{\rm opt} < e/\tau$.

Fig. 1 shows the condition (7) on R^2 as a function of the transmit probability p for different values of τ, and Fig. 2 displays the optimum value of p that maximizes (7) together with the approximation e/τ. Also shown in Fig. 1 are the values for R^2 obtained by letting $p = e/\tau$. It can be seen that the resulting values for the bound on R^2 are very close to the actual maxima.

Let

$$f(x) \triangleq \frac{(1+\log x)x^{1-2/\alpha}}{1-x} \,, \quad 0 < x < 1 \,.$$

So, letting $p = e/\tau$, the typical node can connect to the probe receiver at distance R in at most τ time slots if

$$\lambda\gamma R^2 \le f(\xi) \,, \tag{8}$$

where $\xi = 1 - e/\tau$. For the bound to be positive, $\xi > e^{-1}$ or $\tau > e/(1-e^{-1}) \approx 4.3$, which is close to the obvious condition $\tau > 4$. A simple bound on $f(x)$ for $x \le 1$ that gets asymptotically tight as $x \to 1$ is

$$f(x) \le \frac{x}{1-x} \,.$$

This follows from $(1+\log x)/x^{2/\alpha} \le 1$ for $x \le 1$. Since $e/\tau \ll 1$ for graphs that are not overly sparse, a simple relationship is based on this upper bound:

$$\lambda\gamma R^2 \approx \frac{\tau}{e} - 1 \,. \tag{9}$$

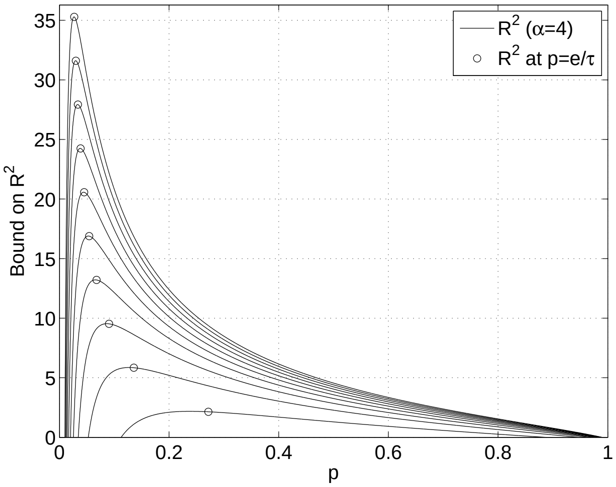


Fig. 1. The bound on R^2 in (7) for $\alpha = 4$, $\lambda = \gamma = 1$, and $\tau \in \{10, 20, \ldots, 100\}$. The lowest curve is the one for $\tau = 10$. The circles indicate the approximate maximum achieved at $p = e/\tau$.

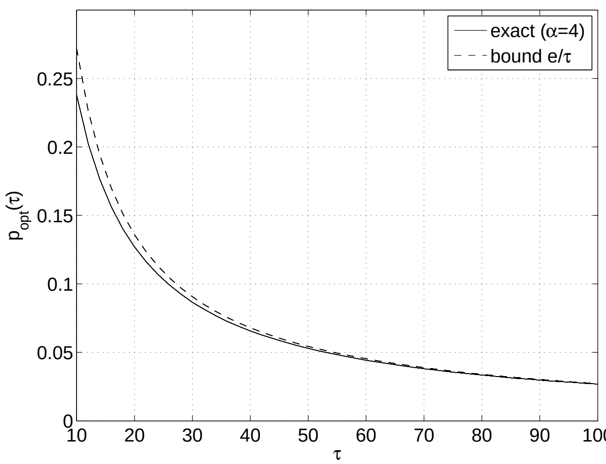


Fig. 2. The optimum p that maximizes the bound on R^2 in (7) for $\alpha = 4$. The solid curve shows the exact p_{opt}, while the dashed curve is the bound e/τ.

Fig. 3 shows delay graphs on the same PPP realization for $p = 0.25$ and various choices of τ.

On average a link distance larger than R means that the edge does not exist. In a single realization of the point process, the distance of the longest edge leaving a node is obviously a random variable, and from Jensen's inequality we can infer that the average node degree in a particular realization is *lower bounded* by the node degree corresponding to the mean distance R. This is confirmed by another simulation, shown in Fig. 4, where $p = 0.05$, and Table I.

Lastly, in Figs. 5 and 6, a smaller simulation result is shown for fixed $\tau = 50$ and variable transmit probability p. In Fig. 5, p is chosen optimally as e/τ, whereas in Fig. 6, p is chosen to be 25% and 50% smaller and larger. As expected, the mean out-degree is largest for $p = p_{\mathrm{opt}}$.

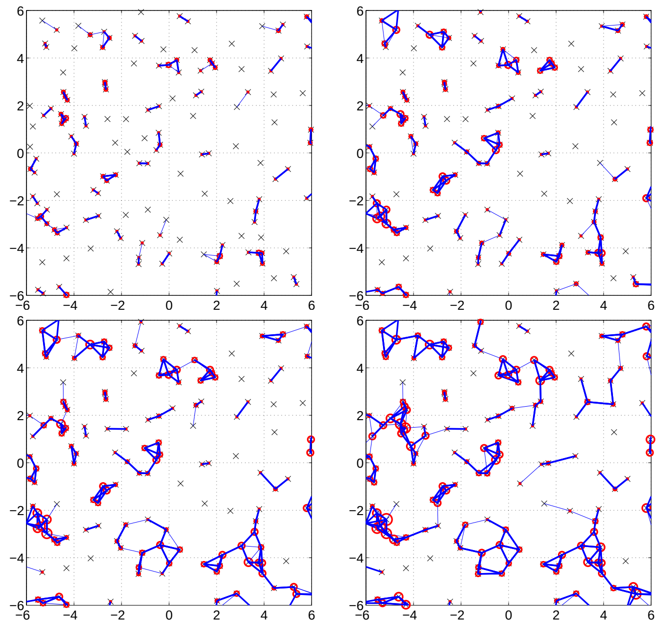


Fig. 3. Delay graphs for $\tau = 20, 50, 100, 200$ for $\lambda = 1$, transmit probability $p = 0.25$, path loss exponent $\alpha = 4$, and SIR threshold $\theta = 10$. Unidirectional edges are shown in gray, whereas bidirectional edges are bold. The mean out-degrees are 1.01, 1.81, 2.28, and 2.89, respectively.

τ	$\bar{\ell}$	$\ell_{\max}$	$\bar{N}_{\mathrm{out}}$	$\mathrm{var}(N_{\mathrm{out}})$	R	N_{Poi}
50	0.750	1.678	3.852	2.918	1.040	3.395
100	1.044	2.322	7.450	7.644	1.395	6.115
200	1.304	3.707	11.31	12.51	1.677	8.835
500	1.607	3.937	16.82	22.69	1.989	12.43

TABLE I

SIMULATION RESULTS FOR $\lambda = 1$, $p = 0.05$, $\alpha = 4$, $\theta = 10$. SIMULATION AREA IS 15×15, SEE FIG. 4. MEAN EDGE LENGTH $\bar{\ell}$, MAXIMUM EDGE LENGTH $\ell_{\max}$, MEAN OUT-DEGREE $\bar{N}_{\mathrm{out}}$, VARIANCE OF OUT-DEGREE $\mathrm{var}(N_{\mathrm{out}})$, THE CORRESPONDING VALUE OF THE RADIUS PER (7), AND THE AVERAGE DEGREE OF THE DISK GRAPH WITH THAT RADIUS.

IV. Concluding Remarks

We have introduced a new and fundamental delay-based notion of connectivity in wireless networks and analyzed some properties of the *delay graph* for Poisson networks.

When averaged also over the point process, a connection can be established between the maximum edge lengths in the disk graph and the delay graph, see (7). This condition can be used to maximize the mean link distance for a given delay threshold τ. It turns out that the ALOHA transmit probability should be chosen as $p = e/\tau$. It is apparent from simulations that this choice maximizes the connectivity of the delay graph in terms of the mean out-degrees.

Many interesting questions remain open. For example, what can be said about the distribution of the edge lengths in the delay graph? More importantly, is there a *critical delay* τ_c, such that the delay graph percolates for $\tau > \tau_c$, whereas it does not for $\tau < \tau_c$? Are there good bounds on τ_c?

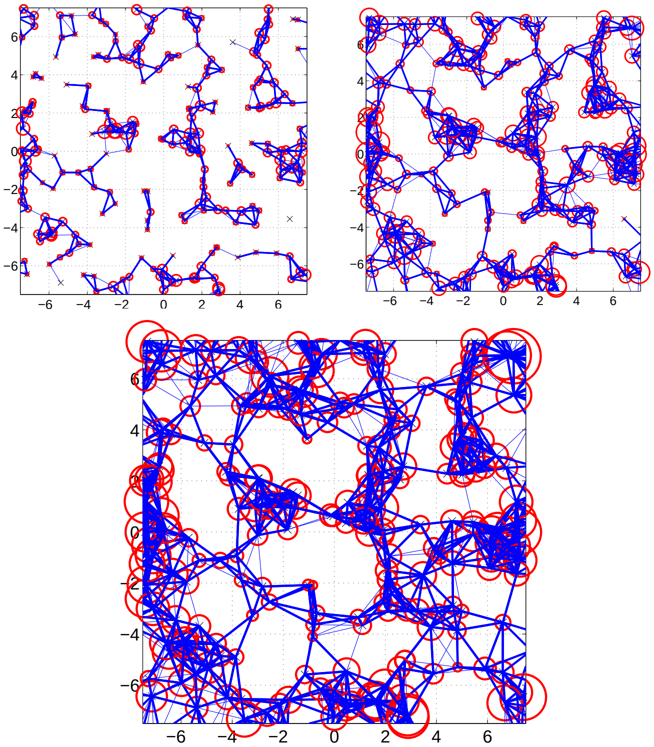
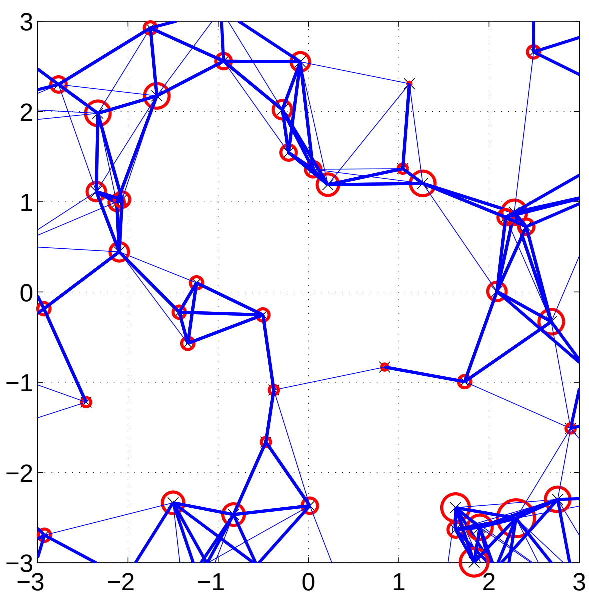
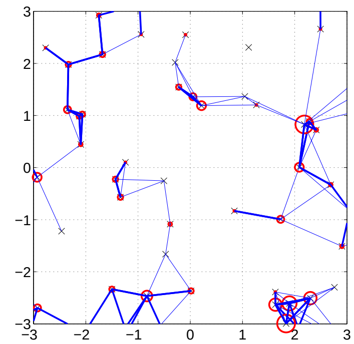


Fig. 4. Delay graphs for $\tau = 50, 100, 200$ for $\lambda = 1$, transmit probability $p = 0.05$, path loss exponent $\alpha = 4$, and SIR threshold $\theta = 10$. The radius of the circles at each node is proportional to the node degree. Some of the properties of these graphs are listed in Table I.

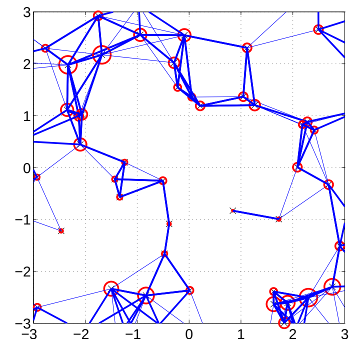


$p = p_{\mathrm{opt}}$. $\bar{N}_{\mathrm{out}} = 5.48$.

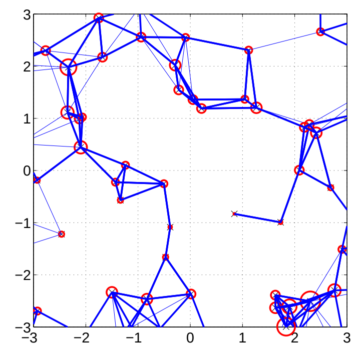
Fig. 5. Delay graph for $\tau = 50$ for $\lambda = 1$, path loss exponent $\alpha = 4$, SIR threshold $\theta = 10$, and $p_{\mathrm{opt}} = e/\tau \approx 0.0544$. Unidirectional edges are shown in gray, whereas bidirectional edges are bold.



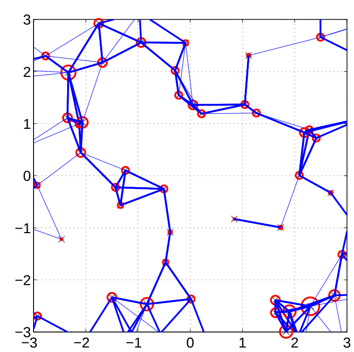
(a) $p = 1/2p_{\mathrm{opt}}$. $\bar{N}_{\mathrm{out}} = 2.93$.



(b) $p = 3/4p_{\mathrm{opt}}$. $\bar{N}_{\mathrm{out}} = 5.27$.



(c) $p = 3/2p_{\mathrm{opt}}$. $\bar{N}_{\mathrm{out}} = 5.00$.



(d) $p = 2p_{\mathrm{opt}}$. $\bar{N}_{\mathrm{out}} = 4.34$.

Fig. 6. Delay graphs for $\tau = 50$ for $\lambda = 1$, path loss exponent $\alpha = 4$, and SIR threshold $\theta = 10$. Unidirectional edges are shown in gray, whereas bidirectional edges are bold. $p_{\mathrm{opt}} = e/\tau \approx 0.0544$. Except for p, the parameters are the same as in Fig. 5.

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