The Meta Distribution of the SINR and Rate in Heterogeneous Cellular Networks

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***Abstract*—This paper focuses on the meta distribution of the signal-to-interference-plus-noise ratio (SINR) and rate in heterogeneous cellular networks (HCNs) with multiple tiers of base stations, where disjoint frequency bands are allocated among tiers and users are associated with each tier with a biased average received power. The meta distribution provides a much sharper version of the "SINR/rate performance" than that merely considered at the typical user through spatial averaging, which gives deep insight into the impacts of heterogeneity, resource coordination, user association, etc., on the performance of individual users. Using tools of stochastic geometry, we develop a general and tractable framework for a fine-grained analysis for HCNs with joint resource partitioning and offloading. With it, we derive exact analytical expressions as well as their asymptotic behaviors for the overall and per-tier moments of the conditional SINR and rate distribution given the point processes, based on which the exact meta distributions are given. We show that although the offloaded users suffer from SINR degradation, the rate performance of all individual users can be improved via load balancing in conjunction with appropriate resource partitioning.**

I. Introduction

Heterogeneous cellular networks (HCNs) are envisioned as a promising approach to address the challenge of the explosive mobile data traffic growth and universal seamless coverage through deploying macro-, pico-, and femto-base stations (BSs) [1]. Due to the load disparity between the macro and small cells, it is desirable to offload users to small cells via flexible cell association and proper spectrum allocation. As a commonly used spectrum allocation scheme, spectrum partitioning has its practical utility since future networks are definitely fusions of multi-standard and multi-band networks and thus different types of BSs are quite likely to operate in non-overlapping bands [2, Chap. 5.2]. It has been established that these techniques are strongly coupled and directly influence the user-perceived rate [3–5], however, the current analysis using stochastic geometry for the HCNs mostly focuses on the typical user by *spatial averaging*, i.e., the evaluation of a certain expectation over the point processes modeling all tiers of BSs.

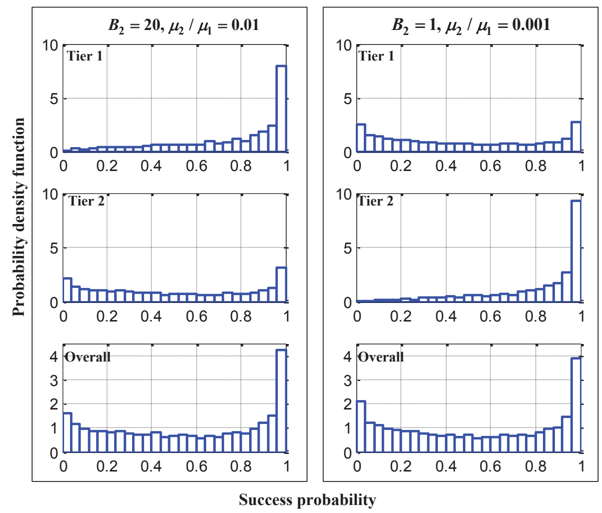

Fig. 1. The histogram of the empirical probability density function of the success probability for two-tier Poisson HCNs with spectrum partitioning, considering the power path loss law with Rayleigh fading and strongest-BS association (on biased average received power), where $\lambda_1 = 1/(\pi 200^2)$, $\lambda_2 = 10\lambda_1$, $\alpha_1 = \alpha_2 = 4$, $B_1 = 1$, $\mu_1 = 46\,\text{dBm}$, $\mu_1 = 100\mu_2$, $\theta = 2$.

While this expected value is certainly important, it cannot reflect the performance variation among the individual users in the same tier or different tiers and how such variation is affected by offloading and resource allocation strategies. For example, Fig. 1 compares the distributions of the success probability among users in each tier and the overall two-tier HCN with different biasing factors and transmit powers. It is shown that the overall distribution of the success probability in each case (i.e., averaging over tier 1 and 2) is almost the same while the per-tier success probability distribution is greatly different. This indicates that sometimes a macroscopic quantity by averaging over all the point processes conceals the actual performance of individual users and its variations influenced by the load balancing and resource partitioning. It is even worse for analyzing the user-perceived rate since the varied load distribution among cells and tiers is also averaged. Thus, it is crucial to study the fine-grained performance for HCNs with joint offloading and resource partitioning.

A fine-grained performance analysis was formally formulated in [6], where the *meta distribution* of the signal-to-interference ratio (SIR) was introduced and analyzed in homogeneous Poisson networks. Since the meta distribution is the *distribution* of the conditional success probability given the point process rather than just the mean, it provides a much sharper version of the "SIR performance" than that most commonly evaluated at the typical link. Since then, the meta distribution has been applied to characterize a series of refined performance metrics in various types of wireless networks [7–10]. However, these studies merely concentrate on the homogeneous (or, equivalently, single-tier) networks. Although a very recent work [11] investigated the SIR meta distribution in HCNs with cell range expansion, the authors did not consider the spectrum allocation and offloading jointly as well as their direct influence on the user-perceived rate.

In this paper, we focus on a fine-grained analysis of a multi-tier HCN with joint resource partitioning and offloading, expecting to get deep insight into the impacts of heterogeneity, resource coordination, user association, etc., on the performance of individual users. Specifically, we propose a general and tractable framework to analyze the overall and per-tier moments of the conditional SINR and rate distribution given the point process as well as their asymptotic behaviors. With them, we derive exact analytical expressions for the meta distribution. The theoretical results leads to microscopic insights for the intricate relationships among the performance (i.e., the SINR and rate) of users in the same tier or different tiers, the biasing factor, and the spectrum allocation strategy.

II. System Model

A. Network Model

We consider a downlink HCN model consisting of K independent network tiers, where the BSs in the k-th tier are spatially distributed according to a homogeneous Poisson point process (PPP) Φ_k with density λ_k, $k = 1, 2, \ldots, K$, with fixed transmit power μ_k. We denote by $\Phi = \bigcup_{k=1}^{K} \Phi_k$ the locations of all BSs in the network. The locations of the users are modeled as another independent homogeneous PPP $\Phi_{\rm u}$ with density $\lambda_{\rm u}$. The channel gain between the transmitter and receiver is modeled by the large-scale path loss and the small-scale fading. A deterministic path loss function $\ell_k(r) = r^{-\alpha_k}$ is adopted, where r is the distance between the transmitter and the receiver, and α_k is the path loss exponent in the k-th tier. The small-scale fading coefficient associated with node $x \in \Phi$ is denoted by h_x, which is an exponential random variable with $\mathbb{E}(h_x) = 1$ (Rayleigh fading), and all h_x are mutually independent and also independent of Φ.

B. User Association and Spectrum Allocation

We assume that different tiers of BSs are allocated separated frequency bands, and the BSs in tier k are allocated the bandwidth $\eta_k W$, where η_k is the resource partitioning fraction of the k-tier and $\sum_{k=1}^{K} \eta_k = 1$, $\eta_k > 0$, and W is the total bandwidth. The orthogonal transmission is considered, where equal time (and/or frequency) slots are allocated to each user.

We consider a flexible user association that each user is associated with the BS that offers the strongest biased average received power. Letting B_k be the association bias for the k-th tier, given a user located at y, the serving BS is given by

$$X(y) = \underset{k\in[K], x\in\Phi_k}{\arg\max}\ \mu_k B_k |x-y|^{-\alpha_k}, \tag{1}$$

where $[K] \triangleq \{1, 2, \ldots, K\}$. Letting $v(x)$ be the index of network tier which the BS x belongs to, the received SINR of the typical user at the origin is

$$\begin{aligned} \mathrm{SINR} &\triangleq \frac{S}{I+\sigma^2} \\ &= \frac{\mu_{x_0} \ell_{v(x_0)}(|x_0|) h_{x_0}}{\sum\limits_{x\in\Phi_{v(x_0)}\setminus x_0} \mu_x \ell_{v(x_0)}(|x|) h_x + \sigma^2}, \end{aligned} \tag{2}$$

where x_0 denotes the serving BS, μ_x is the transmit power of BS x and if $x \in \Phi_k$, $\mu_x = \mu_k$, and σ^2 is the noise power.

C. Meta Distribution

The SINR and the data rate are two fundamental performance metrics for users in cellular networks, and we focus on the fine-grained characterization of these two metrics. Letting T be the (random) data rate of the typical user, the meta distributions represent the complementary cumulative distribution functions (CCDFs) for the random variables in the following form

$$\begin{aligned} P_{\rm s}(\theta) &\triangleq \mathbb{P}^o(\mathrm{SINR} > \theta \mid \Phi), \\ P_{\rm c}(\tau) &\triangleq \mathbb{P}^o(T > \tau \mid \Phi), \end{aligned} \tag{3}$$

where θ and τ are thresholds for the SINR and rate, respectively, and the conditional probability is taken over all the other random effects (such as the fading, the channel access, etc.) given the BS point process, and the randomness of (3) is brought by different realizations of Φ. Therefore, the meta distribution is defined as

$$\bar{F}(y, x) \triangleq \mathbb{P}(P(y) > x), \quad y \in \mathbb{R}^+, \quad x \in [0, 1], \tag{4}$$

where $P(y)$ is $P_{\rm s}(\theta)$ or $P_{\rm c}(\tau)$ corresponding to $y = \theta$ or $y = \tau$. Due to the ergodicity of the point process, the meta distribution can be interpreted as the fraction of links in each realization of the point process that have a

SINR (or rate) greater than θ (or τ) with probability at least x.

By such a definition, the standard success probability (or rate coverage probability) is the mean of $P_{\rm s}(\theta)$ (or $P_{\rm c}(\tau)$), obtained by integrating the meta distribution (6) over $x \in [0,1]$. The standard success probability (or rate coverage probability) answers the questions "Given a threshold θ (or τ), what fraction of users in the whole network can achieve the required SINR or rate on average?", while the meta distributions provide fine-grained information for the individual user and answer more detailed questions such as "What fraction of users achieve a target link reliability given a threshold θ (or τ)?". Since a direct calculation of the meta distribution seems infeasible, we will derive an exact analytical expression through the moments $M_b(\theta) \triangleq \mathbb{E}\left[P_{\rm s}(\theta)^b\right]$ and $S_b(\tau) \triangleq \mathbb{E}\left[P_{\rm c}(\tau)^b\right]$.

III. The Meta Distribution for HCNs

In this section, we first derive the per-tier moments of the conditional success probability, based on which the per-tier and overall moments of the conditional rate coverage are given. Then we give the meta distribution of the rate for HCNs.

A. Moments of the Conditional Success Probability

The following theorem gives the per-tier moments of the conditional success probability.

Theorem 1. *Given that the typical user is served by tier k, the moments of the conditional success probability are*

$$M_{b|k}(\theta)=\frac{1}{A_k}\int_0^\infty \exp\bigg(-{}_2F_1(b,-\delta_k;1-\delta_k,-\theta)r - \frac{b\sigma^2\theta}{\mu_k}\Big(\frac{r}{\pi\lambda_k}\Big)^{\frac{\alpha_k}{2}} - \sum_{i\in[K]^!}\frac{\pi\lambda_i\big(\frac{B_i\mu_i}{B_k\mu_k}\big)^{\delta_i}}{(\pi\lambda_k)^{\alpha_k/\alpha_i}} r^{\frac{\alpha_k}{\alpha_i}}\bigg){\rm d}r, \quad (5)$$

where $b\in\mathbb{C}$, $\delta_k=\frac{2}{\alpha_k}$, $[K]^! = [K]\backslash\{k\}$, ${}_2F_1$ is Gaussian hypergeometric function and

$$A_k=2\pi\lambda_k\int_0^\infty r\exp\bigg(-\pi\sum_{i\in[K]}\lambda_i\Big(\frac{\mu_iB_i}{\mu_kB_k}\Big)^{\delta_i}r^{\alpha_k\delta_i}\bigg){\rm d}r. \quad (6)$$

Proof: Define the nearest-point operator

$${\rm NP}(\Phi) \triangleq \arg\min\{x\in\Phi\colon |x|\} \quad (7)$$

and the reduced point process $\Phi^! \triangleq \Phi \setminus \{{\rm NP}(\Phi)\}$.

Given that the typical user is served by a BS in the k-th tier, we have $x_0 = {\rm NP}(\Phi_k)$, and the conditional success probability of the typical user served by tier k is

$$\begin{aligned}
&\mathbb{P}^o({\rm SINR}>\theta \mid \Phi, x_0\in\Phi_k)\\
&=\mathbb{P}^o\bigg(\frac{\mu_k\ell_k(x_0)h_{x_0}}{\sum_{x\in\Phi_k^!}\mu_k\ell_k(x)h_x+\sigma^2}>\theta \mid \Phi, x_0\in\Phi_k\bigg)\\
&=\mathbb{E}\bigg\{\exp\bigg(-\theta\frac{\frac{\sigma^2}{\mu_k}+\sum_{x\in\Phi_k^!}\ell_k(x)h_x}{\ell_k(x_0)}\bigg) \mid \Phi, x_0\in\Phi_k\bigg\}\\
&=e^{-\frac{\sigma^2}{\mu_k}\theta|x_0|^{\alpha_k}}\prod_{x\in\Phi_k^!}\frac{1}{1+\theta(|x_0|/|x|)^{\alpha_k}}.
\end{aligned} \quad (8)$$

The b-th moment follows as

$$\begin{aligned}
&M_{b|k}(\theta)\\
&=\mathbb{E}\Big[\big(\mathbb{P}^o({\rm SINR}>\theta \mid \Phi, x_0\in\Phi_k)\big)^b\Big]\\
&=\mathbb{E}\Big[e^{-\frac{\sigma^2b\theta}{\mu_k}|x_0|^{\alpha_k}}\prod_{x\in\Phi_k^!}\frac{1}{(1+\theta(|x_0|/|x|)^{\alpha_k})^b}\Big]\\
&\overset{(a)}{=}\int_0^\infty f_k(r)e^{-\frac{\sigma^2b\theta}{\mu_k}r^{\alpha_k}-2\pi\lambda_k\int_r^\infty\big(1-\frac{1}{(1+\theta(\frac{r}{z})^{\alpha_k})^b}\big)z{\rm d}z}{\rm d}r\\
&=\int_0^\infty\frac{2\pi\lambda_kr}{A_k}e^{-\frac{\sigma^2b\theta}{\mu_k}r^{\alpha_k}}\exp\bigg(-\sum_{i\in[K]^!}\pi\lambda_ir^{\alpha_k\delta_i}\Big(\frac{B_i\mu_i}{B_k\mu_k}\Big)^{\delta_i}\\
&\qquad -\pi\lambda_kr^2\Big(1+2\int_0^1\Big(1-\frac{1}{(1+\theta z^{\alpha_k})^b}\Big)z^{-3}{\rm d}z\Big)\bigg){\rm d}r\\
&\overset{(b)}{=}\frac{1}{A_k}\int_0^\infty\exp\bigg(-\frac{b\sigma^2\theta}{\mu_k}\Big(\frac{r}{\pi\lambda_k}\Big)^{\frac{\alpha_k}{2}}-\sum_{i\in[K]^!}\frac{\pi\lambda_i\big(\frac{B_i\mu_i}{B_k\mu_k}\big)^{\delta_i}}{(\pi\lambda_k)^{\alpha_k/\alpha_i}}r^{\frac{\alpha_k}{\alpha_i}}\\
&\qquad -{}_2F_1(b,-\delta_k;1-\delta_k,-\theta)r\bigg){\rm d}r,
\end{aligned} \quad (9)$$

where

$$f_k(r)=\frac{2\lambda_k\pi r}{A_k}\exp\bigg(-\sum_{i\in[K]}\pi\lambda_ir^{\alpha_k\delta_i}\Big(\frac{B_i\mu_i}{B_k\mu_k}\Big)^{\delta_i}\bigg) \quad (10)$$

is the distribution of $|\,{\rm NP}(\Phi_k)|$ given that the typical user is served by tier k [3], A_k is the association probability of the user with tier k, given by

$$A_k=2\pi\lambda_k\int_0^\infty r\exp\bigg(-\pi\sum_{i\in[K]}\lambda_i\Big(\frac{\mu_iB_i}{\mu_kB_k}\Big)^{\delta_i}r^{\alpha_k\delta_i}\bigg){\rm d}r. \quad (11)$$

Step (a) follows from the probability generating functional (PGFL) of the PPP [12] and (b) uses the identity [6]

$$1+2\int_0^1\Big(1-\frac{1}{(1+\theta z^\alpha)^b}\Big)z^{-3}{\rm d}z \equiv {}_2F_1(b,-\delta;1-\delta,-\theta). \quad (12)$$

■

The following corollary gives the asymptotic behavior of the moments of the conditional success probability in the high-reliability ($\theta\to 0$) and high-spectral efficiency ($\theta\to\infty$) regimes, respectively.

Corollary 1. *The asymptotics for the moments of the conditional success probability in the k-tier are*

$$M_{b|k}(\theta)\sim\frac{1}{A_k}\int_0^\infty\exp\bigg(-b\theta\frac{\delta_k}{1-\delta_k}r-\frac{b\sigma^2\theta}{\mu_k}\Big(\frac{r}{\pi\lambda_k}\Big)^{\frac{\alpha_k}{2}}$$

$$-\sum_{i\in[K]}\frac{\pi\lambda_i\left(\frac{B_i\mu_i}{B_k\mu_k}\right)^{\delta_i}}{(\pi\lambda_k)^{\alpha_k/\alpha_i}}r^{\frac{\alpha_k}{\alpha_i}}\Bigg)\mathrm{d}r,\ \theta\to 0, \quad (13)$$

$$M_{b|k}(\theta)\sim\frac{1}{A_k}\int_0^\infty\exp\Bigg(-\theta^{\delta_k}\xi(b,\delta_k)r-\frac{b\sigma^2\theta}{\mu_k}\left(\frac{r}{\pi\lambda_k}\right)^{\frac{\alpha_k}{2}}$$
$$-\sum_{i\in[K]^!}\frac{\pi\lambda_i\left(\frac{B_i\mu_i}{B_k\mu_k}\right)^{\delta_i}}{(\pi\lambda_k)^{\alpha_k/\alpha_i}}r^{\frac{\alpha_k}{\alpha_i}}\Bigg)\mathrm{d}r,\ \theta\to\infty, \quad (14)$$

where $\xi(b,\delta_k)=\int_0^\infty\left(1-\left(1+z^{-1/\delta_k}\right)^{-b}\right)\mathrm{d}z$.

Proof: According to (12), we have

$$\begin{aligned}&{}_2F_1(b,-\delta;1-\delta,-\theta)\\&=1+2\int_0^1\left(1-\frac{1}{(1+\theta z^\alpha)^b}\right)z^{-3}\mathrm{d}z\\&\overset{(a)}{\sim}1+2\int_0^1\big(1-(1-b\theta z^\alpha)\big)z^{-3}\mathrm{d}z,\theta\to 0\\&\sim 1+b\theta\frac{\delta}{1-\delta},\theta\to 0,\end{aligned} \quad (15)$$

where (a) follows from $\frac{1}{(1+x)^b}\sim 1-bx, x\to 0$, and

$$\begin{aligned}&{}_2F_1(b,-\delta;1-\delta,-\theta)\\&=1+2\theta^{\frac{2}{\alpha}}\int_0^{\theta^{\frac{1}{\alpha}}}\left(1-\frac{1}{(1+z^\alpha)^b}\right)z^{-3}\mathrm{d}z\\&\sim 2\theta^{\frac{2}{\alpha}}\int_0^\infty\left(1-\frac{1}{(1+z^\alpha)^b}\right)z^{-3}\mathrm{d}z,\theta\to\infty\\&\overset{(b)}{\sim}\theta^\delta\int_0^\infty 1-\frac{1}{(1+z^{-1/\delta})^b}\mathrm{d}z,\theta\to\infty,\end{aligned} \quad (16)$$

where step (b) follows from $z^{-2}\mapsto z$ and $\delta=2/\alpha$.

Thus, (13) and (14) are obtained by substituting (15) and (16) into (5), respectively. ■

When we consider an interference-limited network, i.e., $\sigma^2=0$, and $\alpha_k=\alpha$, $k\in[K]$, we have

$$M_{b|k}(\theta)=\frac{1}{1+\frac{\lambda_k(B_k\mu_k)^\delta}{\sum_{i\in[K]}\lambda_i(B_i\mu_i)^\delta}\left({}_2F_1(b,-\delta;1-\delta,-\theta)-1\right)}, \quad (17)$$

$$M_{b|k}(\theta)\sim 1-\frac{\lambda_k(B_k\mu_k)^\delta}{\sum_{i\in[K]}\lambda_i(B_i\mu_i)^\delta}\frac{b\delta}{1-\delta}\theta,\ \theta\to 0, \quad (18)$$

$$M_{b|k}(\theta)\sim\frac{\sum_{i\in[K]}\lambda_i(B_i\mu_i)^\delta}{\lambda_k(B_k\mu_k)^\delta\xi(b,\delta)}\theta^{-\delta},\ \theta\to\infty. \quad (19)$$

B. Moments of the Conditional Rate Coverage

According to Thm. 1, we derive the per-tier moments of the conditional rate coverage probability as follows.

Theorem 2. *Given that the typical user is served by tier k, the moments of the conditional rate coverage probability are*

$$S_{b|k}(\tau)=\sum_{n=0}^\infty P_{n,k}M_{b|k}\left(2^{\frac{(n+1)\tau}{\eta_k W}}-1\right),\ b\in\mathbb{C}, \quad (20)$$

where

$$P_{n,k}=\frac{1}{n!}\left(\frac{\lambda_u A_k}{7\lambda_k}\right)^n\frac{(2n+5)!!}{15}\left(1+\frac{\lambda_u A_k}{3.5\lambda_k}\right)^{-3.5-n}, \quad (21)$$

and $n!!$ is the double factorial.

Proof: Given Φ and that the typical user is served by the BS from a certain tier, the conditional rate coverage probability is expressed as

$$P_{\mathrm{c},k}(\tau)=\mathbb{P}^o(T>\tau\mid\Phi,x_0\in\Phi_k). \quad (22)$$

Letting N_k denote the number of users served by the tagged BS (the serving BS of the typical user) in tier k, the transmission bandwidth for each is $\eta_k W/N_k$ and the transmission rate is

$$T=\frac{\eta_k W}{N_k}\log(1+\mathrm{SINR}). \quad (23)$$

Therefore, the moments of the conditional rate coverage probability are

$$\begin{aligned}S_{b|k}(\tau)&=\mathbb{E}\left[\mathbb{P}^o\Big(T>\tau\mid x_0\in\Phi_k,\Phi,N_k\Big)\right]^b\\&\overset{(a)}{=}\sum_{n=1}^\infty P_{n,k}\underbrace{\mathbb{E}\left[\mathbb{P}^o\Big(\mathrm{SINR}>2^{\frac{n\tau}{\eta_k W}}-1\,|\,x_0\in\Phi_k,\Phi\Big)\right]^b}_{\mathcal{X}}\\&=\sum_{n=1}^\infty P_{n,k}M_{b|k}\big(2^{\frac{n\tau}{\eta_k W}}-1\big),\end{aligned} \quad (24)$$

where (a) starts from $n=1$ since N_k includes the typical user, and comparing $\mathcal{X}$ and (9), we have $\mathcal{X}=M_{b|k}(\theta)$ with $\theta=2^{\frac{n\tau}{\eta_k W}}-1$. $P_{n,k}=\mathbb{P}(N_k=n)$ is given as follows.

Letting $N_{o,k}$ be the number of the users except for the typical user, i.e., $N_k=N_{o,k}+1$, we have $\mathbb{P}(N_k=n+1)=\mathbb{P}(N_{o,k}=n)$. As in [3], we assume the probability generating function (PGF) of $N_{k,o}$ to be

$$G_{N_{k,o}}(z)=\left(1-\frac{\lambda_u A_k(z-1)}{3.5\lambda_k}\right)^{-3.5}. \quad (25)$$

Thus, we have

$$\begin{aligned}P_{n,k}&=\frac{1}{n!}G^{(n)}_{N_{k,o}}(0)\\&=\frac{1}{n!}\left(\frac{\lambda_u A_k}{7\lambda_k}\right)^n\frac{(2n+5)!!}{15}\left(1+\frac{\lambda_u A_k}{3.5\lambda_k}\right)^{-3.5-n}.\end{aligned} \quad (26)$$

By substituting (26) into (24), we obtain (20). ■

The following corollary gives the overall moments of the conditional rate coverage for HCNs.

Corollary 2. *For K-tier HCNs with spectrum partitioning among tiers, the moments S_b of the conditional rate coverage probability are*

$$S_b(\tau)=\sum_{k\in[K]}\sum_{n=0}^\infty P_{n,k}M_{b,k}\left(2^{\frac{(n+1)\tau}{\eta_k W}}-1\right),\ b\in\mathbb{C}, \quad (27)$$

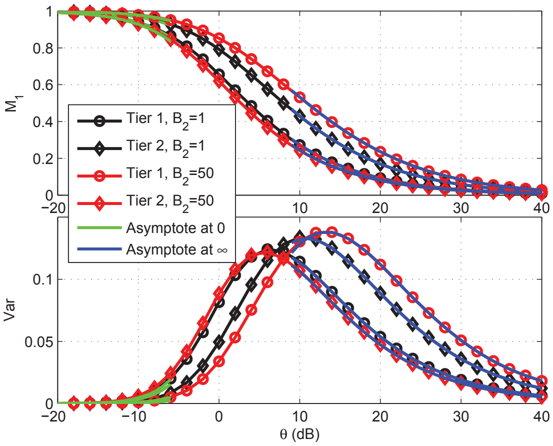

Fig. 2. The standard success probability $M_{1|k}$ and the variance $M_{2|k}-M_{1|k}^2$ for each tier with density ratio $\kappa=5$. From (6), $A_1=0.67$ and $A_2=0.33$ with $B_2=1$ while $A_1=0.22$ and $A_2=0.78$ with $B_2=50$.

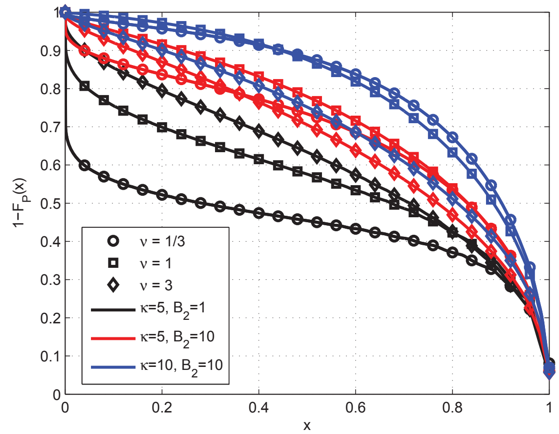

Fig. 3. The meta distribution of the rate for different densities, biasing factors and spectrum partitioning fractions ν.

where

$$M_{b,k}(\theta) = \int_0^\infty \exp\Big(-{}_2F_1(b,-\delta_k;1-\delta_k,-\theta)r - \frac{b\sigma^2\theta}{\mu_k}\Big(\frac{r}{\pi\lambda_k}\Big)^{\frac{\alpha_k}{2}} - \sum_{i\in[K]^!}\frac{\pi\lambda_i\big(\frac{B_i\mu_i}{B_k\mu_k}\big)^{\delta_i}}{(\pi\lambda_k)^{\alpha_k/\alpha_i}} r^{\frac{\alpha_k}{\alpha_i}}\Big)\mathrm{d}r. \quad (28)$$

Proof: Given Φ, the conditional rate coverage probability of the typical user in the entire network is expressed as

$$\begin{aligned} P_{\mathrm{c}}(\tau) &= \mathbb{P}^o(T>\tau \mid \Phi) \\ &= \sum_{k\in[K]} \mathbb{P}^o(T>\tau \mid \Phi)\mathbf{1}_{\{x_0\in\Phi_k|\Phi\}}, \end{aligned} \quad (29)$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. Then, the overall b-th moment can be expressed as

$$\begin{aligned} S_b(\tau) &= \mathbb{E}\Big[P_{\mathrm{c}}(\tau)^b\Big] \\ &= \mathbb{E}\sum_{k\in[K]}\Big(\mathbb{P}^o(T>\tau\mid\Phi)\mathbf{1}_{\{x_0\in\Phi_k|\Phi\}}\Big)^b \\ &= \sum_{k\in[K]} A_k\mathbb{E}\Big(\mathbb{P}^o(T>\tau,\mid x_0\in\Phi_k,\Phi)\Big)^b \\ &= \sum_{k\in[K]} A_k\sum_{n=1}^{\infty}\mathbb{P}(N_k=n)M_{b|k}(2^{\frac{n\tau}{\eta_k W}}-1). \end{aligned} \quad (30)$$

By substituting (5) into (30), we obtain (27). ■

According to the Gil-Pelaez theorem, the meta distribution of the rate is given by

$$\bar{F}_{P_{\mathrm{c}}(\tau)}(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\Im\left(e^{-jt\log x}\mathcal{S}_{jt}\right)}{t}\mathrm{d}t \quad (31)$$

where $\mathcal{S}_{jt}$ can be $S_{jt|k}$ in (20) or S_{jt} in (27) corresponding to the per-tier and overall distributions of the conditional rate coverage probability, respectively, $j \triangleq \sqrt{-1}$, $\Im(z)$ and $\Re(z)$ are the imaginary and real parts of $z\in\mathbb{C}$.

IV. Numerical Results

In this section, we present numerical results of various performance metrics involved in the framework in Section III for HCNs, where $K=2$, $\lambda_1=\frac{1}{\pi 200^2}$, $\lambda_{\mathrm{u}}=50\lambda_1$, $\alpha_1=\alpha_2=4$, $B_1=1$, $\mu_1=46$ dBm, $\mu_2=\mu_1/100=26$ dBm, $\tau=1$ Mbps, $W=10$ MHz, $\sigma^2=0$ and $\lambda_2=\kappa\lambda_1$, $\eta_1=\nu\eta_2$.

Fig. 2 shows the per-tier success probabilities $M_{1|k}$ and the variance of the conditional per-tier success probabilities $M_{2|k}-M_{1|k}^2$ as well as their asymptotics as a function of θ for each tier, respectively. It is observed that the asymptotic curves match the exact results well, especially for large θ, i.e., the asymptotic curve as $\theta\to\infty$ performs extremely well over a large range of θ. The variance has a maximum at some finite value of θ for both tiers, because it necessarily tends to zero for both $\theta\to 0$ and $\theta\to\infty$. Moreover, it can be seen that a larger B_2 offloads more users to small cells (tier 2), causing the association probability for tier 1 to decrease but the success probability for tier 1 to improve. However, an excessively large association bias can cause the small cells to be overly congested with users of poor SINR. That is why for $B_2=1$ and $B_2=50$, the comparative results between tier 1 and tier 2 are just reversed.

Fig. 3 shows the impacts of BS densities, biasing factor, and spectrum partitioning fraction on the overall rate meta distribution. As expected, increasing BS density improves the quality of the individual links due to the decreased load at each BS. In addition, since the bias policy controls the serving distance of users and the load distribution between the two tiers, it influences the received SINR and the available resource at each user, exerting an effect on the corresponding data rate. When $\kappa = 5$ and $B_2 = 1$, the curve with $\nu = 3$ outperforms the other two curves, because in this case $A_1 = 0.67$ and $A_2 = 0.33$, so the average load served by each BS in tier 1 is ten times that in tier 2, hence more spectral resources should be allocated to tier 1. In contrast, when $\kappa = 5$ and $B_2 = 10$, the curve with $\nu = 1$ performs the best, since in this case more users are offloaded to tier 2 and the allocated resource should be adjusted accordingly. Furthermore, it is observed that there is an intersection point between the $\nu = 1/3$ and $\nu = 3$ with $B_2 = 10$, which verifies that the meta distribution provides more information on the rate performance, i.e., for different link reliability constraints, the fraction of users that have a rate greater than the target rate is quite different for different resource partitioning fractions, and this cannot be reflected by the average ergodic rate or standard rate coverage probability.

V. Conclusions

In this paper, we have provided an analytical framework for a fine-grained analysis of HCNs with joint offloading and resource partitioning. Although it is established that these two techniques play an important role in radio resource management, our work is the first to analyze the meta distribution of the SINR and rate in a HCN, while incorporating resource partitioning and a user association policy with biasing. The availability of a functional form for the refined SINR and rate performance as functions of system parameters unlocks a plethora of avenues to gain deep design insights. Using the developed analysis, the importance of combining load balancing with resource partitioning in terms of the meta distribution was clearly established. The meta distribution is a key refined metric for studying these techniques, and insights based on just the mean SINR distribution are inconclusive.

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