Bounds on Information Propagation Delay in Interference-Limited ALOHA Networks

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***Abstract*—In a wireless network the set of transmitting nodes changes frequently because of the MAC scheduler and the traffic load. Analyzing the connectivity of such a network using static graphs would lead to pessimistic performance results. In this paper, we consider an ad hoc network with half-duplex radios that uses multihop routing and slotted ALOHA for the network MAC contention, and introduce a random dynamic multi-digraph to model its connectivity. We first provide analytical results about the degree distribution of the graph. Next, defining the path formation time as the minimum time required for a causal path to form between the source and destination on the dynamic graph, we derive the distributional properties of the connection delay using techniques from first passage percolation and epidemic processes. We show that the delay scales linearly with the distance and provide asymptotic results (with respect to time) for the positions of the nodes which are able to receive information from a transmitter located at the origin. We also provide simulation results to support the theoretical results.**

I. Introduction

In a multihop ad hoc network, bits, frames or packets are transferred from a source to a destination in a multihop fashion with the help of intermediate nodes. Decoding, storing, and relaying introduces a delay that, measured in time slots, generally exceeds the number of hops. For example, a five-hop route does not guarantee a delay of only five time slots. In a general setting, each node can connect to multiple nodes. So a large number of paths may form between the source and the destination. Each path may have taken a different time to form with the help of different intermediate nodes. Consider a network in which each node wants to transmit to its destination in a multihop fashion. In general in such a network, a relay node queues the packets from other nodes and its own packets and transmits them according to some scheduling algorithm. If one introduces the concept of queues, the analysis of the system becomes extremely complicated because of the intricate spatial and temporal dependencies between various nodes. In this paper we take a different approach. We are concerned only with the physical connections between nodes, i.e., we do not care when a node i transmits a particular packet to a node j (which depends on the scheduler), but we analyze when a (physical) connection (maybe over multiple hops) is formed between the nodes i and j. This delay is a lower bound on the delay with any queueing scheduler in place.

We assume that the nodes are distributed as a Poisson point process (PPP) on the plane. In each time slot, every node decides to transmit or receive using ALOHA. Any transmitting node can connect to a receiving node when a modified, noiseless version of the protocol model criterion introduced in [1] is met. Since at each time instant, the transmit and receive nodes change, the connectivity graph changes dynamically. We analyze the time required for a causal path to form between a source and a destination node. The system model is made precise in Section II.

This problem is similar in flavor to the problem of First-Passage Percolation (FPP) [2]–[4], and the process of dynamic connectivity also resembles a simple epidemic process [5]–[7] on a Euclidean domain. In a spatial epidemic process, an infected individual infects a certain (maybe random) neighboring population, and this process continues until the complete population is infected or the spreading of the disease stops. In the literature cited above, the time of spread of the epidemic is analyzed for different models of disease spread. We draw many ideas from this theory of epidemic process and FPP. The main difference between an epidemic process and the process we consider is that the spreading (of packets) depends on a subset of the population (due to interference) and is not independent from node to node. In [8], the latency for a message to propagate in a sensor network is analyzed using similar tools. They consider a Boolean connectivity model with randomly weighted edges and derive the properties of first-passage paths on the weighted graph. Their model does not consider interference and thus allows the use of Kingman's subadditive ergodic theorem [9] while ours does not. Percolation in signal-to-interference ratio graphs was analyzed in [10] where the nodes are assumed to be full-duplex. In practice, radios do not transmit and receive at the same time (at the same frequency), and hence the instantaneous network graph is always disconnected. Connectivity between nodes far apart occurs because of the dynamic nature of the MAC protocol. In this paper, we first introduce a dynamic graph process to model and analyze connectivity and then derive the properties of this graph process for ALOHA.

In Section II, we introduce the system model. In Section III, we study the connectivity properties of the random geometric graph formed at any time instant. In Section IV, we derive the properties of the delay and the average number of paths between a source and destination and show that the delay increases linearly with increasing source-destination distance or, equivalently, that the propagation speed is constant, i.e., the distance of the farthest nodes to which the origin can connect increases linearly with time.

II. System Model

The location of the wireless nodes (transceivers) is assumed to be a Poisson point process (PPP) ϕ of intensity λ on the plane. We assume that time is slotted and the MAC protocol used is slotted ALOHA. At every time slot each node transmits with probability p. Nodes are half-duplex, and they act as receivers if they are not transmitting. We use an interference-based model to decide if the communication between a transmitter and a receiver is successful in a given time slot: A transmitting node located at x can connect to a receiver located at y if the disk $B(y, \beta\|x-y\|), \beta > 0$, does not contain any other transmitting nodes. $B(x, r)$ denotes a disk of radius r centered around x and $B^c(x, r) = \mathbb{R}^2 \setminus B(x, r)$. β is a system parameter and captures the resilience of the receiver against interference. This is a variant of the protocol model [1] that does not include the power constraint. The standard SIR model of communication can be related to the protocol model easily when there is no fading. A detailed discussion about the protocol model can be found in [11]. We shall use $\mathbf{1}(x \to y, \Delta)$ to represent a random variable that is equal to one if a transmitter at x is able to connect to a receiver y when the transmitting set is Δ, i.e., the interfering set is $\Delta \setminus \{x\}$. We will drop Δ if there is no ambiguity. At any time instant k, we denote the set of transmitters (decided by ALOHA) by $\phi_t(k)$ and the set of receivers by $\phi_r(k)$. So we have $\phi_t(k) \cup \phi_r(k) = \phi$ and $\phi_t(k) \cap \phi_r(k) = \emptyset$, where $\emptyset$ denotes the empty set.

The connectivity at time k is captured by a directed and weighted random geometric graph $g(k) = (\phi, E_k)$ with vertex set ϕ and edge set

$$E_k = \{(x, y) : \ \mathbf{1}\left(x \to y, \phi_t(k)\right) = 1, x \in \phi_t(k), y \in \phi_r(k)\}. \tag{1}$$

Each edge in this graph $g(k)$ is associated with a weight k that represents the time slot in which the edge was formed. Let $G(m, n)$ denote the weighted directed multigraph (multiple edges with different time stamps are allowed between two vertices) formed between times m and $n > m$, i.e.,

$$G(m, n) = \left(\phi, \ \bigcup_{k=n}^{m} E_k\right).$$

So $G(m, n)$ is the *edge-union* of the graphs $g(k)$, $m \leq k \leq n$.

Definition 1: A directed path $x_0, e_0, x_1, \ e_1, \ldots, e_{q-1}, x_q$ between the nodes $x_0 \in \phi$ and $x_q \in \phi$ where $e_i = (x_i, x_{i+1})$ denotes an edge in the multigraph is said to be a *causal path* if the weight of the edges e_i are *strictly increasing* with i.

This means that the edge e_{i-1} was formed before e_i for $0 < i < q$. For the rest of the paper, we always mean causal path when speaking about a path.

We observe that the random graph $g(k)$ is a snapshot of the ALOHA network at time instant k. The random graph process $G(0, m)$ captures the entire connectivity history up to time m. The graph $g(k)$ has the flavor of the interference graph analyzed in [10] where the authors consider only bi-directional links (full-duplex radios). They proved that such a graph percolates with respect to the density of the nodes if the processing gain is high enough. In the graph $G(0, m)$ there is a notion of time and causality, i.e., packets can propagate only on a causal path.

We make the following assumption which we shall use in Section IV. *We assume that the interference at different time instants is independent*. More precisely we assume the following, $\forall m \neq n$ and $\forall \ a, b, c, d \in \phi$,

$$\mathbb{E}[\mathbf{1}_{E_m}\left((a, b)\right) \mathbf{1}_{E_n}\left((c, d)\right)] = \mathbb{E}[\mathbf{1}_{E_m}\left((a, b)\right)]\mathbb{E}[\mathbf{1}_{E_n}\left((c, d)\right)] \tag{2}$$

where the expectation is taken with respect to ALOHA and the point process ϕ. E_m is the edge set defined in (1) and, $\mathbf{1}_{E_m}\left((a, b)\right)$ is the indicator function of the edge set E_m, which is equal to 1 if and only if the edge (a, b) belongs to E_m.

Assumption (2) is true if $B(b, \beta\|a-b\|) \cap B(d, \beta\|c-d\|) = \emptyset$ or if the node set ϕ is not random (or if we condition on the location of the nodes) since the ALOHA protocol chooses independent transmitter sets across time. In reality interference is not independent in time but almost because of the MAC protocol.

III. Properties of the snapshot graph $g(k)$

In this section, we will analyze the properties of the random graph $g(k)$. We first observe that the graphs $g(k)$ are identically distributed for all k. So for this section we will drop the time index unless otherwise indicated. g a planar Euclidean graph *even with straight lines* as edges [12, Lemma 2]. We first characterize the distribution of the in-degree of a receiver node and the out-degree of a transmit node.

A. Node degree distributions

Let $N_t(x)$ denote the number of receivers a transmitter located at x can connect to, i.e., the out-degree of a transmitting node. Similarly, let $N_r(x)$ denote the number of transmitters that can connect to a receiver at x, i.e., the in-degree of a receiver node. We first calculate the average out-degree of a transmitting node.

Proposition 1: $\mathbb{E}\left[N_t(x)\right] = \frac{1-p}{p}\beta^{-2}$.

Proof: By stationarity of ϕ, we have $N_t(x) \stackrel{d}{=} N_t(o)$ where $\stackrel{d}{=}$ stands for equality in distribution. So it is sufficient to consider the out-degree of a transmitter placed at the origin, which is given by $\sum_{x \in \phi_r} \mathbf{1}(o \to x, \phi_t)$. So the average degree is

$$\begin{aligned}
\mathbb{E}[N_t(o)] &= \mathbb{E}\left[\sum_{x \in \phi_r} \mathbf{1}(o \to x, \phi_t)\right] \\
&\stackrel{(a)}{=} \lambda(1-p)\int_{\mathbb{R}^2} \mathbb{E}_{\phi_t}\left[\mathbf{1}(o \to x, \phi_t)\right] \mathrm{d}x \\
&\stackrel{(b)}{=} \lambda(1-p)\int_{\mathbb{R}^2} \exp\left(-\lambda p \pi \beta^2 \|x\|^2\right) \mathrm{d}x \\
&= \frac{1-p}{p}\beta^{-2},
\end{aligned}$$

where (a) follows from Campbell's theorem [13] and the independence of ϕ_r and ϕ_t. (b) follows from the fact that $\mathbf{1}(o \to x, \phi_t)$ is equal to one if and only if the ball $B(x, \beta\|x\|)$ does not contain any interferers. ■

We observe that $\mathbb{E}[N_t(x)] \to \infty$ when $p \to 0$. This is because the interference reduces as p becomes smaller. This behavior is a modelling artifact; if the interference vanished, a power constraint would have to be introduced.

Proposition 2: The probability distribution of N_t is given by

$$\mathbb{P}\left(N_t = m\right) = \sum_{k=m}^{\infty} \frac{(-1)^{k+m}}{k!} \left(\frac{1-p}{p}\right)^k V_k(\beta), \quad (3)$$

where $V_k(\beta) = \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \exp\left(-\text{vol}\left(\cup_{i=1}^k B(x_i, \beta\|x_i\|)\right)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k$.

Proof: We provide the complete characterization of N_t using the Laplace transform $\mathcal{L}_{N_t}(s)$ and is given by

$$\begin{aligned}
&= \mathbb{E}\left[\exp\left(-sN_t\right)\right] \\
&= \mathbb{E}\left[\exp\left(-s\sum_{x\in\phi_r}\mathbf{1}(o\to x,\phi_t)\right)\right] \\
&\overset{(a)}{=} \mathbb{E}_{\phi_t}\exp\left[-\lambda(1-p)\int_{\mathbb{R}^2}1-\exp(-s\mathbf{1}(o\to x,\phi_t))\mathrm{d}x\right] \\
&= \mathbb{E}_{\phi_t}\exp\left[-ap\int_{\mathbb{R}^2}\mathbf{1}(o\to x,\phi_t)\mathrm{d}x\right], \quad (4)
\end{aligned}$$

where (a) follows from the probability generating functional of a PPP and $a = \frac{1-p}{p}(1-\exp(-s))$. The distribution of $\mathbf{1}(o \to x)$ does not change if x is scaled by $\sqrt{\lambda p}$ and the density of ϕ_t is reduced by λp. So, letting ν denote a two dimensional Poisson point process of density 1, we have $\mathcal{L}_{N_t}(s)$ is equal to $\mathbb{E}_\nu \exp\left[-a\int_{\mathbb{R}^2}\mathbf{1}(o\to x,\nu)\mathrm{d}x\right]$. Then $\mathcal{L}_{N_t}(s)$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\frac{(-a)^k}{k!}\mathbb{E}_\nu\left(\int_{\mathbb{R}^2}\mathbf{1}(o\to x,\nu)\mathrm{d}x\right)^k \\
&= 1+\sum_{k=1}^{\infty}\frac{(-a)^k}{k!}\int_{\mathbb{R}^2}\cdots \\
&\qquad \int_{\mathbb{R}^2}\exp\left(-\text{vol}\left(\cup_{i=1}^k B(x_i,\beta\|x_i\|)\right)\right)\mathrm{d}x_1\cdots\mathrm{d}x_k \quad (5)
\end{aligned}$$

By comparison of coefficients (replace e^{-s} with z), we obtain (3). ■

We next evaluate the in-degree distribution of a receive node. Since the point process is stationary, the distribution of $N_r(x)$ is the same for all receivers x.

Proposition 3: The average in-degree $\mathbb{E}[N_r(x)]$ of a node in g is β^{-2}. When $\beta > 1$, N_r is distributed as a Bernoulli random variable with mean β^{-2}.

Proof: We have $N_r(x) \overset{d}{=} N_r(o)$ and hence,

$$\begin{aligned}
\mathbb{E}[N_r(o)] &= \mathbb{E}\left[\sum_{y\in\phi}\mathbf{1}_{\phi_t}(y)\mathbf{1}(y\to o,\phi_t)\right] \\
&= \lambda p\int_{\mathbb{R}^2}\mathbb{E}_{\phi_t}\left[\mathbf{1}(y\to o,\phi_t)\right]\mathrm{d}y = \beta^{-2}.
\end{aligned}$$

If $\beta > 1$, at most one transmitter can connect to any receiver, so N_r is Bernoulli. Since $\mathbb{E}\left[N_r(x)\right] = \beta^{-2}$, we have $N_r(x) \sim$ Bernoulli(β^{-2}). ■

Observe that the in-degree $N_r(x)$ does not depend on p. This is because of the homogeneity of the protocol model and the point process. Also observe that $\mathbb{E}[N_t(x)]$ and $\mathbb{E}[N_r(x)]$ are spatial averages and not time averages. We have $p\mathbb{E}\left[N_t\right] = (1-p)\mathbb{E}\left[N_r\right]$.

IV. The Time Evolution Graph $G(0, n)$

In the previous section we analyzed the connectivity graph formed at a particular time instant. In this section we will consider the superposition of these graphs and study how the connectivity evolves over time.

A. Asymptotic analysis of $G(0, n)$

We first define the connection time between two nodes. For $x, y \in \phi$, we denote the *path formation time* between x and y as

$$T(x,y) = \min\left\{k \,:\, G(0,k) \text{ has a path from } x \text{ to } y\right\}.$$

For general $x, y \in \mathbb{R}^2$, define $T(x, y) = T(x^*, y^*)$ where x^* (resp. y^*) is the point in ϕ closest to x (resp. y). Since the point process is isotropic, it is sufficient for most cases to consider destinations along a given direction. For notational convenience we define for $y \in \mathbb{R}$, $T(x, y) = T(x, (y, 0))$. This path formation time is the minimum time required for a packet to propagate from a source x to its destination y in an ALOHA network. In this section we show that this propagation delay increases linearly with the source-destination distance. Similar to $T(x, y)$ we define

$$T_n(x,y) = \min_{k>n}\left\{k-n \,:\, G(n,k) \text{ has a path from } x \text{ to } y\right\}.$$

Let

$$\tilde{B}_t = \left\{x\colon x\in\mathbb{R}^2,\ T(o,x)\le t\right\}$$

denote the set of points which can be reached from the origin by time t. The evolution of the graph $G(0, n)$ is similar to the growth of an epidemic on the plane and one can relate this problem to the theory of Markovian contact processes [7] which was used to analyze the growth of epidemics. We now provide bounds on the path formation time between two points.

Direct connection: By assumption (2), we have that the time taken for a direct connection between two points x and y is a geometric random variable with parameter

$$\eta(x,y) = p(1-p)\mathbb{E}\left[\exp\left(-\lambda p\pi\beta^2\|x^*-y^*\|^2\right)\right]$$

where the average is with respect to the distribution of $\|x^* - y^*\|$. *For most of the analysis we assume $\|x-y\|$ to be large so that $\|x^* - y^*\| \approx \|x-y\|$. Henceforth we shall not distinguish between x and x^** .

Lemma 1: For large $x \in \mathbb{Z}^+$, the tail probability of $T(o, x)$ is bounded as

$$\mathbb{P}(T(o,x) > k) \le I_{1-\eta(o,a)}(k+1,m)$$

for any $1/\sqrt{\lambda} < a < x$, where $m = \lceil x/a \rceil$ and

$$I_{1-\eta(o,a)}(k+1,m) = \frac{(m+k+1)!}{m!(k+1)!}\int_0^{\eta(o,a)} t^m(1-t)^k dt$$

is the regularized beta function.

Proof: We imposed $1/\sqrt{\lambda} < a$ so that $\|(0, a)^*\| \approx a$. Let $t_1(a)$ be the time for an edge to form between o and $(a, 0)$ and $t_2(a)$ be the time required for a direct connection to form between $(a, 0)$ and $(2a, 0)$ after the first edge is formed. Similarly define $t_k(a)$ to be the time required for a connection to form between $((k-1)a, 0)$ and $(ka, 0)$ after all the previous $k - 1$ connections are formed. See Figure 1. By assumption

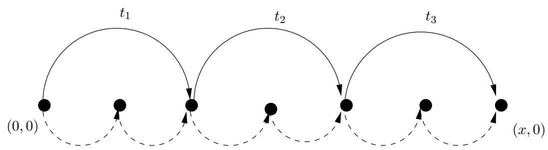

Figure 1. The node at the origin can transfer packets to a node at $(x, 0)$ by using the shorter hops (indicated by dashed line) or using longer hops (solid lines). Longer hops are difficult to form but only few are required to reach the destination. Shorter hops are easy to form but a higher number is required to reach the destination.

(2), we have $t_i, 1 \leq i \leq m$, to be independent. So we have

$$T(o, x) \leq \sum_{i=1}^{m} t_i(a) \tag{6}$$

The t_i are iid geometrically distributed with parameter $\eta(o, a)$. Hence we have

$$\begin{aligned} \mathbb{P}(T(o,x) > k) &\leq \mathbb{P}\left(\sum_{i=1}^{m} t_i(a) > k\right) \\ &\stackrel{(a)}{=} I_{1-\eta(o,a)}(k+1, m), \end{aligned} \tag{7}$$

where (a) follows from the fact that the sum of geometric random variables follows a negative binomial distribution. ■

In the following arguments we rely on the spatial subadditivity of $T(o, x)$ to analyze the asymptotic properties. Subadditivity of random variables is a powerful tool which is often used to prove results in percolation and geometric graph theory. The problem of finding the minimum delay path is similar to the problem of first-passage percolation. From the definition of $T(o, y)$, we observe that

$$T(o, y) \leq T(o, x) + T_{T(o,x)}(x, y). \tag{8}$$

We also have that $T_{T(o,n)}(x, y) \stackrel{d}{=} T(x, y)$ from the way the graph process is defined. Observe that (8) resembles the triangle inequality (specially if $T_{T(o,y)}(x, y)$ was $T(x, y)$) and thus provides a pseudo-metric, which holds in FPP problems and is the reason that the shortest paths in FPP are called geodesics. In the next two lemmata we show that the average time for a path to form between two nodes scales linearly with the distance between them.

Lemma 2: The time constant defined by

$$\mu = \lim_{x\to\infty} \frac{\mathbb{E}T(o, x)}{x}$$

exists when $x \in \mathbb{Z}^+$.

Proof: Let $y \in \mathbb{Z}^+$. From (8), we have

$$T(o, y + x) \leq T(o, y) + T_{T(o,y)}(y, y + x). \tag{9}$$

From the definition of the graph, E_k does not depend on E_i, $i < k$. Hence we have that $T_{T(o,y)}(y, y + x)$ has the same distribution as $T(y, y + x)$. Also from the invariance of the point process ϕ, we have $T(y, y + x) \stackrel{d}{=} T(o, x)$. Taking expectations of (9), we obtain

$$\mathbb{E}T(o, y + x) \leq \mathbb{E}T(o, y) + \mathbb{E}T(o, x),$$

and the result follows from the basic properties of subadditive sequences. ■

We do not require assumption (2) to prove Lemma 2. Consistent with the FPP terminology we will call μ the time constant of the process. We now prove that the time constant for the modified protocol model is always greater than zero and finite.

Lemma 3: For the modified protocol model

$$\frac{\beta\sqrt{p\pi\lambda}}{\sqrt{\ln(1 + p(1 - p))}} \leq \mu \leq \frac{\beta\sqrt{2\pi\lambda}\exp(1/2)}{(1 - p)\sqrt{p}} \tag{10}$$

Proof: Upper bound: Taking expectation on both sides of (6) and since $t_i(a)$ are identically distributed for all i (we drop the i in the subscript for notational convenience), we have

$$\mathbb{E}T(o, x) \quad \leq \quad \left\lceil \frac{x}{a} \right\rceil \mathbb{E}t(a) \leq \left(\frac{x}{a} + 1\right) \mathbb{E}t(a).$$

Dividing both sides by x and taking the limit we obtain

$$\lim_{x\to\infty} \frac{\mathbb{E}T(o, x)}{x} \quad \leq \quad \frac{\mathbb{E}t(a)}{a}.$$

Assuming $\|(0, a)^*\| \approx a$, $t(a)$ is a geometric random variable with mean $p(1 - p) \exp(-p\lambda\pi\beta^2 a^2)$. So we get

$$\mu \leq \frac{\exp(p\lambda\pi\beta^2 a^2)}{ap(1 - p)}.$$

The upper bound is obtained by using $a = 1/(\beta\sqrt{2p\lambda\pi})$ for which the right hand side of the above equation is minimized. *Lower bound*: Taking large hops to reach the destination requires fewer hops but the success probability for each hop would be small and hence it takes more time to connect. On the other hand taking smaller hops will result in a higher probability of success for each hop and result in a smaller time of connection, but we require a large number of hops to get to the destination. We will use the tradeoff between the hopping distance versus time to show that

$$\mathbb{P}(T(o, x) < cx) \to 0$$

as $x \to \infty$ for some positive c and $x \in \mathbb{Z}^+$. This implies $\mathbb{E}T(o, x)/x > c$ for some $c > 0$ and hence $\mu > 0$. For the sake of notational convenience let cx be identified with $\lceil cx \rceil$. So to evaluate the event $\{T(o, x) \leq cx\}$, we consider only those paths which have a maximum of cx hops. By the union bound we have

$$\mathbb{P}(T(o, x) < cx) \quad \leq \quad \sum_{i=1}^{cx} p_i \tag{11}$$

where $p_i = \mathbb{P}(T(o, x) < cx \mid$ there is a path from o to x with i hops). The time to form any single direct link between two nodes o and y is a geometric random variable with parameter $\eta(o, y) = p(1 - p) \exp(-c_1\|o - y\|^2)$, where $c_1 = \lambda p\pi\beta^2$. So the times to form the hops in a k-hop path between $o, x_1, x_2 \ldots, x_{k-1}, x$ are a series of geometric random variables t_i with parameters $\eta(x_{i-1}, x_i)$ which are independent because they occur in different time slots, see (2). Let $\xi > 0$. So we have $p_k \leq \mathbb{P}(\sum_{i=1}^{k} t_i < cx)$. We also have that if $t_1, ..., t_i$ are independent geometric random variables with parameters p_i, then

$$\mathbb{P}(\sum_{i=1}^{k} t_i < a) \leq \exp(\xi a) \left(\frac{e^{-\xi}}{1 - e^{-\xi}}\right)^k \prod_{i=1}^{k} p_i \tag{12}$$

for any $\xi > 0$ (follows from Chernoff bound). So

$$\begin{aligned} p_k &\stackrel{(a)}{\leq} \exp(\xi c x) \frac{1}{(\exp(\xi)-1)^k} \prod_{i=1}^{k} \eta(x_{i-1}, x_i) \\ &= \exp(\xi c x) \left(\frac{p(1-p)}{\exp(\xi)-1} \right)^k \exp(-c_1(\|o - x_1\|^2 \\ &\quad + \|x_2 - x_1\|^2 + \ldots + \|x_{k-1} - x\|^2)) \\ &\stackrel{(b)}{\leq} \exp(\xi c x) \left(\frac{p(1-p)}{\exp(\xi)-1} \right)^k \exp\left(-c_1 \frac{x^2}{k} \right). \end{aligned}$$

(a) follows from (12) and (b) follows from the fact that the minimum value of $\|x_1\|^2 + \|x_2 - x_1\|^2 + \ldots + \|x_{k-1} - x\|^2$ is x^2/k. So from (11), we have $\mathbb{P}(T(o, x) < cx)$

$$\begin{aligned} &\leq \sum_{k=1}^{cx} \exp(\xi c x) \left(\frac{p(1-p)}{\exp(\xi)-1} \right)^k \exp\left(-c_1 \frac{x^2}{k} \right) \\ &\stackrel{(a)}{\leq} cx \exp(\xi' c x) \exp\left(-\frac{c_1}{c} x \right), \end{aligned}$$

where (a) follows by choosing $\xi = \xi'$ such that $p(1-p)/(\exp(\xi')-1) < 1$ and using $k = cx$ for all the terms. The right hand side goes to 0 if $c < \sqrt{c_1/\xi'}$. Hence we have $\mathbb{E}[T(o, x)/x] > c$ which implies $\mu > c$. We can choose $\xi' = (1+\epsilon)\ln(1 + p(1-p))$ for any $1 > \epsilon > 0$ and we then have the lower bound $c \geq (1-\epsilon)\sqrt{\frac{\lambda p \pi \beta^2}{(1+\epsilon)\ln(1+p(1-p))}}$. ■

In the modified protocol model we are considering, we do not have any power constraint. So any node can potentially connect to any receiver no matter how far it is but the probability decreases exponentially with distance and hence $\mu < \infty$. This is in contrast to standard first-passage percolation on a lattice where the probability distribution (CDF) on each edge should have a mass less than P_c at zero for $\mu < \infty$, where P_c is the bond percolation threshold of the lattice. If we had considered a power constraint, for example by putting a hard limit on the maximum link distance, $\|x - y\| < R$ (original protocol model), then there is no guarantee that the time constant $\mu < \infty$. We conjecture that if R is chosen so that the disk graph formed by placing disks of radius R around each node of ϕ percolates, i.e., for $R > \sqrt{1.435/\lambda}$ [14] then $\mu < \infty$. In deriving the lower bound we have used assumption (2). In practice the constants may change but the scaling with respect to the different parameters would remain the same. From the lower bound on μ we have that $\mu > 0$ when $p \to 0$, but as noted previously, this is an observation that is of mathematical interest only, since the noise-free assumption does not hold when $p \to 0$. We also observe that the lower bound on the time constant increases with p. From the upper and lower bounds we observe that μ scales like $\beta\sqrt{\lambda}$.

Since we do not have $T(o, x + y) \leq T(o, x) + T(o, y)$, Kingman's subadditive ergodic theorem [9] cannot be directly applied to (8). But since $T_{T(o,x)}(x, y) \stackrel{d}{=} T(x, y)$, there is hope that such a result holds. In the next lemma, we prove that this is indeed the case.

Lemma 4: Let μ be the time constant of the process,

$$\frac{T(o, x)}{x} \longrightarrow \mu, \quad x \to \infty \tag{13}$$

$x \in \mathbb{Z}^+$ and where the convergence is in L^2 and hence in probability.

Proof: From (8), and $T_{T(o,x)}(x, x+y) \stackrel{d}{=} T(o, y)$ and the fact that $T_{T(o,y)}(y, x+y)$ is independent of $T(o, y)$ (because of assumption (2)), we have

$$F_{x+y}(\xi) \geq (F_x * F_y)(\xi),$$

where F_x is the CDF of $T(o, x)$. $\mathbb{E}(T(o, x)^2) < \infty$ follows from Lemma 1. So we have a superconvolutive sequence and hence by Kesten's lemma [15], [16], [17, p. 120] holds[1]. ■

This result shows that with high probability, the delay required for a packet propagation scales linearly with distance.

V. Simulation Results

In this section we illustrate the results using simulation results. For the purpose of simulation we consider a PPP of unit density in the square $[-50, 50]^2$. For most of the simulations, we use $\beta = 1.2$, and we average over 200 independent realizations of the point process. In Figure 2, $\mathbb{E}T(o, x)$ is plotted with respect to x for different values of p. The time constant μ is plotted as a function of p in Figure 3. We make the following observations:

1) The time constant increases with the ALOHA parameter p.
2) In Figure 2, we observe that $\mathbb{E}T(o, x) \approx \mu(p)x + C(p)$, where $C(p)$ is a decreasing function of p and $\mu(p)$ is increasing. For smaller values of p, the time taken for a node to become a transmitter is large, but the probability of a successful transmission is also high because of the low density of transmitters. This results in a large $C(p)$ and smaller $\mu(p)$ for small p.
3) Figure 2 also implies that the presence of interfering transmitters causes the delay to increase when the packet has to be transmitted over longer distances. So when the packet transmission distance is large, it is beneficial to decrease the density of contending transmitters.
4) For each x, there is an optimal p which minimizes the delay, and the optimum p is a decreasing function of x.

For two nodes located at o and x and $\|x\|$ large, there will in general be many paths between o and x which form by time $\mu\|x\|$. From such an ensemble of delay-optimal paths, we will consider paths which have the minimum number of hops and call them *fastest paths*. In Figure 4, we show the average hop in these paths. We observe that for a given p, the average hop length decreases as the source-destination distance x increases. This shows that for larger source-destination distance, it is beneficial to use shorter hops since they are more reliable and form faster than longer hops. Also from Figure 3, we observe that for larger x, it is beneficial to be less aggressive in terms of spatial reuse and use a smaller p.

VI. Conclusions

Connectivity in a wireless network is dynamic and directed because of the MAC scheduler and the half-duplex radios. Since these properties are not captured in static graph models that are usually used, we have introduced a dynamic connectivity graph and analyzed its properties for ALOHA. We have shown that the time taken for a causal path to form between a source and a destination on this dynamic ALOHA graph scales

[1]To prove the a.e. convergence using Kesten's lemma, we would require that $T(o, n)$ be a monotone sequence, which is not true in our case.

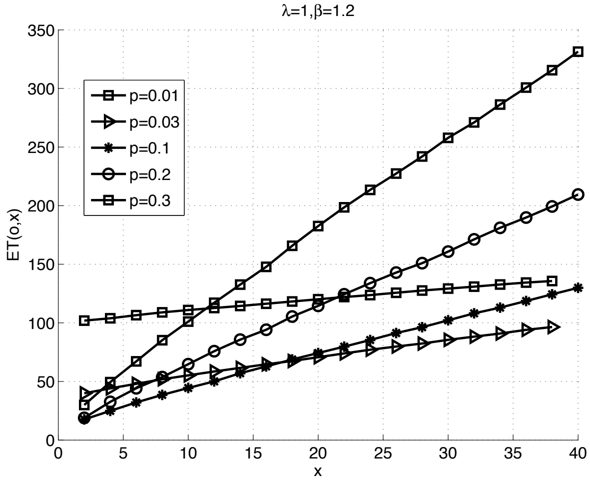

Figure 2. $\mathbb{E}T(o,x)$ as a function of x, for $\beta = 1.2$. We first observe the linear scaling of $\mathbb{E}T(o,x)$ with the distance x and that the slope increases with p. Also for small values of x we observe that $\mathbb{E}T(o,x) \approx p^{-1}$ since for small x the path delay time is dominated by the MAC contention time. For small values of p, once the source is a transmitter, long edges form due to the low interference.

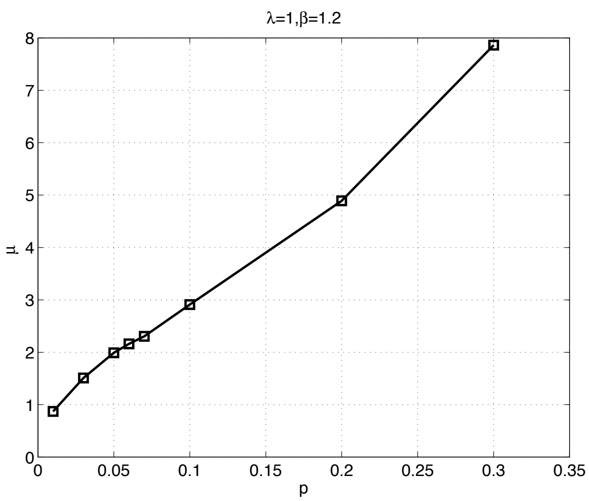

Figure 3. The time constant μ as a function of p, for $\beta = 1.2$

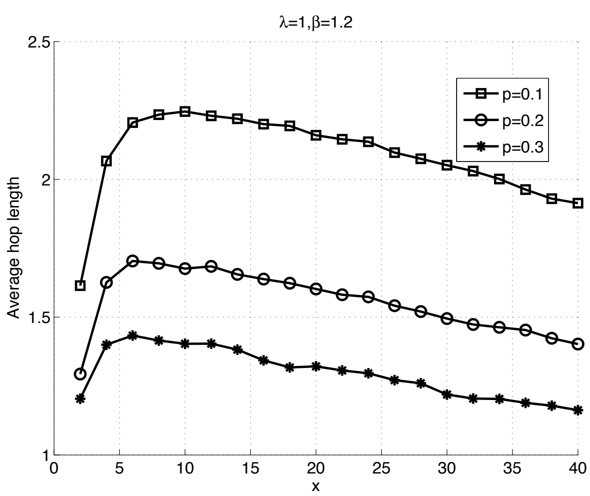

Figure 4. Average hop length in the fastest path versus the source-destination distance.

linearly with the source-destination distance and have derived bounds on the pre-constants. This implies that every node can be reached in a time that is linear with the distance. The result shows that one does not require full connectivity in a single instant; hence the requirement of a giant connected component (percolation) in a network with interference [10] is greatly relaxed. Hence, e.g., in a route discovery flooding algorithm, the time to find the route scales linearly with the diameter of the network. By simulations we showed that it is beneficial to use higher values of the ALOHA contention parameter for smaller source-destination distances and lower values for large distances, and that the average hop length of the fastest paths first increases rapidly but then decreases slowly as a function of the source-destination distance. This observation provides some insight how to choose the hop length for efficient routing in ad hoc networks.

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