Continuum percolation with holes

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Abstract

We analyze a mathematical model of a cognitive radio network introduced in Yemeni et al. (2016). Our analysis reveals several surprising features of the model. We explain some of these features using ideas from percolation theory and stochastic geometry.

Keywords: percolation, Poisson process, Gilbert model, cognitive networks

1. Introduction

Percolation on the standard disc graph (Gilbert's disc model) is a well-studied topic since the seminal work by Gilbert (1961). It has applications in wireless ad hoc or sensor networks (Haenggi, 2012), where it is assumed that the network is composed of a single class of transceivers with a fixed transmission radius. In an important emerging class of networks, the so-called cognitive networks, however, there exist two classes of transceivers, where the so-called secondary users are only allowed to be active if they are not too close to any of the primary users (Lee and Haenggi, 2012). In these networks, the primary users are allowed unrestricted access to their licensed radio spectrum, while the secondary users are prohibited from causing harmful interference to the primary users, i.e., they need to respect a guard zone around the primary users.

We focus on percolation in the network formed by the secondary users. Assuming that primary and secondary users form independent Poisson point processes, the subset of secondary users who are allowed to be active is a *Poisson hole process*, since the guard zones around the primary users create holes in the point process of active secondary users. This point process was introduced in Lee and Haenggi (2012) and further studied in Yazdanshenasan et al. (2016).

The problem of joint percolation in both the primary and secondary networks was proposed and studied in Yemeni et al. (2016). Our main contribution in this paper is threefold. First, we introduce a re-parametrization of the problem, reducing the number of parameters from the five in (Yemeni et al., 2016) to three. This enables us to summarize the behavior of the full model in a single

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plot, from which one can easily read off information about the original model. Second, we present simulation results on the critical radius for the existence of a left-right crossing, which approximates the critical radius for percolation; we also indicate the necessary steps towards a better approximation of the latter parameter. Finally, these simulation results, shown in Figure 1, suggest several mathematical results on the dependence of the critical radius on the other parameters of the model. We state these results, together with sketches of some of their proofs, in the last main section. Rigorous proofs will appear in a forthcoming paper.

2. Mathematical background

We will use several facts and methods from continuum percolation. Most of these relate to the *Gilbert graph*, which was first defined and studied in Gilbert (1961), and we include a brief description of that model here. We also provide a short explanation of the methods that have been used to study the model, some of which were used in (Yemeni et al., 2016), and some of which we will use ourselves. For more information, and rigorous proofs, the reader is encouraged to consult the books by Meester and Roy (1996), Penrose (2003), Bollobás and Riordan (2006), Haenggi (2012), and the survey article by Balister et al. (2009).

In Gilbert's model, we start with a Poisson process $\mathcal{P}$ of intensity one in the plane. These points form the vertices of an infinite graph G_r. The edges of G_r are obtained by joining two points of $\mathcal{P}$ if they lie at (Euclidean) distance less than r, where r is a fixed parameter.

The main quantity of interest for the Gilbert model is the *critical radius for percolation*. To define this, imagine fixing $\mathcal{P}$ and slowly increasing r, starting from $r = 0$. Initially, the graph G_r will consist of small components, whose vertices happen to lie close together, and isolated vertices. (Here, we use standard graph-theoretic terminology, so that a *component*, by definition, means a connected component.) As we increase r, these components will grow and merge, and at some point an infinite component I, containing a positive fraction $\theta(r) > 0$ of all the vertices in G_r, will appear. When this happens, we say that *percolation* occurs, or that the model G_r *percolates*. The fraction $\theta(r)$ of vertices in I can also be interpreted as the probability that a fixed vertex of G_r belongs to I, and, as r increases, $\theta(r)$ will naturally increase towards 1.

From a rigorous mathematical perspective, Kolmogorov's 0-1 law on tail events implies that, for any fixed value of r, the probability that G_r percolates (and also $\theta(r) > 0$) is either zero or one. In other words, if we consider several different instances of $\mathcal{P}$, and simultaneously increase r in each of them, at the same rate, then percolation occurs *at the same time* in each instance. Consequently, if we define r_{crit} as

$$r_{\text{crit}} = \sup\{r : \theta(r) = 0\},$$

then, for $r < r_{\text{crit}}$, G_r does not percolate (almost surely), and, for $r > r_{\text{crit}}$, G_r percolates (again, almost surely, i.e., with probability 1). As it happens, when $r = r_{\text{crit}}$, G_r does not percolate; this was established in Alexander (1996).

Given this, the next step is to obtain good bounds on r_{crit}. Currently the best known rigorous bounds, due to Hall (1985), are

$$0.833 < r_{\text{crit}} < 1.836.$$

These bounds are only slight improvements of Gilbert's original bounds from 1961, and were obtained using refinements of Gilbert's original methods. The lower bounds were obtained using branching processes, while the upper bounds come from comparison with a lattice percolation model, specifically, face percolation on a hexagonal lattice. More recently, Balister et al. (2005) used dependent percolation to show that, with confidence 99.99%,

$$1.1978 < r_{\text{crit}} < 1.1989.$$

(In detail, Balister, Bollobás and Walters showed that, subject to a certain bound on a certain multidimensional integral, the stated bounds on r_{crit} hold; the integral itself was estimated using Monte Carlo methods, resulting in the stated confidence level.)

For more complicated models, such as the secrecy graph model (Sarkar and Haenggi, 2013), and the model considered in Yemeni et al. (2016), comparison with (dependent or independent) lattice percolation remains the main tool for bounding the various thresholds (indeed, it is used extensively in (Yemeni et al., 2016)). These comparisons work by superimposing an appropriately-sized lattice on the plane, and declaring a face F of the lattice "open" if F contains a point of $\mathcal{P}$. If the lattice spacing has been chosen correctly, then face percolation in the lattice implies percolation in the original model. Therefore, we can use classical bounds for lattice percolation thresholds to deduce that percolation occurs in the original model, for certain parameter values. The method can also be used to show that, for certain other parameter values, percolation does not occur; occasionally, one has to use dependent percolation to make the comparisons work, and this usually results in very weak bounds. Recent innovations include the *rolling ball method* proposed in Balister and Bollobás (2016), and the *high confidence* method introduced in Balister et al. (2005), referred to above. Both these newer methods can also be adapted to other models; for instance they were used in Sarkar and Haenggi (2013) to study the secrecy graph.

The Gilbert model is primarily a model of a random geometric *graph*. However, there is a related *coverage process*, which we will make heavy use of in this paper. To define this coverage process $\mathcal{C}_r$, known as the *Gilbert disc model*, we start with a unit-intensity Poisson process $\mathcal{P}$ as before, but this time we place an open disc $B(p, r)$ of radius r around each point $p \in \mathcal{P}$. The connection between the Gilbert disc model and the Gilbert graph G_r is that graph-theoretic components in G_r correspond exactly to topological components of $\mathcal{C}_{r/2}$. If $\mathcal{C}_{r/2}$ has an infinite (topological) component, we extend our earlier terminology by saying that $\mathcal{C}_{r/2}$ percolates, which, by the above, occurs if and only if G_r percolates as well.

There are several quantities related to the Gilbert disc model which can be conveniently expressed in terms of the connection radius r. First, there is the

average coverage level $\alpha_r = \pi r^2$, which represents both the average number of times a point of $\mathbb{R}^2$ is covered by $\mathcal{C}_r$ and also the average degree in G_r. Then there is the *reduced coverage level* $\alpha'_r = \alpha_{r/2} = \alpha_r/4$, representing the average coverage level of the "reduced" Gilbert model $\mathcal{C}_{r/2}$. Finally, there is the *covered area fraction* β_r, defined to be the proportion of $\mathbb{R}^2$ which is covered by $\mathcal{C}_r$, and which can also be interpreted as the probability that a fixed point $x \in \mathbb{R}^2$ is covered by $\mathcal{C}_r$. Since x is not covered by $\mathcal{C}_r$ if and only if no $p \in \mathcal{P}$ lies within distance r of x, which happens with probability $e^{-\pi r^2}$, we see that $\beta_r = 1 - e^{-\pi r^2}$. (There is also the covered area fraction for the reduced model $\beta_{r/2}$, but we will not use this parameter in our analysis.) To summarize:

$$\text{Reduced coverage level} = \alpha'_r = \frac{1}{4}\pi r^2$$
$$\text{Covered area fraction} = \beta_r = 1 - e^{-\pi r^2}$$

The main facts we will use about $\mathcal{C}_r$ are the precise results from Balister et al. (2010) on the distribution of the regions left uncovered by $\mathcal{C}_r$ when r is large. Since these results are somewhat technical, we will postpone detailed discussion of them until we need them (in Section 7).

3. The work of Yemeni et al.

Yemeni et al. (2016) considers the following five-parameter model of two interlinked random geometric graphs, which they term the *heterogeneous* model. In their model, there are two networks, a *primary* network, and a *secondary* one. The primary network consists of *primary nodes*, distributed according to a Poisson process $\mathcal{P}_1$ of intensity λ_p in the plane; two nodes x and y of the primary network are connected if they lie at Euclidean distance at most (or, equivalently, less than) D_t. The secondary network consists of *secondary nodes*, distributed according to a Poisson process $\mathcal{P}_2$ of intensity λ_s in the plane, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are independent point processes. Two nodes u and v of the secondary network are connected if they lie at Euclidean distance at most d_t from each other, and if *also* there is no primary node within distance D_f of either u or v. The parameters D_t and d_t are the *transmission ranges* of the primary and secondary networks respectively, while D_f is the radius of the *guard zone* of the primary nodes.

This is a complicated and very general model, so one should perhaps not expect a complete analysis of every aspect of it. The first result in (Yemeni et al., 2016) concerns the region of the parameter space for which both networks percolate, which the authors call the *simultaneous connectivity region*. The result is that this region forms a connected subset of $\mathbb{R}^5$. After that, the authors modify an argument from Meester and Roy (1996) (who had in turn modified the classic "trifurcation" argument in Burton and Keane (1989)) to show that, with probability one, there is at most one infinite component in the primary network, and at most one infinite component in the secondary network. Then, they use a "Peierls" (circuit-counting) argument to show that, for all fixed D_t

and d_t, and for λ_p and λ_s above the respective separate percolation thresholds, there exists $D_f > 0$ such that both networks percolate; intuitively this follows by taking D_f sufficiently small so that the primary network has a negligible effect on the secondary one. Finally, they use comparison with site percolation to derive necessary conditions and (much stronger) sufficient conditions for both networks to percolate.

4. Re-parametrization

Our first remark is that, for percolation to occur in the primary network, it is necessary and sufficient that

$$D_t > {\lambda_p}^{-1/2} r_{\text{crit}}. \tag{1}$$

This condition, which appears in the form $\lambda_p > D_t^{-2}\lambda_c(1)$ in each theorem of Sections 5 and 6 of Yemeni et al. (2016), is separate from the rest of the model. In other words, to check whether percolation occurs in the primary network, we just check whether (1) is satisfied, and this just depends on D_t and λ_p. Consequently, we can ignore the parameter D_t, as well as the issue of percolation in the primary network. What is at stake is percolation in the secondary network.

Second, there is no loss of generality in assuming that $\lambda_p = 1$. This is because the model with parameters

$$(\lambda_p, \lambda_s, D_t, d_t, D_f) \tag{2}$$

can be re-scaled to have parameters

$$(1, \lambda_s/\lambda_p, D_t\sqrt{\lambda_p}, d_t\sqrt{\lambda_p}, D_f\sqrt{\lambda_p}). \tag{3}$$

In detail, given the model with the parameters in (2), we can magnify the plane $\mathbb{R}^2$ by a linear factor of $\sqrt{\lambda_p}$ in both directions. This has the effect of dividing both intensities by λ_p and multiplying all distances (including the transmission ranges and guard zone radii) by the factor $\sqrt{\lambda_p}$, yielding the parameters in (3). This special case therefore captures all the essentials of the original model and is the only case we need consider.

From now on, we will write $\lambda = \lambda_s, d = d_t$ and $D = D_f$ for the remaining three parameters. The model can be described as follows. The primary nodes land in the plane, and each primary node creates a "hole" of radius D around itself. These holes cover a fraction

$$\beta_D = 1 - e^{-\pi D^2}$$

of the plane. Then the secondary nodes land. Although these have intensity λ, only the ones landing outside the holes can transmit. Therefore, the average intensity of *active* secondary nodes, i.e., secondary nodes which can actually transmit, is given by

$$\lambda' = \lambda(1 - \beta_D) = \lambda e^{-\pi D^2}.$$

The active secondary nodes do not form a Poisson process but a Poisson hole process as introduced in Lee and Haenggi (2012). Our analysis will be in terms of the parameter

$$\alpha_{\text{sec}}(d, D, \lambda) = \lambda' \alpha'_d = \frac{1}{4}\lambda\pi d^2 e^{-\pi D^2},$$

which represents the average coverage level when discs of radius $d/2$ are placed around each active secondary node. (This corresponds to the reduced coverage level introduced earlier, except that the underlying point process is no longer Poisson.) Specifically, if d_{crit} is the critical radius for percolation in the model (whose existence follows from Kolmogorov's 0-1 law), then we write

$$\alpha_\lambda(\beta) = \alpha_{\text{crit}}(D, \lambda) = \alpha_{\text{sec}}(d_{\text{crit}}, D, \lambda) = \frac{1}{4}\lambda\pi d_{\text{crit}}^2 e^{-\pi D^2} = \frac{1}{4}\lambda\pi d_{\text{crit}}^2(1-\beta).$$

Our goal will be to understand the dependence of $\alpha = \alpha_{\text{crit}}(D, \lambda)$ on $\beta = \beta_D$, for different values of λ. In other words, we will essentially consider α as a function $\alpha_\lambda(\beta)$ of β. First we present some simulation results, followed by some theorems suggested by these results.

5. Simulation method

In the simulation, we focus on left-right (LR) crossings in a large square region of the network. For given values of λ and β (or D), we adjust the parameter α until there exists an LR crossing in between 45 and 55 out of 100 realizations of the model. This experiment is repeated 50 times, and the average of the 50 resulting values of α is taken.

For the initial guess for d_{crit} (and thus α), we use

$$d_{\text{init}} = \frac{r_{\text{crit}}}{\sqrt{\lambda}} e^{\pi D^2/2}(1 + \lambda D^2/12).$$

Depending on the fraction of LR crossings found in 100 network realizations, the initial guess for d_{crit} is adjusted up or down, until it falls in the 45-55% range. The network is simulated on a square of side length $2s$, where $s = \max\{24, 10d_{\text{init}}\}$.

This simulation process is relatively efficient for most of the parameter space but gets time-consuming as $\lambda \to 0$ and $\beta \to 1$ (for $D > 1.3$). Although the existence of an LR crossing does not guarantee percolation, the resulting values for α give some insight into the behavior of $\alpha_\lambda(\beta)$ and also provide supporting evidence for various conjectures on its shape. In principle we can get arbitrarily good approximations to the actual curves using the high-confidence method of Balister et al. (2005), except that this would take a very long time, since the method necessitates a laborious computational process for each value of λ and β; in any case, we expect these results to strongly resemble those in Figure 1 below.

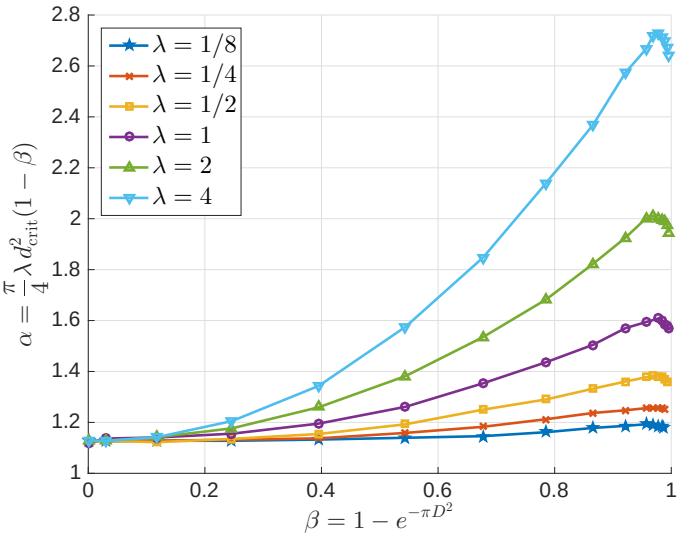

Figure 1: Critical average coverage levels $\alpha_\lambda(\beta)$ for different values of λ.

6. Results

The simulation results for $\lambda = 1/8, 1/4, 1/2, 1, 2$ and 4 are shown in Figure 1. Note that, when $D = 0$, so that $\beta_D = 0$, we recover the original Gilbert model, for which

$$\alpha_{\text{crit}} := \alpha_\lambda(0) = \frac{1}{4}\pi r_{\text{crit}}^2 \approx 1.13.$$

Note also that when $\lambda = 0$ there is no secondary network, so that we cannot meaningfully talk about a critical radius d_{crit} in this case.

Several features are immediately apparent from Figure 1. Here are some of them.

Observation 1. For each λ, and every $\beta \in [0, 1)$, $\alpha_\lambda(\beta) \geq \alpha_{\text{crit}}$.

Observation 2. For each fixed β, $\alpha = \alpha_\lambda(\beta)$ is monotonically increasing with λ.

Observation 3. For each fixed λ, the curve $\alpha = \alpha_\lambda(\beta)$ is unimodal. Also, as λ increases, the peak of $\alpha_\lambda(\beta)$ rises and moves further to the right.

Observation 4. For each fixed λ, $\alpha_\lambda(\beta) \to \alpha_\lambda(0) = \alpha_{\text{crit}}$ as $\beta \to 1$. (Note: it took several hours of computer time to confirm this.)

Observation 5. We have $\alpha_\lambda(\beta) \to \alpha_{\text{crit}}$ uniformly as $\lambda \to 0$.

We can prove some, but not all, of these observations. In the next section, we explain as many of them as we can.

7. Analysis

For Observations 1, 2 and 5, we have only heuristic explanations. First, fix $\beta \in [0,1)$ and $\lambda > 0$. The holes $B(p,D)$ around each point $p \in \mathcal{P}_1$ of the primary network constrain the locations of the active secondary nodes $q \in \mathcal{P}_2 \setminus \bigcup_{p\in\mathcal{P}_1} B(p,D)$, causing the discs $B(q,d/2)$ to overlap more than in the basic Gilbert model, so that the average coverage level required for percolation increases, explaining Observation 1. This overlap will be more pronounced for large λ, explaining Observation 2. Also, when λ is very small, the locations of the active secondary nodes still form an approximate Poisson process, regardless of the value of β, so that our model approximates the basic Gilbert model, which explains Observation 5.

For Observations 3 and 4, we need to describe the geometry of the union of discs $\bigcup_{p\in\mathcal{P}_1} B(p,D)$ in some detail, following the approach in (Balister et al., 2010). The idea is to consider the boundaries $\partial B(p,D)$ of the discs $B(p,D)$, rather than the discs themselves. Consider a fixed disc boundary $\partial B(p,D)$. This boundary intersects the boundaries $\partial B(p',D)$ of all discs $B(p',D)$ whose centers p' lie at distance less than $2D$ from p. There are an expected number $4\pi D^2$ of such points $p' \in \mathcal{P}_1$, each contributing two intersection points $\partial B(p,D) \cap \partial B(p',D)$, and each intersection is counted twice (once from p and once from p'). Therefore we expect $4\pi D^2$ intersections of disc boundaries per unit area over the entire plane; note that these intersections *do not* form a Poisson process, since they are constrained to lie on various circles.

The next step is to move from intersections to regions. The disc boundaries partition the plane into small "atomic" regions. Drawing all the disc boundaries in the plane yields an infinite plane graph, each of whose vertices (disc boundary intersections) has four curvilinear edges emanating from it. Each such edge is counted twice, once from each of its endvertices, so there are almost exactly twice as many edges as vertices in any large region R. It follows from Euler's formula $V - E + F = 2$ for plane graphs (Bollobás, 1998) that the number of atomic regions in R is asymptotically the same as the number of intersection points in R. Moreover, each vertex borders four atomic regions, so that the average number of vertices bordering an atomic region is also four. Note that this last figure is just an average, and that many atomic regions will have less than, or more than, four vertices on their boundaries.

The third step is to return to the discs themselves and calculate the expected number of *uncovered* atomic regions per unit area. It is most convenient to calculate this in terms of uncovered intersection points. A fixed intersection point is uncovered by $\bigcup_{p\in\mathcal{P}_1} B(p,D)$ with probability $e^{-\pi D^2}$ (using the independence of the Poisson process), so we expect $4\pi D^2 e^{-\pi D^2}$ uncovered intersections, and so $\pi D^2 e^{-\pi D^2}$ uncovered regions, per unit area in R. (It turns out that the average number of sides of an *uncovered* region is also four.) Consequently, for large D, these uncovered regions are rare, and, moreover, their distribution in the plane is approximately Poisson. This last assertion can be made precise, and proved using the Chen-Stein method of Poisson approximation (see (Balister et al., 2010) for details).

How large are these uncovered atomic regions? To answer this, recall that the uncovered area fraction in $\mathbb{R}^2$ is $e^{-\pi D^2}$. Since the uncovered regions form an approximate Poisson process of intensity $\pi D^2 e^{-\pi D^2}$, the expected area of each uncovered region is $(\pi D^2)^{-1} = (-\log(1-\beta))^{-1}$. Again, this is just an average; some uncovered regions will be much larger than this, and others will be much smaller.

These observations lead to the following theorem, in which we let $\beta \to 1$ with $\lambda = \lambda(\beta)$ depending on β. We use (almost) standard asymptotic notation, so that $f(x) \sim g(x)$ as $x \to c$ means $f(x)/g(x) \to 1$ as $x \to c$.

Theorem 1. *Suppose that $\beta \to 1$ with $\lambda(\beta) = -f(\beta)\log(1-\beta)$. Then, if $f(\beta) \to \infty$, $\alpha_{\lambda(\beta)}(\beta) \sim f(\beta)\alpha_{\text{crit}}$ as $\beta \to 1$, while if $f(\beta) \to 0$, $\alpha_{\lambda(\beta)}(\beta) \to \alpha_{\text{crit}}$ as $\beta \to 1$.*

Proof. (Sketch) Suppose that the hypotheses of the theorem hold, and note that the disc radius D (corresponding to the covered area fraction β) satisfies $\pi D^2 = -\log(1-\beta)$. Write U for the random variable representing the number of active secondary nodes in an uncovered atomic region and V for the random variable representing the area of an uncovered atomic region. Then, by the tower law of conditional expectation and basic properties of the Poisson process,

$$\mathbb{E}(U) = \mathbb{E}(\mathbb{E}(U|V)) = \mathbb{E}(\lambda(\beta)V) = \lambda(\beta)\mathbb{E}(V) = \frac{\lambda(\beta)}{\pi D^2} = \frac{-\lambda(\beta)}{\log(1-\beta)} = f(\beta).$$

If $f(\beta) \to \infty$, then almost every uncovered atomic region will contain some active secondary nodes q_i, whose discs $B(q_i, d/2)$ will almost exactly coincide, since the uncovered regions are very small and very far apart. On average, for every uncovered atomic region, we will see $f(\beta)$ overlapping discs—sometimes more and sometimes less, depending on the size of the region. This means that the average coverage level required for percolation will be $f(\beta)\alpha_{\text{crit}}$ instead of just α_{crit}.

If, however, $f(\beta) \to 0$, then almost every uncovered atomic region will have no active secondary nodes, and those regions which *do* contain such nodes will contain just one of them. Therefore, when $f(\beta) \to 0$, the active secondary nodes form an approximate Poisson process, so that the average coverage level required for percolation tends to α_{crit} as $\beta \to 1$. □

From this, Observation 4 follows trivially, since if λ is fixed then $f(\beta) \to 0$ as $\beta \to 1$. Theorem 1 is also consistent with Observation 3. It is less clear what happens when $f(\beta) \to c$, and still less clear what happens close to the maximum of $\alpha_\lambda(\beta)$ for each fixed λ, so that most of Observation 3 remains unexplained. We hope to return to these questions in a future paper.

8. Conclusions

In this paper we have analyzed a mathematical model of a cognitive radio network with two classes of user. This model, proposed in Yemeni et al. (2016),

is related to the *Poisson hole process* introduced in Lee and Haenggi (2012). We have simplified the original model, while retaining every one of its essential features, and then studied the new model both computationally and analytically. Our simulation results have revealed several surprising phenomena, some of which are easier to explain than others. We hope that our work will stimulate further research on this problem.

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