The Transport Capacity of a Wireless Network is a Subadditive Euclidean Functional

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Abstract

The transport capacity of a dense ad hoc network with n nodes scales like $\sqrt{n}$. We show that the transport capacity divided by $\sqrt{n}$ approaches a non-random limit with probability one when the nodes are i.i.d. distributed on the unit square. We prove that the transport capacity under the protocol model is a subadditive Euclidean functional and use the machinery of subadditive functions in the spirit of Steele to show the existence of the limit.

I. Introduction

Consider a wireless network of n nodes in a unit square on the plane. Finding the capacity region of this setup is an unsolved problem. Transport capacity is a metric which, in a loose sense, indicates the sum rate of the network while incorporating the notion of distance. It was shown in [1] and [2] that the transport capacity (TC) is $\Theta(\sqrt{n})$. More precisely when *no cooperative communication techniques are used* (except for pure relaying of packet), the transport capacity T is bounded by [1], [2]

$$C_2\sqrt{n} < T(X_n) < C_1\sqrt{n}$$

when $X_n = \{x_1, \cdots, x_n\}$ are n nodes uniformly distributed on the unit square and n is large. The lower bound is provided by Franceschetti etal. using percolation theory. When cooperative communication techniques are used, the transport capacity scales like n [3]. When one restricts the network to act like a packet network without any cooperative techniques (except packet relaying), TC exhibits a nice geometric behavior. While it has been proved that TC scales like $\sqrt{n}$, the question whether the limit

$$\lim_{n\to\infty} \frac{T(X_n)}{\sqrt{n}} \tag{1}$$

exists remained open when the n nodes $x_i,\ 1 \leq i \leq n$ are i.i.d distributed in a unit square. In this paper we show that (1) converges to a constant with probability one. This technique can be easily extended to show that

$$\lim_{n\to\infty} T(X_n)/n^{(d-1)/d} = A_d \text{ a.s.}$$

when the nodes x_i are distributed i.i.d in $[0,1]^d,\ d \geq 2$ and A_d is a constant depending only on the system parameters and the dimension d. We show that transport capacity has a geometric flavor similar to the minimum spanning trees (MST), Euclidean matching (EM) problem and Euclidean travelling salesman problem (TSP). The existence of a limit is more of a mathematical interest, but the techniques used in proving the limit will help in a better understanding of scheduling and routing mechanisms.

The paper is organized as follows. In Section II, we introduce the communication model and the definition of TC. In Section III, we present the geometrical properties of TC and derive the limit. In Theorem 2 we prove the convergence result when the nodes are i.i.d uniformly distributed on a unit square. Theorem 3 provides a similar result when the nodes are i.i.d distributed with a general PDF $f(x)$.

II. System Model

We assume the protocol model [1] for communication between two nodes, i.e., a node located at x_i can communicate successfully to a node located at x_j if the ball centered around x_j with radius $\beta|x_i - x_j|$, $\beta > 1$, does not contain any other transmitter. When the communication is successful, we assume one packet of information is transmitted[1]

[1]Basically we are neglecting noise. Neglecting noise can make the achievable rate unbounded. So we cap the link capacity to unity. Alternatively we can assume a packet of information transmitted.

Definition 1: Transport Capacity: For n nodes $\{x_1, x_2, \cdots, x_n\} \subset \mathbb{R}^2$, the transport capacity of these n nodes is defined as

$$T(\{x_1, x_2, \cdots, x_n\}) = \sup_{\mathcal{S}} \left[\sum_{(i,j)\in[1,2..n]^2} \lambda_{ij} |x_i - x_j| \right]$$

where the supremum is taken over the supportable rate pairs $\mathcal{S}$. The set $\mathcal{S}$ can also be thought of as the set of all scheduling and routing algorithms. The set $\mathcal{S}$ contains scheduling algorithm with fixed source and destination pairs. λ_{ij} denotes the information rate that node x_i can communicate to x_j (we don't count the relaying nodes). Observe that *the definition of* $T(\{x_1, \cdots, x_n\})$ *depends only on the location of the nodes* $x_i, 1 \le i \le n$. We make the following assumptions:

1) Time is discretized.
2) Message set for each source destination pair is independent.
3) *No cooperative communication techniques are used.*
4) $T(\{x_1\}) = 0$

We will consider two cases. One with no constraint on λ_{ij} and the other with the following constraint.

Constraint 1: $\lambda_{ij} > 0$ for some j for every i, i.e., $\max_j \lambda_{ij} > 0$, $\forall i$

Notation: Let $B(x, r)$ denote a ball of radius r centered around x. For a set A, the complement is denoted by the set A^c. For a finite set A, $|A|$ denotes the cardinality of the set A. We will use $(A \to B)$ to denote the set of transmissions with transmitters in A and receivers in B.

III. Limit Theorems

In this section we show the existence of the limit (1) using tools from subadditive sequences. A sequence $\{a_m\}$ is subadditive if $a_{m+n} \le a_m + a_n$. By a theorem of Fekete, we have that $\lim a_m/m = \inf(a_m/m)$ exists. Similar results hold when the sequence is superadditive. Most of the geometrical quantities like the length of a minimum spanning tree on n points, or a Euclidean matching of n points are not strictly subadditive. They have a small correction factor, i.e., of the form $a_{m+n} \le a_m + a_n + c(m, n)$. If the growth of $c(m, n)$ can be controlled, the existence of the limit can be proved. When the underlying sequences are random variables, the existence of the limit is provided by a classical result of Kingman [4]. Steele has used such a frame work to prove the existence of the limit of a weakly subadditive sequences in the geometrical setting [5]. The geometrical quantities which exhibit such subadditivity are coined "Subadditivie Euclidean functionals". We will use the framework of Steele to prove the existence of the limit (1). For doing so, we first establish the weak subadditivity of TC and other required properties. We start by introducing the following bound on TC which was proved in [6]. We state it for convenience.

Lemma 1: [Sphere packing bound] The transport capacity of n nodes $\{x_1, x_2, \cdots, x_n\}$ located in a square $[0, t]^2$ is bounded by $Ct\sqrt{n}$, where C is a constant not depending on the location of nodes or n.

Proof: See Section 2.5 in [6] ■

A. Basic properties of TC

In this subsection, unless indicated, $X_n = \{x_1, x_2, \cdots, x_n\}$ are deterministic points on the plane. From the definition of T, we can consider T as a functional on finite subsets of $\mathbb{R}^2$. We then have

(A0) $T(X_n)$ is a continuous function of $\{x_1, x_2, \cdots, x_n\}$ and hence measurable.

(A1) $T(aX_n) = aT(X_n)$ for all $a > 0$.

(A2) $T(X_n + x) = T(X_n)$ for all $x \in \mathbb{R}^2$ where $X_n + x = \{x_1 + x, x_2 + x, \cdots, x_n + x\}$

(A1) and (A2) imply T is a Euclidean functional.

(A3) *Monotone property*: $T(X_n \cup \{x\}) \ge T(X_n)$. *The above monotone relation does not hold true with constraint* 1.

(A4) Finite variance:

$$\text{Var}\, T(\{x_1, x_2, \cdots, x_n\}) < \infty$$

when x_i are independently and uniformly distributed on $[0, 1]$. This follows from Lemma 1.

The next lemma provides an estimate, which is used to bound the correction factor in the subadditivity of TC.

Lemma 2: Consider the scenario in which nodes in a square $S = [0, t]^2 \subset \mathbb{R}^2$ can only be transmitters that have to communicate with receivers outside the square S in a single hop. If we restrict the maximum Tx-Rx distance to be $c_1 t$, then the transport capacity in this setup is upper bounded by $c_2 t$.

Proof: For a transmitter receiver pair (x_k, y_k) denote

$$D_k = \cup_{x\in \text{line}(x_k, y_k)} B\left(x, \frac{(\beta - 1)}{2} |x_k - y_k|\right)$$

i.e., the $\frac{(\beta-1)}{2}|x_k - y_k|$ neighborhood of the line joining x_k and y_k . See Figure 1. For all the successful Tx-Rx pairs, the regions D_k are disjoint. The proof of the above is identical to Theorem 3.3 in [6]. In our case we have that the transmitters are inside the square $[0, t]^2$. Let the contending transmitter-receiver distances be $\{r_1, r_2, \cdots, r_n\}$. Since the receivers are outside the box and each transmitter-receiver pair cuts the boundary, we have

$$2\frac{\beta - 1}{2}(r_1 + r_2 + \cdots + r_n) \le 4t$$

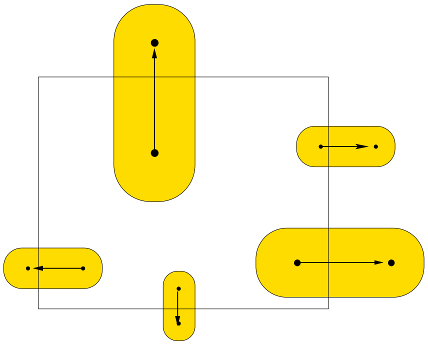

Figure 1. Illustration of the Proof. The coloured regions represent D_k

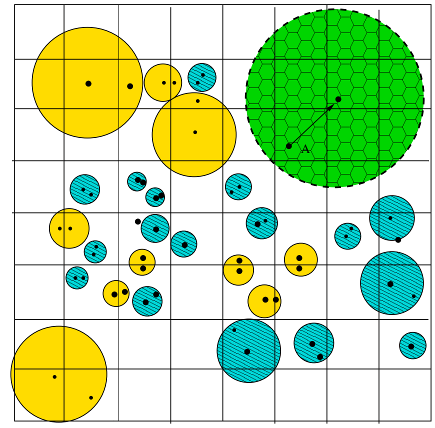

Figure 2. Proof technique: The blue hashed circles (dark hashed) correspond to $(A_i \to A_i)$ and the TC contribution can be bounded by $T(A_i)$. The yellow unhashed circles corresponds to $(A_i \to A_i^c)$. However these cannot contribute much to the TC by Lemma 2. The maximum contribution from them is $cm^2t/m = cmt$. There is a trade-off between $(A_i \to A_i^c)$ and the large transmissions denoted by hashed yellow region on the top corner. Observe that when the Tx-Rx distance is greater than $a = 2\sqrt{2}t/(m(\beta-1))$, there can be a maximum of one transmission per square (as in the green comb circle on the right corner).

Hence the single hop transport capacity in this case is upper bounded by $4t/(\beta-1)$ ■

From the previous lemma we observe that the TC is constrained by the perimeter of the domain A which contains the nodes, when the transmissions are from the set $(A \to A^c)$. In some sense this indicates that TC is maximized when the communication is local, i.e., short hops. In the next lemma we prove that the bottleneck in a multihop network for achieving TC is the maximum packing of scheduling on a plane. Loosely speaking *unconstrained TC metric is more suitable for a single-hop network.*

Lemma 3: Multihop to single-hop conversion [*Flattening the network*]: Any scheme which achieves the TC consists of only single hops, i.e., every packet reaches the destination from source in a single hop.

Proof: Suppose a flow λ_{ij} is helped by n nodes. Now instead of assisting this flow, each of these n nodes send their own independent packets for a single hop they serve. By simple triangle inequality this procedure guarantees a single hop scheme that achieves the same or larger TC. ■

In the next lemma we prove a form of subadditivity. We use the fact that the network can be visualized of as a single-hop network and the idea that the TC is maximized by local communications. See Figure 2, for a graphical illustration of the proof.

Lemma 4: [*Cutting Lemma*]: Consider a square $A = [0,t]^2 \subset \mathbb{R}^2$ and let $X = \{x_1 \dots x_k\} \subset A$ denote a set of k nodes. Divide A into m^2 squares of equal sides with length t/m and denote each square by A_i. We then have

$$T(X) \leq \sum_{i=1}^{m^2} T(X \cap A_i) + Cmt$$

Proof: Let some scheme achieve the TC of X. By Lemma 3 the scheme that achieves TC is a single hop scheme. We now focus on a single square A_i. There are three types of transmissions, $(A_i \to A_i)$, $(A_i \to A_i^c)$ and $(A_i^c \to A_i)$. See Figure 2. The contribution of transmissions from A_i into A_i to the TC, can be upper bounded by $T(X \cap A_i)$. Hence the total contribution by $(A_i \to A_i)$, $1 \leq i \leq m^2$ is upper bounded[2] by $\sum_{i=1}^{m^2} T(X \cap A_i)$. The only transmissions which involve A_i, to be accounted are $(A_i \to A_i^c)$ and $(A_i^c \to A_i)$. Denote the contribution of these transmissions to the TC by $\tilde{T}$. Let $F(A_k)$ denote the set of feasible transmitters in square A_k with receivers in A_k^c. By the sphere packing bound we have

$$\sum_{k=1}^{m^2} \sum_{(x,y)\in F(A_k)} |x-y|^2 \leq Ct^2$$

Let $b_k = \sum_{(x,y)\in F(A_k)} |x-y|$. So we require to bound $\tilde{T} = \sup\left\{\sum_{k=1}^{m^2} b_k\right\}$ where the supremum is taken over all the feasible transmissions. Let the number of squares with all of their transmission distance less than $a = 2\sqrt{2}t/(m(\beta-1))$ be η. Denote this set of squares by $C_a \subset \{1, \cdots, m^2\}$. So we have $|C_a| = \eta$ and $\tilde{T} = \sup\left\{\sum_{k\in C_a} b_k + \sum_{k\in C_a^c} b_k\right\}$. Let $A_k \in C_a^c$. We then have $|F(A_k)| = 1$. Hence $\sum_{k\in C_a^c} b_k$ is upper bounded by (since the maximum number of transmitters is $m^2 - \eta$)

$$c_1 t\sqrt{m^2-\eta}$$

For the other set C_a with Tx-Rx distances less than a, by Proposition 2, the contribution $\sum_{k\in C_a} b_k$ to the transport capacity is upper bounded by

$$c_2 \frac{t}{m}\eta$$

[2] This is true since we consider $T(X \cap A_i)$ as only a function of $X \cap A_i$.

So we have

$$\tilde{T} \leq c_1 t\sqrt{m^2-\eta} + c_2\frac{t}{m}\eta,\ 0 \leq \eta \leq m^2$$

The maximum value of the right hand side for the given range of η is ctm. ■

Theorem 1: Let $\left\{Q_i : 1 \leq i \leq m^2\right\}$ be a partition of the square $[0,1]^2$ into squares with edges parallel to the axis and length m^{-1}. Let $tQ_i = \{x; x = ty, y \in Q_i\}$.

(A5) Subadditivity: Let $X = \{x_1, x_2 \cdots x_n\}$. We then have

$$T(X \cap [0,t]^2) \leq \sum_{i=1}^{m^2} T(X \cap tQ_i) + Ctm \qquad (2)$$

Proof: This follows immediately from Lemma 4. ■

Equation (2), does not imply subadditivity, but only a weaker form of it. Nevertheless it is denoted as subadditive property for convenience.

Theorem 2: Let $x_i, 1 \leq i \leq n$, and x_i are i.i.d uniformly distributed in $[0,1]^2$. If λ_{ij} is not constrained then

$$\lim_{n\to\infty} \frac{T\left(\{x_1, x_2, \cdots, x_n\}\right)}{\sqrt{n}} = A_2 \qquad (3)$$

with probability one. A_2 is a constant depending only on β.

Proof: The conditions (A1) to (A5) indicate that T is a monotone, Euclidean functional with finite variance and satisfies subadditivity. (3) follows from the subadditive Euclidean convergence theorem by Michael Steele [5, Thm 1]. ■

Observe that in the above theorem, monotonicity of T is necessary. Hence it does not hold with constraints on λ_{ij}, i.e., constraint 1. To overcome this we require to prove the *smoothness* of T.

Let $Q_i, i \in \{1,2,3,4\}$ be a partition of the unit square into 4 equal squares. By Theorem 2 we have

(A6)

$$T(F) \leq \sum_{i=1}^{4} T(F \cap Q_i) + C$$

where F is any finite set in $[0,1]^2$. The above result follows from (A5) but we numbered it for convenience. In the next Lemma we prove the smoothness of $T(A)$ with respect to the cardinality of A. Observe that this sense of continuity is different from (A0).

Lemma 5: (A7) [Smoothness]: For finite point sets $F, G \subset [0,1]^2$ (observe F and G need not be disjoint), we have

$$|T(F \cup G) - T(G)| < c\sqrt{|F|} \qquad (4)$$

where c is a constant that does not depend on F and G.

Proof: We use the same trick as we did in Theorem 4. We flatten the network of $F \cup G$. The transmissions can be partitioned into $(G \to G)$, $(F \to F)$, $(G \to F)$, $(F \to G)$. The contribution of the transmissions $(G \to G)$ to TC can be upper bounded by $T(G)$. Observe that the maximum cardinality of the remaining transmissions can be $|F|$. So we have

$$T(F \cup G) \quad < \quad T(G) + c\sqrt{|F|}$$

If we do not assume any constraint on λ_{ij}, then we are done by the monotonicity. If Constraint 1 has to be satisfied, we n have to prove

$$T(F \cup G) \quad \geq \quad T(G) - c\sqrt{|F|}$$

We use time sharing to prove this. By Lemma 1, we have $T(F) < c_1\sqrt{|F|}$. So we can assume $T(G) > T(F)$ (otherwise there is nothing to be proved). We use time sharing between the set of nodes, G and F. By time sharing the constraint that each node transmits some data of its own is satisfied. So we obtain a transport capacity of

$$\lambda T(G) + (1-\lambda)T(F) \qquad (5)$$

Choose

$$1 - \lambda \quad = \quad \frac{1}{\frac{T(G)}{T(F)} - 1}$$

So if $T(G) > 2T(F)$, we have $(1-\lambda) < 1$ and

$$\begin{aligned} & T(G) - (T(G) - T(F))(1-\lambda) \\ = \quad & T(G) - T(F) \end{aligned}$$

Otherwise we have $0 < T(G) - T(F) \leq T(F)$. So from (5), we have

$$\begin{aligned} & T(G) - (T(G) - T(F))(1-\lambda) \\ \geq \quad & T(G) - T(F)(1-\lambda) \\ \geq \quad & T(G) - T(F) \end{aligned}$$

i.e., any time sharing will give a transport capacity greater than $T(G) - T(F)$. So by time sharing we have constructed a scheme which obeys constraint 1 and has a TC of at least $T(G) - T(F)$. Since $T(F \cup G)$ is the supremum over all such schemes we have,

$$\begin{aligned} T(F \cup G) \quad & \geq \quad T(G) - T(F) \\ & \overset{(a)}{\geq} \quad T(G) - c\sqrt{|F|} \end{aligned}$$

where (a) follows from the sphere packing bound on the set F. ■

(B-1) We also have the following. Let F and G be any finite subsets of $[0,1]^2$. Then

$$\begin{aligned} & |T(F) - T(G)| \\ \overset{(a)}{\leq} \quad & |T(F) - T(F \cap G)| + |T(G) - T(F \cap G)| \\ \overset{(b)}{\leq} \quad & c\left\{\sqrt{|F \setminus (F \cap G)|} + \sqrt{|F \setminus (F \cap G)|}\right\} \\ \leq \quad & \sqrt{2}c\left\{\sqrt{|F \setminus (F \cap G)| + |G \setminus (F \cap G)|}\right\} \\ = \quad & \sqrt{2}c\sqrt{|F \triangle G|} \end{aligned}$$

where (a) follows from triangle inequality and (b) follows from Lemma 5.

We now use the theorem from Rhee [7] to prove the existence of the limit when Condition 1 is satisfied. From the conditions (A1) to (A8) we have the following convergence of the mean and concentration around the mean. This result holds even with Constraint 1 unlike Theorem 2.

Lemma 6: Let $X_n = \{x_1, x_2, \cdots, x_n\}$ denote n i.i.d nodes in $[0,1]^2$. For the transport capacity we have that

$$\lim_{n\to\infty} \frac{\mathbb{E}T(X_n)}{\sqrt{n}} = A_2$$

and

$$\mathbb{P}\left(|T(X_n) - \mathbb{E}T(X_n)| \geq t\right) \quad \leq \quad C\exp\left(-C_1\frac{t^4}{n}\right) \quad (6)$$

Proof: Follows from [7, Thm 1]. Here we do not require monotonicity and the complete subadditive hypothesis. Conditions (A6) and (A7) replace those two. ■

If we choose t to be $t\sqrt{n}$, we have the right hand side of (6) is $\exp(-C_1 t^4 n)$. Equation (6) also implies complete convergence i.e., for all $\epsilon > 0$

$$\sum_{n>1} \mathbb{P}\left(\left|\frac{T(x_n)}{\sqrt{n}} - A_2\right| > \epsilon\right) < \infty$$

B. Non uniform distribution of nodes

In the previous subsection, we have proved the existence of the limit when the nodes are uniformly distributed on an unit square. In this subsection we prove the existence of the limit and show its relation to A_2 when the nodes are distibuted with a PDF $f(x)$. In Lemma 4, we proved an upperbound to $T(X_n)$ by the transport cpacity of disjoint subsets of x_n. We now prove a lower bound to $T(X_n)$ by similar subsets of X_n.

Lemma 7: [*Asymptotic Glueing Lemma*] Consider two bounded disjoint sets $A, B \subset \mathbb{R}^2$ and an infinite sequence of nodes $\{x_i\}$. Let $X_n = \{x_1, x_2, \cdots, x_n\}$ be a subset of the sequence. We then have

$$\begin{aligned} & T(X_n \cap A) + T(X_n \cap B) & (7)\\ \leq \quad & T(X_n \cap (A \cup B)) + o(\sqrt{n}) & (8) \end{aligned}$$

Proof: Consider the flattened networks of A and B which achieve the TC of A and B respectively. Wlog we can assume we can assume $T(A) = \Theta(\sqrt{n})$ and $T(B) = \Theta(\sqrt{n})$ (otherwise there is nothing to prove). We have to find a scheme such that (7) is satisfied. Consider the following. At any time, neglect all transmissions with transmitter receiver distance greater than $\sqrt{\log(n)/n}$. The loss in TC by removing these transmissions is $\sqrt{n/\log(n)}$. This is because, the loss is given by $\max\left(\sum_{(i,j)\in\mathcal{T}} d_{ij}\right)$ with the following constraints

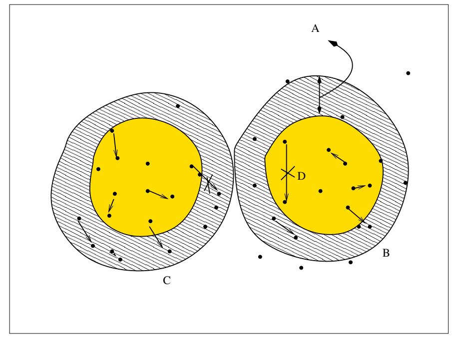

Figure 3. The hashed region is the boundary with thickness $2\sqrt{\log(n)/n}$. We neglect all transmissions in the inside region with length greater than $\sqrt{\log(n)/n}$.

$$\begin{cases} \sum_{(i,j)\in\mathcal{T}} d_{ij}^2 < A \\ d_{ij} > \sqrt{\frac{\log(n)}{n}} \end{cases}$$

where $\mathcal{T}$ is the set of all feasible transmissions with Tx-Rx distance greater than $\sqrt{n/\log(n)}$. The solution to the above problem is $\sqrt{An/\log(n)}$. See Figure 3. Now neglect all the nodes along the boundary of A and B in a strip of width $2\sqrt{\log(n)/n}$. The maximum penalty because of this is

$$c\sqrt{\frac{\log(n)}{n}}\sqrt{n} \quad = \quad c\sqrt{\log(n)}$$

Now operate A and B networks together except for the nodes in the strip as mentioned above and the transmissions with Tx Rx lengths greater than $\sqrt{\log(n)/n}$. So we are still left with a transport capacity of (that can be achieved by the union).

$$\begin{aligned} & T(A) + T(B) - c\sqrt{\frac{n}{\log(n)}} - c_2\sqrt{\log(n)} \\ = \quad & T(A) + T(B) - o(\sqrt{n}) \end{aligned}$$

We can operate the neglected strips of A and B, the neglected transmissions and the others in a time sharing fashion with time shares

$$\left(1 - \frac{1}{n}, \frac{1}{3n}, \frac{1}{3n}, \frac{1}{3n}\right)$$

In the resulting network Constraint 1 is satisfied. ■

We have the following lemma required to prove the limit when the nodes are not uniformly distributed. We can generalize the previous Lemma to s disjoint squares to prove the following.

Lemma 8: (A-9) Let Q_i, $1 \leq i \leq s$ be a finite collection of disjoint squares with edges parallel to the axes and let $x_i \in \mathbb{R}^2$, $1 \leq i < \infty$ an infinite sequence. Let $X_n = \{x_1, x_2, \cdots, x_n\}$. We then have

$$\sum_{i=1}^{s} T\left(X_n \cap Q_i\right) \quad \leq \quad T\left(X_n \cap \cup_{i=1}^{s} Q_i\right) + o(\sqrt{n})$$

Proof: Follows from Lemma 7. ■

We now prove the limit theorem when the nodes are i.i.d. distributed with a *blocked distribution.* A blocked distribution is of the form $\phi(x) = \sum_{i=1}^{s} 1_{Q(i)}(x)$ where $Q(i)$ are disjoint squares with edges parallel to the axes. We use the homogeneous property of TC and the glueing lemma to prove the next lemma. Also observe that $\phi(x)$ looks like a simple function. Extending the result to general distributions is of more technical nature and is stated in Theorem 3.

Lemma 9: Let $Y_i,\ 1 < i \leq n$ be a sequence of i.i.d random variables with bounded support and no singular part [8]. Let the absolutely continuous part be given by $\phi(x) = \sum_{i=1}^{s} 1_{Q(i)}(x)$ where $Q(i)$ are disjoint cubes with edges parallel to the axes. Let $\mathcal{Y}_n = \{Y_1, \cdots, Y_n\}$ One then has

$$\lim_{n\to\infty} \frac{T(\mathcal{Y}_n)}{\sqrt{n}} = A_2 \int_{R^d} \sqrt{\phi(x)} dx$$

Proof: We follow the method provided in [5]. Without loss of generality, we assume that the support of RV Y_i lies in $[0,1]^2$. Since the $Q(i)$ are disjoint we have by Theorem 2,

$$T(\mathcal{Y}_n) \quad \leq \quad \sum_{i=1}^{s} T(\mathcal{Y}_n \cap Q(i)) + Cs \tag{9}$$

We have that $\mathcal{Y}_n \cap Q(i)$ is uniform on $Q(i)$ except for the un-normalized measure $m(Q(i))$. Using (A-1) and Theorem 2, we have

$$\lim_{n\to\infty} \quad \frac{T(\mathcal{Y}_n \cap Q(i))}{\sqrt{\sum_{j=1}^{n} 1_{Q(i)}(y_j)}} \quad = A_2 \sqrt{m(Q(i))}$$

By the law of large numbers we have,

$$\sum_{j=1}^{n} 1_{Q(i)}(y_j) \sim n\alpha_i m(Q(i)) \ a.s$$

So

$$\lim_{n\to\infty} \frac{T(\mathcal{Y}_n \cap Q(i))}{\sqrt{n}} = A_2 \sqrt{\alpha_i} m(Q(i))$$

So using (9), we obtain

$$\lim \sup_{n\to\infty} \frac{T(\mathcal{Y}_n)}{\sqrt{n}} \geq A_2 \int \sqrt{\phi(x)} dx$$

By Lemma 8,

$$T(\mathcal{Y}_n) \geq \sum_{i=1}^{s} T(\mathcal{Y}_n \cap Q(i)) + o(\sqrt{n}) \tag{10}$$

By using a similar procedure on (10), we have a similar result on $\lim\inf$ and hence the lemma follows. ■

The next theorem characterizes the limiting behavior of TC when the nodes are not uniformly distributed.

Theorem 3: Let y_i be i.i.d random variables, with PDF $f(x)$ (i.e., no singular part w.r.t Lebesgue measure) and bounded support. We then have

$$\lim_{n\to\infty} \frac{T\left(y_1, y_2, \cdots, y_n\right)}{\sqrt{n}} = A_2 \int_{\mathbb{R}^2} \sqrt{f(x)} dx$$

Proof: Follows from (B-1), Lemma 9 and Theorem 3 in [9] (Observe the above theorem is not proved when the measure of y_i has singular support). ■

We immediately observe that the constant $A_2 \int_{\mathbb{R}^2} \sqrt{f(x)} dx$ is maximized when y_i are uniformly distributed.

IV. Conclusion

In this paper we have shown that the transport capacity of n nodes distributed i.i.d with bounded support, when scaled by $\sqrt{n}$ approaches a non-random limit. The existence of a limit is more of a mathematical interest, but the techniques used in proving the limit will help in a better understanding of scheduling and routing mechanisms.

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References

[1] P. Gupta and P. Kumar, "The capacity of wireless networks," *Information Theory, IEEE Transactions on*, vol. 46, no. 2, pp. 388–404, 2000.

[2] M. Franceschetti, O. Dousse, D. Tse, and P. Thiran, "Closing the gap in the capacity of random wireless networks," *Information Theory, 2004. ISIT 2004. Proceedings. International Symposium on*.

[3] A. ÖzgÜr, O. LÉvÊque, and D. Tse, "Hierarchical Cooperation Achieves Optimal Capacity Scaling in Ad Hoc Networks," *Information Theory, IEEE Transactions on*, vol. 53, no. 10, pp. 3549–3572, 2007.

[4] J. Kingman, "The Ergodic Theory of Subadditive Stochastic Processes," *Journal of the Royal Statistical Society. Series B (Methodological)*, vol. 30, no. 3, pp. 499–510, 1968.

[5] J. Steele, "Subadditive Euclidean functionals and nonlinear growth in geometric probability," *Ann. Prob*, vol. 9, pp. 365–376, 1981.

[6] P. Kumar and F. Xue, *Scaling Laws for Ad-Hoc Wireless Networks: An Information Theoretic Approach*. Now Publishers Inc, 2006.

[7] W. Rhee, "A matching problem and subadditive Euclidean functionals," *Ann. Appl. Probab*, vol. 3, no. 3, pp. 794–801, 1993.

[8] G. B. Folland, *Real Analysis, Modern Techniques and Their Applications*. Wiley, 2 ed., 1999.

[9] J. Steele, "Growth rates of Euclidean minimal spanning trees with power weighted edges.," *Ann. Probab*, vol. 16, no. 4, pp. 1767–1787, 1988.