

Efficient Calculation of Meta Distributions and the Performance of User Percentiles

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Abstract—Meta distributions (MDs) are refined performance metrics in wireless networks modeled using point processes. While there is no known method to directly calculate MDs, the moments of the underlying conditional distributions (given the point process) can often be expressed in exact analytical form. The problem of finding the MD given the moments has several solutions, but the standard approaches are inefficient and sensitive to the choices of a number of parameters. Here we propose and explore the use of a method based on binomial mixtures, which has several key advantages over other methods, since it is based on a simple linear transform of the moments.

Index Terms—Wireless networks, stochastic geometry, point process, meta distribution.

I. INTRODUCTION

Meta distributions. A meta distribution (MD) is the distribution of a conditional probability conditioned on the underlying point process(es) Φ , *i.e.*, the distribution of

$$P_t = \mathbb{P}(X > t \mid \Phi), \quad (1)$$

where X is a metric of interest, such as the SIR, SINR, rate, or energy. Hence P_t is averaged over all randomness in the network except for the spatial locations of the transceivers (fading, channel access, etc.). Upon averaging over the point process, the MD of X at the typical link or user is the cumulative distribution function (cdf)

$$F(x) = \mathbb{P}(P_t \leq x),$$

with the understanding that F is a function of t also. Depending on the network model, the probability measure needs to be replaced by the Palm measure given that a transmitter or receiver resides at a given location. For ergodic point processes Φ , $1 - F(x)$ is the fraction of links or users that achieve $X > t$ with probability at least x in each realization of Φ . Hence a key advantage of the MD over the commonly used mean metric $\mathbb{P}(X > t) = \int_0^1 (1 - F(x)) dx$ is that it reveals the performance of user percentiles. For example, $F^{-1}(0.05)$ gives the reliability x that 95% of the users achieve but 5% do not, *i.e.*, the reliability of the “5% user”.

The MD for the SIR was introduced in [1] and evaluated for two basic Poisson network models. This refined SIR analysis was extended to cellular networks with D2D underlay in [2], to base station cooperation in [3], to power control for up- and downlink in [4], and to non-orthogonal multiple access (NOMA) in [5]. Further, [6] extended the SIR to the SINR for mm-wave D2D networks and introduced the MD

of the transmission rate, where $X = W \log_2(1 + \text{SINR})$ for bandwidth W , and [7] explores the MD of the secrecy rate. Lastly, [8] defined the energy MD for a wireless network with energy harvesting capabilities.

Calculation of the meta distributions. Since there is no known way to calculate the MD directly, a “detour” is needed via the moments of P_t . With the moments in hand, the calculation of the MD is an instance of the *Hausdorff moment problem* since the support of P_t is bounded to $[0, 1]$. The standard method to calculate the MD is to use the Gil-Pelaez theorem [9], which requires the integration of the imaginary part of $e^{-it \log x} M_{it}/t$ over $t \in \mathbb{R}^+$ for each value of x and t , where $i^2 = -1$ and $M_b = \mathbb{E}(P_t^b)$. By its nature, the Gil-Pelaez approach requires a careful selection of the range of the numerical integration (depending on the rate of convergence of the integrand) and its step size. Moreover, there is no simple way to bound the error of the resulting approximation.

Another approach is to use only the first and second moments and use the corresponding beta distribution as an approximation. This method has proven surprisingly accurate but has its limitations if the actual distribution falls outside the class of beta distributions.

Very recently, the work [10] proposed to use Fourier-Jacobi expansion to express the MD as an infinite sum of shifted Jacobi polynomials. Truncations to a finite sum yield approximations, such as the beta approximation above. The method is promising but its convergence properties are unclear.

Notation. $[n] \triangleq \{1, 2, \dots, n\}$, $[n]_0 \triangleq \{0\} \cup [n]$. \mathbb{N} is the set of (positive) natural numbers, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

II. THE BINOMIAL MIXTURE METHOD

For a cdf F with bounded support $[0, T]$, let \mathcal{M} be the operator that yields the moments

$$M_n = (\mathcal{M}F)_n \triangleq \int_0^T x^n dF(x), \quad n \in \mathbb{N},$$

with $M_0 = 1$. The Hausdorff moment problem is to retrieve F from $M = (M_n)_{n \in \mathbb{N}}$, *i.e.*, to find \mathcal{M}^{-1} . Here we focus on $T = 1$, since the random variables of interest are conditional probabilities. The map \mathcal{M}^{-1} is unique since T is bounded.

Necessarily, the sequence of moments M of any distribution on $[0, 1]$ is completely monotonic, *i.e.*,

$$(-1)^k (\Delta^k M)_n \geq 0, \quad (2)$$

where Δ^k is the iterated difference operator, with $(\Delta M)_n = (\Delta^1 M)_n = M_{n+1} - M_n$. For example, $(\Delta^3 M)_3 = M_6 - 3M_5 + 3M_4 - M_3$. (2) follows from

$$(-1)^k (\Delta^k M)_n = \int_0^1 x^n (1-x)^k dF(x),$$

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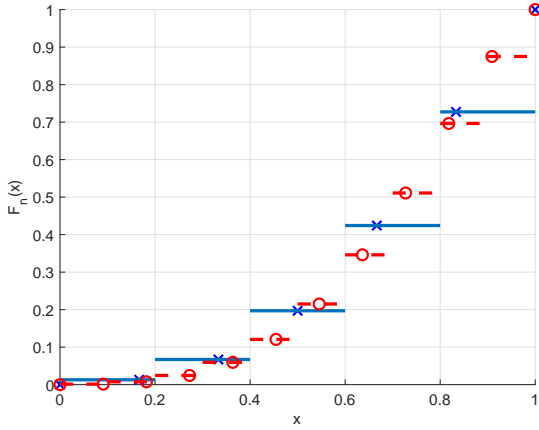


Fig. 1. $F_5(x)$ (solid) and $F_{10}(x)$ (dashed) for a beta distribution with parameters $\alpha = 5$ and $\beta = 2$. The \times denote the samples $F_5(k/6)$, $k \in [6]_0$, and the \circ denote the samples $F_{10}(k/11)$, $k \in [11]_0$.

where the right hand side is non-negative.

In [11], a piecewise approximation based on binomial mixtures is proposed for F . It is defined as follows.

Definition 1 (Piecewise approximation) For any $n \in \mathbb{N}_0$, define the approximate cdf F_n

$$F_n(x) \triangleq \sum_{k=0}^{\lfloor nx \rfloor} \sum_{j=k}^n \binom{n}{j} \binom{j}{k} (-1)^{j-k} M_j, \quad x \in (0, 1], \quad (3)$$

and $F_n(0) = 0$, where $\lfloor u \rfloor$ is the largest integer smaller than or equal to u .

The key property of this approximation is that $F_n(x) \rightarrow F(x)$ as $n \rightarrow \infty$ for each x at which F is continuous [11], [12], i.e., (3) converges to the map \mathcal{M}^{-1} .

III. NUMERICAL RECIPE

A. Sampling F_n

For the numerical calculation of (3), $x \in [0, 1]$ needs to be sampled at discrete values. We choose $x_k = k/(n+1)$, for $k \in [n+1]_0$, which is the densest uniform sampling such that $F_n(x_k) \neq F_n(x_j)$, $i \neq j$. Moreover, if F is the cdf of a uniform random variable, the linear interpolation of $F_n(x_k)$ gives the exact cdf for any $n \in \mathbb{N}_0$. Fig. 1 gives an example of F_5 and F_{10} for a beta distribution and shows the sampling values.

B. Approximate Probability Density Function

Letting

$$h_k \triangleq \sum_{j=k}^n \binom{n}{j} \binom{j}{k} (-1)^{j-k} M_j, \quad k \in [n]_0, \quad (4)$$

we have

$$F_n(x_k) = \sum_{m=0}^{k-1} h_m, \quad k \in [n+1]_0,$$

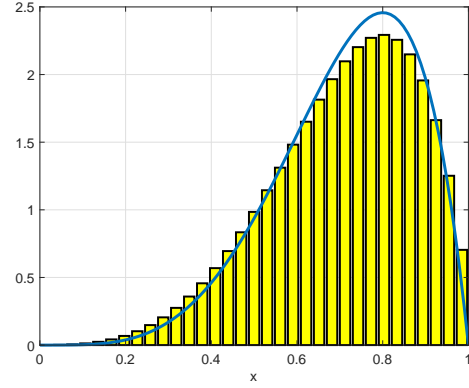


Fig. 2. $f_{32}(x)$ for a beta distribution with parameters $\alpha = 5$ and $\beta = 2$. The solid curve is the exact beta pdf.

and $f_n(x_k) \triangleq (n+1)h_k$ is an approximation of the probability density function (pdf) since $F_n(x_k)$ can be interpreted as a Riemann sum

$$\frac{1}{n+1} \sum_{m=0}^{k-1} f_n(x_k),$$

which converges to $F(x_k)$ as $n \rightarrow \infty$.

Fig. 2 shows an exact beta pdf and the approximation f_{32} .

C. The Method as a Linear Transform

Eqn. (4) shows that h_k , $k \in [n]_0$, is just a linear combination of the moments M_j , $j \in [n]_0$. Accordingly, we can write $\mathbf{h} = (h_k)_{k=0}^n$ as a linear transformation of $\mathbf{m} = (M_j)_{j=0}^n$, i.e.,

$$\mathbf{h} = \mathbf{A}\mathbf{m}, \quad (5)$$

where \mathbf{m} and \mathbf{h} are understood as column vectors and the transform matrix $\mathbf{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ is, from (4), given by

$$A_{ij} \triangleq \binom{n}{j} \binom{j}{i} (-1)^{j-i} \mathbf{1}(j \geq i), \quad i, j \in [n]_0,$$

where $\mathbf{1}$ is the indicator function. For example, for $n = 4$,

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In addition to being (right) upper triangular, \mathbf{A} is also symmetric with respect to the antidiagonal. Hence only about $n^2/4$ entries need to be calculated¹.

The key benefit of the linear mapping (5) is that the transform matrix needs to be calculated only once for the desired level of accuracy n . This can be done “offline”, which reduces the calculation of the approximate meta distribution to a simple matrix-vector multiplication requiring $(n^2 + 3n)/2$ multiplications (once the moments are known).

¹The exact number is $\lfloor (n+2)^2 - 1 \rfloor / 4$.

D. Choosing the Required Accuracy

The default accuracy of standard mathematical software such as Matlab[®] is insufficient to calculate F_n for n larger than about 30. Here we give an estimate of the number of decimal digits b that enable the calculation F_n with sufficient precision, for arbitrary n .

The maximum of $|\mathbf{A}|$ occurs on the antidiagonal $\mathbf{d} = (A_{0n}, A_{1,n-1}, \dots, A_{n0})$. Indexing \mathbf{d} from 0 to n , $d_{\lceil n/3 \rceil}$ is the largest entry in absolute value, where $\lceil u \rceil$ is rounding u to the nearest integer. Stirling's approximation yields

$$\max |\mathbf{A}| \sim \frac{\sqrt{27} 3^n}{2\pi n}, \quad n \rightarrow \infty.$$

With $\sqrt{27}/(2\pi) < 1$ and $\log_{10}(3) < 1/2$, about $n/2 - \log_{10} n$ decimal digits are needed to calculate \mathbf{A} .

For the moment vector \mathbf{m} , it depends how quickly the moments go to zero. The two extreme cases are the uniform distribution, where $M_n = 1/(n+1)$, and the (degenerate) step function, where $M_n = \nu^n$, where $\nu < 1$ is the constant value of the random variable. This exponential decay of the moments is not of practical interest, as a comparison of M_1^2 and M_2 would immediately reveal that the distribution is degenerate.

This leaves two qualitatively different cases of interest.

1) *Case 1: Superpolynomial decay:* In this case, $-\log M_n / \log n \rightarrow \infty$ but there exists $c > 0$ and $0 < \delta < 1$ such that

$$M_n \leq 10^{-cn^\delta}, \quad \forall n \geq 0.$$

As a result,

$$b = n/2 + cn^\delta - \log_{10} n$$

decimal digits are sufficient.

2) *Case 2: Polynomial decay:* In this case, there exists $s \geq 1$ such that

$$M_n \leq (n+1)^{-s}, \quad \forall n \geq 0,$$

and

$$b = n/2 + (s-1) \log_{10} n$$

decimal digits suffice.

The parameters c , δ , or s can be estimated by inspection of the moment sequence. As a simple rule of thumb, $b = n/2 + 16$ is a sensible choice. Assuming $c = 1$ and $\delta = 1/2$ in the superpolynomial case, this is sufficient for n up to 350. In the polynomial case, $s = 6$ can be accommodated up to $n = 1000$, which may be considered the practical limit of this approach.

IV. ACCURACY OF THE METHOD

A detailed study of the convergence properties of F_n to F is provided in [12, Theorem 2]. Letting $f(x) = F'(x) = dF(x)/dx$ be the exact pdf and $f'(x)$ its derivative, the theorem asserts that as $n \rightarrow \infty$,

$$\|F_n - F\| \leq \frac{1}{n+1} \left(\|f\| + \frac{\|f'\|}{2} \right) + o(n^{-1}), \quad (6)$$

where $\|f\| = \sup_{x \in [0,1]} |f(x)|$. So if f' is bounded, convergence is uniform, and the maximum error decreases with $1/n$.

Fig. 4 shows how the maximum error decreases with n in the case of a beta distribution for $\alpha = 5$, $\beta = 2$, for which

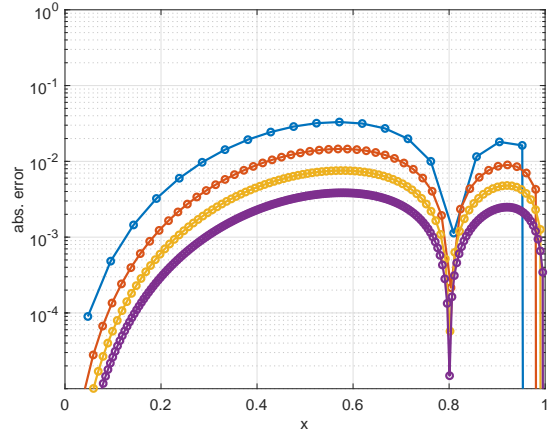


Fig. 3. Absolute error $|F_n(x_k) - F(x_k)|$ for $n = 20, 50, 100, 200$ for a beta distribution with parameters $\alpha = 5$ and $\beta = 2$.

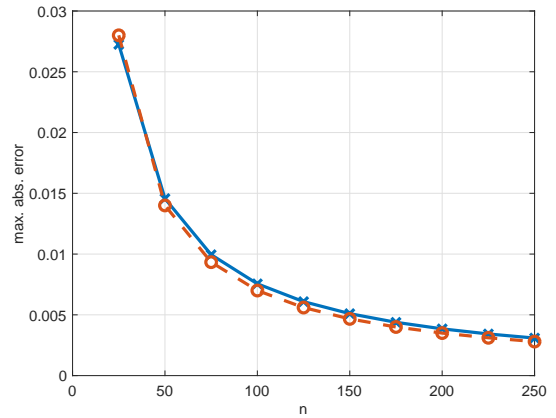


Fig. 4. The solid line shows the maximum error $\max_{k \in [n]} |F_n(x_k) - F(x_k)|$ as a function of n for a beta distribution with parameters $\alpha = 5$ and $\beta = 2$. The dashed line shows $0.7n^{-1}$.

$\|f\| + \|f'\|/2 = 10911/625 \approx 17.5$. The pre-constant 0.7 in the reference curve in the figure is about 25 times smaller than that, indicating that the bound is conservative in some cases. The figure also shows that the $1/n$ scaling of the maximum error starts at modest n already.

For the beta distribution with $\alpha < 1$ or $\beta < 1$, the $1/n$ scaling does not hold since $\|f'\|$ is not bounded.

V. RATE-RELIABILITY TRADE-OFF FOR USER PERCENTILES

As an application, we explore the rate-reliability trade-offs for users that are in a certain percentile of all users. In this case, the underlying random variable X in (1) is the SIR, and t is the SIR threshold required for successful reception, henceforth denoted by θ . We are interested in the pairs (θ, x) for which $F_n(x) = p$, where p is the user percentile. For example, setting $p = 0.05$, choosing $\theta > 0$ and solving for x yields the (θ, x) pairs the 5% user achieves.

We use the standard downlink Poisson cellular model with nearest-base station association, power-law path loss and Rayleigh fading, for which the moments of the conditional success probability are given in [1] as

$$M_n = ({}_2F_1(n, -\delta; 1 - \delta, -\theta)^{-1},$$

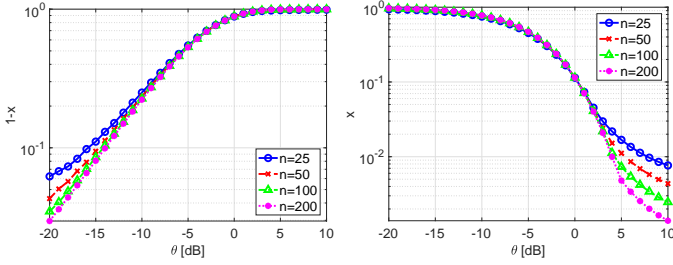


Fig. 5. Pairs $(\theta, 1-x)$ (left plot) and (θ, x) (right plot) for which $F_n(x) = 0.1$ for $n \in \{25, 50, 100, 200\}$.

where ${}_2F_1$ is the Gauss hypergeometric function and $2/\delta$ is the path loss exponent. Fig. 5 explores the effect of n on the trade-off between θ and x for the 10% user. As can be seen, for $\theta \in [-10, 0]$ dB, $n = 25$ is sufficiently accurate. For $\theta < -10$ dB, the reliability x increases with n , indicating that all curves are (increasingly tight) lower bounds, while for $\theta > 10$ dB, the reliability decreases with n , i.e., the curves are upper bounds.

In Fig. 6, the (θ, x) pairs for the 5%, 10%, 20%, and 50% user percentiles are shown, and Fig. 7 presents the spectral efficiency-reliability trade-off ($\log_2(1+\theta), x$) for the same percentiles, both for $n = 400$. These figures involve the evaluation of more than 16,000 values of the MD ($n = 400$ times 41 values of θ), which takes less than 30 seconds with Matlab[®] on a standard desktop computer (not including the time to calculate the transform matrix \mathbf{A} , which is done only once, and the moments for each value of θ).

Interestingly, the curves in Fig. 6 for the different percentiles run parallel asymptotically for both $\theta \rightarrow 0$ and $\theta \rightarrow \infty$. Moreover, the gap is approximately the same on both sides. Between the 5% and the 50% users, it is about 10 dB. This asymptotic gap can be used to quantify the fairness between users.

VI. CONCLUSIONS

The binary mixture method is very efficient for the numerical computation of meta distributions, because it is a simple linear transform of the moment sequence. Only a single parameter needs to be chosen, namely the number of points n to be calculated. The approximated cdf F_n converges uniformly to the exact F at a rate of $1/n$, and the required number of decimal digits is about $n/2$.

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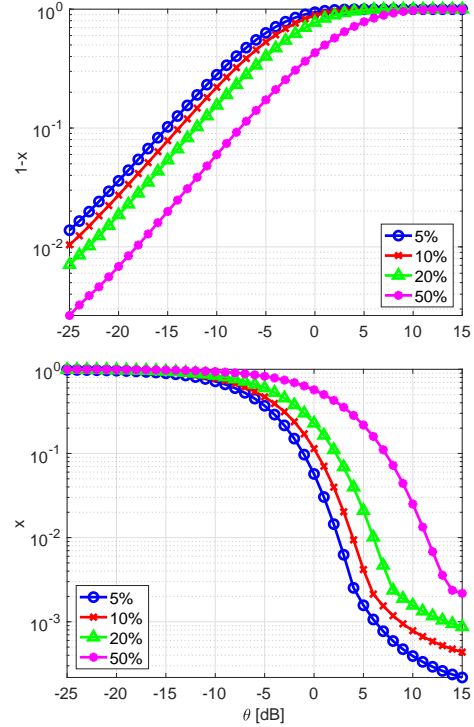


Fig. 6. Pairs $(\theta, 1-x)$ (top plot) and (θ, x) (bottom plot) for which $F_n(x) = 0.05, 0.1, 0.2, 0.5$ for $n = 400$.

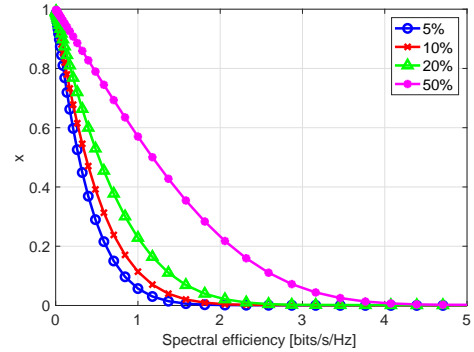


Fig. 7. Rate-reliability trade-off (linear scales) for user percentiles.