On the Location-Dependent SIR Gain in Cellular Networks

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***Abstract*—In wireless networks, the distances from a user to its desired transmitters and undesired interferers play a critical role in its channel quality. In this paper, we study this location-dependence in a cellular network, where a user is always served by its nearest base station. For any stationary and ergodic base station process, we partition its associated Voronoi cells into the cell centers and the cell boundaries. We show that in Poisson networks, the top fraction x of users enjoy a signal-to-interference ratio (SIR) gain of $-5\alpha\log_{10} x$ dB relative to the typical user for Rayleigh fading and the power-law path loss with the exponent α. For the cell boundary users, we give both the exact and asymptotic form of the SIR distribution. As such, this paper permits the grouping of users and the analysis of different groups of users.**

***Index Terms*—Cellular networks, location-dependence, SIR gain, Poisson networks, stochastic geometry.**

I. Introduction

In the downlink orthogonal frequency-division multiplexing (OFDM) systems, each base station (BS) serves users within its cell while causing interference to users in other cells using the same resource block. The distances between a user to its transmitting/interfering BSs shape the link quality in the long term [1]. In the literature, "the cell boundary users" typically refers to users who are almost equally close to the serving and nearest interfering BS and "the cell center users" refers to those who are much closer to the serving BS than to interfering ones. The former type is often the bottleneck of the network while the latter type benefits from good locations. To optimize resource allocation and improve the fairness, it is important to distinguish these two types of users and study their gain/loss relative to the typical user.

We define the region of a location u by how much closer u is to its serving BS than to its nearest interfering BS following [1], [2]. Let $\Phi \subset \mathbb{R}^2$ be an ergodic and stationary BS point process and $x_i(u) \in \Phi$ be the i-th nearest BS to u. For $\gamma \in [0, 1]$ and $\rho \triangleq 1 - \gamma$ we define

$$\begin{aligned} \mathcal{C}_1 &\triangleq \{u \in \mathbb{R}^2\colon\ \|u - x_1(u)\| \leq \rho\, \|u - x_2(u)\|\} \\ \mathcal{C}_2 &\triangleq \{u \in \mathbb{R}^2\colon \rho\, \|u - x_2(u)\| < \|u - x_1(u)\|\}. \end{aligned} \tag{1}$$

γ controls the area fraction of each region.

In the case when Φ is a homogeneous Poisson point process (PPP) with intensity λ, the area fraction of each region equals the probability that the origin falls into each region [2]:

$$\mathbb{P}(o \in \mathcal{C}_1) = \rho^2, \quad \mathbb{P}(o \in \mathcal{C}_2) = 1 - \rho^2. \tag{2}$$

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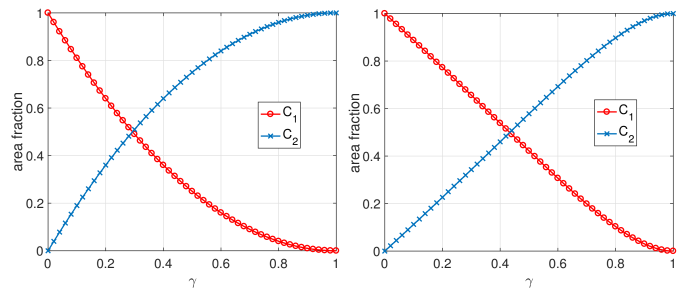


(a) Poisson networks. (b) Triangular lattice networks.

Fig. 1. The area fraction of each region.

In the case when Φ is a lattice network, the calculation of the area fractions is straightforward but the result is unwieldy. Fig. 1 shows the area fraction of each region in Poisson networks and triangular networks as γ increases from 0 to 1.

II. Poisson Networks

The success probability is defined as the probability of the SIR exceeding a threshold θ

$$\bar{F}(\theta) \triangleq \mathbb{P}(\mathrm{SIR} > \theta). \tag{3}$$

When BSs forms a PPP $\Phi \subset \mathbb{R}^2$, the success probability of the typical user is [3]

$$\bar{F}_{\mathrm{PPP}}(\theta) = \frac{1}{{}_2F_1(1, -\delta; 1 - \delta; -\theta)} \tag{4}$$

with $\delta \triangleq 2/\alpha$. ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function.

Using the geometric partition, we can express the success probability as

$$\bar{F}_{\mathrm{PPP}}(\theta) = \sum_{i=1}^{2} \mathbb{P}(\mathrm{SIR} > \theta \mid o \in \mathcal{C}_i)\mathbb{P}(o \in \mathcal{C}_i). \tag{5}$$

In the next two subsections, we will study the success probability conditioned on the typical user being in the two regions. $r_i = \|x_i(o)\|$ denotes the distance of the i-th nearest BS to o.

A. The Cell Center Region

Theorem 1. *The success probability conditioned on the typical user lying in $\mathcal{C}_1$ is*

$$\begin{aligned} \mathbb{P}(\mathrm{SIR} > \theta \mid o \in \mathcal{C}_1) &= \bar{F}_{\mathrm{PPP}}(\theta\rho^\alpha) \\ &= \frac{1}{{}_2F_1(1, -\delta; 1 - \delta; -\rho^\alpha\theta)}. \end{aligned} \tag{6}$$

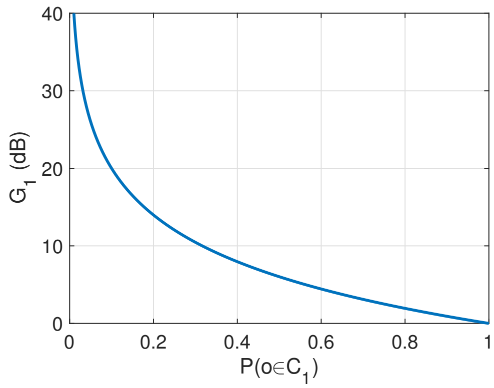


Fig. 2. The SIR gain (in dB) as a function of the area fraction of $\mathcal{C}_1$, $\alpha = 4$.

In particular, for $\alpha = 4$, we have

$$\mathbb{P}(\mathrm{SIR} > \theta \mid o \in \mathcal{C}_1) = \frac{1}{1+\rho^2\sqrt{\theta}\arctan(\rho^2\sqrt{\theta})}. \tag{7}$$

Proof.

$$\begin{aligned}\mathbb{P}(\mathrm{SIR} > \theta \mid o \in \mathcal{C}_1) &= \mathbb{P}(S > \theta I \mid o \in \mathcal{C}_1)\\ &= \mathbb{E}\Big[\prod_{i=2}^{\infty}\frac{1}{1+\theta(\frac{r_1}{r_i})^\alpha} \mid o \in \mathcal{C}_1\Big]\\ &= \mathbb{E}\Big[\prod_{i=2}^{\infty}\frac{1}{1+\theta\rho^\alpha(\frac{r_1/\rho}{r_i})^\alpha} \mid o \in \mathcal{C}_1\Big]\\ &\overset{(a)}{=} \mathbb{E}\Big[\prod_{i=2}^{\infty}\frac{1}{1+\theta\rho^\alpha(\frac{r_1}{r_i})^\alpha}\Big]\\ &= \bar{F}_{\mathrm{PPP}}(\theta\rho^\alpha),\end{aligned}$$

where (a) is due to the fact that the region $\mathcal{C}_1$ is equivalent to $\{r_1/r_2 \le \rho\} = \{r_1/\rho \le r_2\}$. Put differently, the probability law of $r_1/\rho, r_2, \ldots$ conditioned on $r_1/r_2 \le \rho$ is the same as the law of $r_1, r_2, \ldots$ without conditioning. This can be shown by establishing that $f_{\frac{r_1}{\rho}}(x \mid \frac{r_1}{r_2} \le \rho) = f_{r_1}(x)$ in the following derivation and using the independence property of the PPP:

$$\begin{aligned}\mathbb{P}\left(r_1 \le x \mid \frac{r_1}{r_2} \le \rho\right) &= \frac{\mathbb{P}(r_1 \le x, r_1/r_2 \le \rho)}{\mathbb{P}\big(r_1/r_2 \le \rho\big)}\\ &= \frac{\int_0^x\int_{\frac{u}{\rho}}^{\infty}(2\lambda\pi)^2 uv\exp(-\lambda\pi v^2)\mathrm{d}v\mathrm{d}u}{\rho^2}\\ &= 1 - \exp\left(-\lambda\pi\frac{x^2}{\rho^2}\right),\end{aligned} \tag{8}$$

and the pdf

$$f_{r_1}\left(x \mid \frac{r_1}{r_2} \le \rho\right) = \frac{2\lambda\pi x}{\rho^2}\exp\left(-\lambda\pi\frac{x^2}{\rho^2}\right). \tag{9}$$

Now

$$f_{\frac{r_1}{\rho}}(x \mid r_1/r_2 \le \rho) = 2\pi\lambda x\exp(-\lambda\pi x^2) = f_{r_1}(x).$$

□

Remark 1 Theorem 1 shows the SIR gain (in dB) conditioned on the typical user being in $\mathcal{C}_1$ is

$$G_1 = -10\log_{10}\rho^\alpha. \tag{10}$$

(10) is remarkably simple and directly shows that the top fraction $x = \rho^2$ of users enjoy an SIR gain of $-5\alpha\log_{10} x$ dB relative to the typical user. Here, the "top" users are those with the highest distance ratio of the nearest interferer and the serving BS. Fig. 2 shows the SIR gain G_1 as a function of the area fraction of users in $\mathcal{C}_1$. For instance, there are 31.5% of the users that enjoy an average gain of 10 dB over the typical user, and 10% achieve a gain of 20 dB.

Remark 2 It is interesting to compare this result with the success probability of a BS silencing scheme that mutes all the BSs within r_1/ρ for the typical user. We have

$$\begin{aligned}&\mathbb{P}(\mathrm{SIR} > \theta)\\ &= \int_0^\infty 2\pi\lambda x\exp\Big(-\pi\lambda x^2 - \int_{\frac{x}{\rho}}^{\infty}\big(1-\frac{1}{1+\theta(\frac{x}{t})^\alpha}\big)2\pi\lambda t\mathrm{d}t\Big)\mathrm{d}x\\ &= \frac{1}{1-\rho^{-2}+\rho^{-2}\,{}_2F_1(1,-\delta;1-\delta;-\rho^\alpha\theta)}.\end{aligned} \tag{11}$$

It is easy to show that (11) is smaller than (6) for any $\theta > 0$ and $\rho \in [0,1]$. This is expected since muting the interfering BSs in r_1/ρ does not affect the ratio of r_1/r_i for $r_i > r_1/\rho$.

Corollary 1. *The gain of the typical user being in $\mathcal{C}_1$ is the same as the gain when all interferers are $1/\rho$ times more distant*

$$r_i' = r_i/\rho, \quad i > 1, \tag{12}$$

or, equivalently, the interference power is ρ^α times smaller, i.e., $I' = I\rho^\alpha$.

Proof. Trivial. □

Theorem 1 leads to the evaluation of the conditional success probability, denoted by $P_\mathrm{s}(\theta)$, and the SIR meta distribution [4] conditioned on the typical user being in $\mathcal{C}_1$.

Corollary 2. *The b-th moment of the conditional success probability conditioned on the typical user lying in $\mathcal{C}_1$ is*

$$\mathbb{E}[P_\mathrm{s}(\theta)^b \mid o \in \mathcal{C}_1] = \frac{1}{{}_2F_1(b,-\delta;1-\delta;-\rho^\alpha\theta)}, \quad b \in \mathbb{C}. \tag{13}$$

and the SIR meta distribution conditioned on the typical user lying in $\mathcal{C}_1$ satisfies

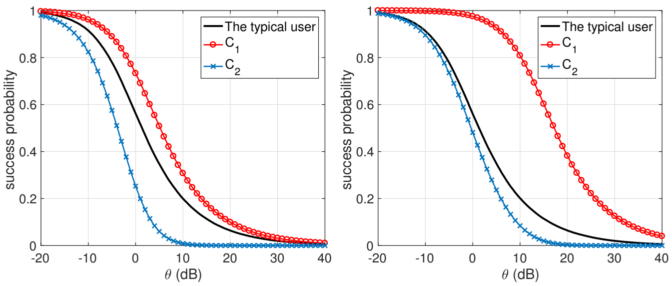
$$\mathbb{P}(P_\mathrm{s}(\theta) > x \mid o \in \mathcal{C}_1) = \mathbb{P}(P_\mathrm{s}(\rho^\alpha\theta) > x), \quad x \in [0,1]. \tag{14}$$

Proof.

$$\begin{aligned}\mathbb{E}[P_\mathrm{s}(\theta)^b \mid o \in \mathcal{C}_1] &= \mathbb{E}\Big[\prod_{i=2}^{\infty}\frac{1}{(1+\theta(r_1/r_i)^\alpha)^b} \mid \Phi, o \in \mathcal{C}_1\Big]\\ &= \mathbb{E}\Big[\prod_{i=2}^{\infty}\frac{1}{(1+\rho^\alpha\theta(r_1/r_i)^\alpha)^b} \mid \Phi\Big]\\ &= \frac{1}{{}_2F_1(b,-\delta;1-\delta;-\rho^\alpha\theta)}.\end{aligned}$$

Since this holds for any $b \in \mathbb{C}$, it holds for the SIR meta distribution [5]. □

Remark 3 (14) shows that for the same target reliability and percentile, the typical user in $\mathcal{C}_1$ achieves an SIR that is $\rho^{-\alpha}$ times higher than that of the typical user.



(a) $\gamma = 0.2$.

(b) $\gamma = 0.6$.

Fig. 3. The conditional success probability for the two regions, $\alpha = 4$. The curves for $\mathcal{C}_1$ (red circle) correspond to the curve for the typical user (black) shifted by 3.876 dB and 15.92 dB, respectively.

B. The Cell Boundary Region

Corollary 3. *The success probability conditioned on the typical user being in $\mathcal{C}_2$ is*

$$\mathbb{P}(\mathrm{SIR} > \theta \mid o \in \mathcal{C}_2) = \frac{\bar{F}_{\mathrm{PPP}}(\theta) - \rho^2 \bar{F}_{\mathrm{PPP}}(\theta\rho^\alpha)}{1-\rho^2} \tag{15}$$

Proof. Combining (5), $\mathbb{P}(o \in \mathcal{C}_2) = 1 - \rho^2$ and the result in Theorem 1, we obtain Corollary 3. □

From (15) we notice that the horizontal gain within $\mathcal{C}_2$ is not constant but depends on θ. Fig. 3 shows the success probability in the two regions plotted using (6) and (15). The success probability of the typical user is the weighted average of them.

Taking the limit $\rho \to 1$ of (15) we obtain the success probability for the typical edge user, *i.e.*, the typical user that lies on the edges of the Voronoi cells,

$$\begin{aligned}
&\mathbb{P}(\mathrm{SIR} > \theta \mid x_1(o) = x_2(o)) \\
&= \frac{1}{{}_2F_1(1,-\delta;1-\delta;-\theta)} - \frac{\theta}{1-\delta}\frac{{}_2F_1(2,1-\delta;2-\delta;-\theta)}{{}_2F_1(1,-\delta;1-\delta;-\theta)^2} \\
&= \frac{1}{(1+\theta)\ {}_2F_1(1,-\delta;1-\delta;-\theta)^2} \\
&= \frac{\bar{F}_{\mathrm{PPP}}(\theta)^2}{1+\theta}.
\end{aligned}$$

In contrast, for the typical vertex user [6], *i.e.*, the user lying on the vertex of the Voronoi cells,

$$\mathbb{P}(\mathrm{SIR} > \theta \mid x_1(o) = x_2(o) = x_3(o)) = \frac{\bar{F}_{\mathrm{PPP}}(\theta)^2}{(1+\theta)^2}.$$

There is an extra factor $1 + \theta$ in the denominator due to the third equidistant BS.

We now calculate the asymptotic SIR gain (the SIR gain as $\theta \to 0$) of the users in $\mathcal{C}_2$. Denote by G_2 the asymptotic SIR gain of the users in $\mathcal{C}_2$ relative to the typical user. Note that $G_2 \leq 1$. We can write the asymptotic form of the success probability of the users in $\mathcal{C}_2$ as [7]

$$\mathbb{P}(\mathrm{SIR} > \theta \mid o \in \mathcal{C}_2) \sim \bar{F}_{\mathrm{PPP}}(\theta/G_2), \quad \theta \to 0, \tag{16}$$

where

$$G_2 = \frac{\mathrm{MISR}_{\mathrm{PPP}}}{\mathrm{MISR}_{\mathcal{C}_2}} = \frac{1-\rho^2}{1-\rho^{\alpha+2}}. \tag{17}$$

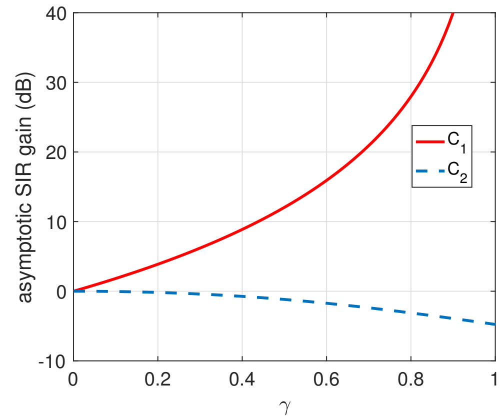


Fig. 4. The asymptotic SIR gain conditioned on the typical user being in the two regions, $\alpha = 4$.

(17) follows from the the mean interference-to-signal ratio (MISR) of the typical user $\mathrm{MISR}_{\mathrm{PPP}} = 2/(\alpha - 2)$, and the MISR conditioned on the typical user being in $\mathcal{C}_2$ as

$$\begin{aligned}
\mathrm{MISR}_{\mathcal{C}_2} &= \mathbb{E}\Big[\sum_{i=2}^{\infty}\Big(\frac{r_1}{r_i}\Big)^\alpha \mid o \in \mathcal{C}_2\Big] \\
&= \mathbb{E}\Big[\Big(\frac{r_1}{r_2}\Big)^\alpha \mid o \in \mathcal{C}_2\Big]\Big(1 + \mathbb{E}\Big[\sum_{i=3}^{\infty}\Big(\frac{r_2}{r_i}\Big)^\alpha\Big]\Big) \\
&= \frac{2(1-\rho^{\alpha+2})}{(\alpha+2)(1-\rho^2)}\Big(1 + \frac{4}{\alpha-2}\Big) \\
&= \frac{2(1-\rho^{\alpha+2})}{(\alpha-2)(1-\rho^2)}.
\end{aligned}$$

Fig. 4 shows the asymptotic SIR gain of the two types of users as a function of γ.

Alternatively, we can express the asymptotic success probability using the success probability of the typical edge user and the typical vertex user as baselines. Denote by $G_{2,\mathrm{e}}$, $G_{2,\mathrm{v}}$ the asymptotic gains compared to the typical edge user and the typical vertex user. By combining the typical edge user $\mathrm{MISR}_{\mathrm{e}} = (\alpha+2)/(\alpha-2)$, and the typical vertex user $\mathrm{MISR}_{\mathrm{v}} = 2\alpha/(\alpha-2)$ [7], we have $G_{2,\mathrm{e}} = (\alpha+2)(1-\rho^2)/(2(1-\rho^{\alpha+2}))$, $G_{2,\mathrm{v}} = \alpha(1-\rho^2)/(1-\rho^{\alpha+2})$ and

$$\begin{aligned}
\mathbb{P}(\mathrm{SIR} > \theta \mid o \in \mathcal{C}_2) &\sim \bar{F}_{\mathrm{PPP}}(\theta/G_2) \\
&\sim \frac{\bar{F}_{\mathrm{PPP}}(\theta/G_{2,\mathrm{e}})^2}{1+\theta/G_{2,\mathrm{e}}} \\
&\sim \frac{\bar{F}_{\mathrm{PPP}}(\theta/G_{2,\mathrm{v}})^2}{(1+\theta/G_{2,\mathrm{v}})^2}, \quad \theta \to 0.
\end{aligned} \tag{18}$$

C. Spectral Efficiency

We determine the spectral efficiency in units of nats/s/Hz in an interference-limited scenario assuming rate adaption. Letting $R_i = \mathbb{E}\big[\ln(1+\mathrm{SIR}) \mid o \in \mathcal{C}_i\big]$, $i = 1, 2$, we have

$$R = \mathbb{E}\ln(1+\mathrm{SIR}) = R_1\mathbb{P}(o \in \mathcal{C}_1) + R_2\mathbb{P}(o \in \mathcal{C}_2). \tag{19}$$

Corollary 4. *The spectral efficiencies conditioned on the typical user being in $\mathcal{C}_1$ and $\mathcal{C}_2$ are*

$$R_1 = \int_0^\infty \frac{1}{{}_2F_1(1,-\delta;1-\delta;-\rho^\alpha(e^t-1))}\mathrm{d}t, \tag{20}$$

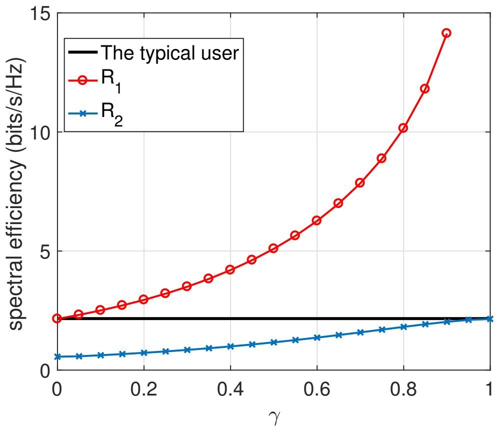


Fig. 5. The spectral efficiency conditioned on the typical user being in the two regions in comparison with that of the typical user $R = 2.163$ bits/s/Hz (the black line), $\alpha = 4$.

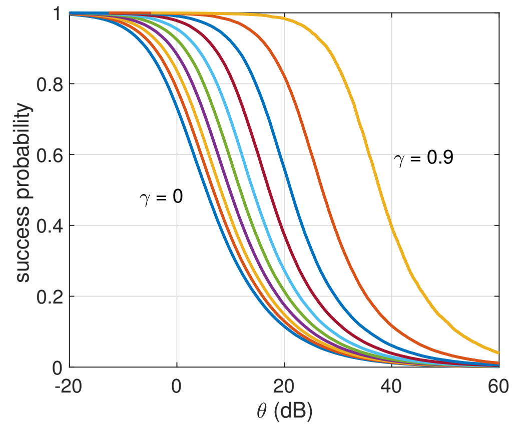


Fig. 6. The success probability in $\mathcal{C}_1$ in triangular lattice networks for $\gamma = 0,\ 0.1,\ \ldots,\ 0.9$.

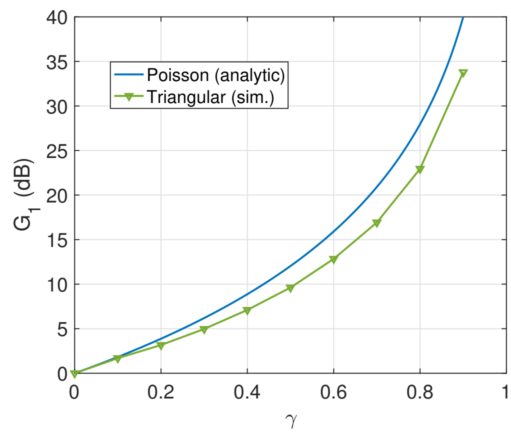


Fig. 7. The asymptotic SIR gain in Poisson networks and lattice networks.

and

$$R_2 = \frac{1}{1-\rho^2}\Big(R - \rho^2 R_1\Big). \tag{21}$$

Proof.

$$\begin{aligned} R_1 &= \mathbb{E}\big[\log(1+\mathrm{SIR}) \mid o \in \mathcal{C}_1\big] \\ &= \int_0^\infty \mathbb{P}(\log(1+\mathrm{SIR}) > t \mid o \in \mathcal{C}_1)\mathrm{d}t \\ &= \int_0^\infty \mathbb{P}(\mathrm{SIR} > e^t - 1 \mid o \in \mathcal{C}_1)\mathrm{d}t \\ &= \int_0^\infty \frac{1}{{}_2F_1(1, -\delta; 1-\delta; -\rho^\alpha(e^t-1))}\mathrm{d}t. \end{aligned} \tag{22}$$

The second part is trivial. □

Fig. 5 shows the spectral efficiency conditioned on the typical user being in each region in bits/s/Hz. For instance, when $\alpha = 4$ and $\gamma = 0.4$, $R_1 \approx 4.2$ bits/s/Hz, so 36% of the users achieve almost double of R as a result of good locations. In contrast, the other 64% achieve only $R_2 \approx 0.995$ bits/s/Hz, which is less than half of R as a result of bad locations. R_1 approaches infinity as $\gamma \to 1$ due to the singularity of the power-law path loss function.

III. Lattice Networks

In this section, we study the success probability and the asymptotic SIR gain of the cell center region defined in (1) in triangular lattice networks. Fig. 6 shows the simulated result of the success probability in $\mathcal{C}_1$ with different γ. Fig. 7 shows the asymptotic SIR gain G_1 in Poisson networks and triangular lattice networks. The former is plotted using (10), and the latter using simulation results evaluated at $p_{\mathrm{s}}(\theta) = 0.95$ with the success probability of the typical user in triangular lattices as the baseline. The latter is smaller since a user is more likely to have other nearby interfering BSs when fixing the distance ratio between the nearest two BSs.

IV. Conclusions

This paper compares the SIR distribution and the related performance metrics for the cell center users and the cell boundary users in cellular networks. We show a surprisingly simple relationship between the SIR performance of the cell center users and the typical user in Poisson networks. The idea of grouping users and analyzing the corresponding performance applies to general networks.

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