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Dynamic Connectivity and Path Formation Time in Poisson Networks

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Abstract The connectivity of wireless networks is commonly analyzed using static geometric graphs. However, with half-duplex radios and due to interference, static or instantaneous connectivity cannot be achieved. It is not necessary, either, since packets take multiple time slots to propagate through the network. For example, if a packet traverses a link in one time slot, it is irrelevant if the next link is available in that time slot also, but it is relevant if the next hope exists in the next time slot.

To account for half-duplex constraints and the dynamic changes in the transmitting set of nodes due to MAC scheduling and traffic loads, we introduce a random multi-digraph that captures the evolution of the network connectivity in a dynamic fashion. To obtain concrete results, we focus on Poisson networks, where transmitters form a Poisson point process on the plane at all time instants. We first provide analytical results for the degree distribution of the graph and derive the distributional properties of the end-to-end connection delay using techniques from first-passage percolation and epidemic processes. Next, we prove that under some assumptions, the delay scales linearly with the source-destination distance even in the presence of interference. We also provide simulation results in support of the theoretical results.

Keywords Ad hoc networks · Connectivity · Interference · Delay · Percolation

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1 Introduction

In a multihop ad hoc network, bits, frames or packets are transferred from a source to a destination in a multihop fashion with the help of intermediate nodes. Decoding, storing, and relaying introduces a delay that, measured in time slots, generally exceeds the number of hops. For example, a five-hop route does not guarantee a delay of only five time slots. Due to the broadcast nature of the wireless medium, a large number of paths may form between the source and the destination, and each path may have taken a different time to form with the help of different relay nodes. In general, a relay node queues the packets from other nodes and its own packets and transmits them according to some scheduling algorithm. If one introduces the concept of queues, the analysis of the system becomes extremely complicated because of the intricate spatial and temporal dependencies between the queues. In this paper we take a different approach. We are concerned only with the physical connections between nodes, i.e., we do not care when a node i transmits a particular packet to a node j (which depends on the scheduler), but we analyze how long it takes until a (physical) connection (maybe over multiple hops) is formed between the nodes i and j, assuming all nodes have packets buffered. This delay is a lower bound on the delay with any traffic model and scheduler in place.

We assume that the transmitting nodes are distributed as a Poisson point process (PPP) on the plane in each time slot, which implies that the total set of nodes forms a PPP and slotted ALOHA is used as the MAC scheme. Any transmitting node can connect to a receiving node when a modified version of the protocol model criterion introduced in [11] is met. Since at each time instant, the transmitting and receiving nodes change, the connectivity graph changes dynamically. We analyze the time required for a causal path to form between a source and a destination node. The system model is made precise in Section 2.

This problem is similar to the problem of First-Passage Percolation (FPP) [15,13,2], and the process of dynamic connectivity also resembles an epidemic process [7,20,21] on a Euclidean domain. In a spatial epidemic process, an infected individual infects a certain (maybe random) neighboring population, and this process continues until the complete population is infected or the spreading of the disease stops. In the literature cited above, the spreading time of the epidemic is analyzed for different models of disease spread. We draw many ideas from this theory of epidemic process and FPP. The main difference between an epidemic process and the process we consider is that the spreading (of packets) depends on a subset of the population (due to interference) and is not independent from node to node. In [6], the latency for a message to propagate in a sensor network is analyzed using similar tools. They consider a Boolean connectivity model with randomly weighted edges and derive the properties of first-passage paths on the weighted graph. Their model does not consider interference and thus allows the use of Kingman's subadditive ergodic theorem [16] while ours does not. Percolation in signal-to-interference ratio graphs was analyzed in [5] where the nodes are assumed to be full-duplex. In practice, radios do not transmit and receive at the same time (at the same frequency), and hence the instantaneous network graph is always disconnected. In [14, 1, 17, 23] look at sparse disconnected networks and provide bounds on the speed of information propagation in the network. In [9][8], we have introduced the concept of dynamic connectivity graphs, and we proved that the average delay scales linearly with source-destination distance but the temporal correlation between interference was neglected. A similar type of dynamic graph was introduced in [3] based on a SINR-based (or physical) model of connectivity. Without noise, they proved that below a certain ALOHA parameter p^*, the average delay of connectivity between nodes scales linearly with the distance. Above this threshold, the average path formation time becomes infinity. When noise is considered, the average path formation time is always infinity for finite average transmit power. In this paper we prove similar results for the protocol model of communication. In contrast to [3], we show that the time of connectivity scales linearly with the source-destination distance irrespective of the ALOHA parameter on the giant percolative component of the geometric disc graph, where the radius of the disc is proportional to the thermal noise.

In Section 2, we introduce the system model. In Section 3, we study the connectivity properties of the random geometric graph formed at any time instant, the so-called snapshot graph. In Section 4, we derive the properties of the delay and the average number of paths formed between a source and destination, and we show that the delay increases linearly with increasing source-destination distance or, equivalently, that the propagation speed is constant, i.e., the distance of the farthest nodes to which the origin can connect increases linearly with time. Section 5 presents simulation results, and Section 6 concludes the paper.

2 System Model

The locations of the wireless nodes (transceivers) are assumed to form a Poisson point process (PPP) ϕ of intensity λ on the plane. We assume that time is slotted and the MAC protocol used is slotted ALOHA, so that in every time slot each node transmits with probability p or remains silent with probability $1-p$. Nodes are half-duplex, i.e., they act as receivers only if they are not transmitting. We use the protocol model [11] to decide if the communication between a transmitter and a receiver is successful in a given time slot: A transmitting node located at x can connect to a receiver located at y if and interference and a noise condition are both met:

1. *Interference:* The disk $B(y, \beta\|x-y\|), \beta > 0$, does not contain any other transmitting nodes.
2. *Noise:* $\|x-y\| < \eta$.

$B(x, r)$ denotes a disk of radius r centered around x and $B^c(x, r) = \mathbb{R}^2 \setminus B(x, r)$. β is a system parameter and captures the resilience of the receiver

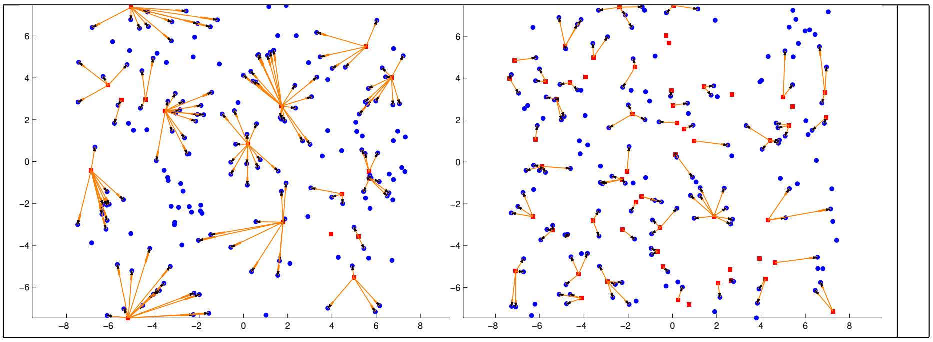

Fig. 1 Illustration of a snapshot graph g for $p = 0.2$ (left) and $p = 0.3$ (right) for different realizations of ϕ. The squares represent the transmitters and the circles the receivers.

against interference. The standard physical SINR model of communication can be related to the protocol model easily when there is no fading. A detailed discussion of the protocol model can be found in [19]. The interference-limited regime can be modeled by ignoring Condition 2, and the noise-limited scenario can be modeled by ignoring Condition 1.

We shall use $\mathbf{1}(x \to y, \Delta, \eta)$ to represent a random variable that is equal to one if a transmitter at x is able to connect to a receiver y when the transmitting set is Δ, i.e., the interfering set is $\Delta \setminus \{x\}$. We will drop Δ if there is no ambiguity. At any time instant k, we denote the set of transmitters (decided by ALOHA) by $\phi_t(k)$ and the set of receivers by $\phi_r(k)$. So we have $\phi_t(k) \cup \phi_r(k) = \phi$ and $\phi_t(k) \cap \phi_r(k) = \emptyset$, where $\emptyset$ denotes the empty set.

The connectivity at time k is captured by a directed and weighted random geometric graph $g(k) = (\phi, E_k)$ with vertex set ϕ and edge set

$$E_k = \{(x, y) \colon \ \mathbf{1}\left(x \to y, \phi_t(k), \eta\right) = 1, x \in \phi_t(k), y \in \phi_r(k)\}\,. \tag{1}$$

See Figure 1 for an illustration of $g(0)$ and $g(1)$. Each edge in this graph $g(k)$ has a weight k that represents the time slot in which the edge was formed. Let $G(m, n)$ denote the weighted directed multigraph (multiple edges with different time stamps are allowed between two vertices) formed between times m and $n > m$, i.e.,

$$G(m, n) = \left(\phi, \bigcup_{k=n}^{m} E_k\right). \tag{2}$$

So $G(m, n)$ is the *edge-union* of the graphs $g(k)$, $m \le k \le n$. See Figure 2.

Definition 1 A directed path $x_0, e_0, x_1,\ e_1, \ldots, e_{q-1}, x_q$ between the nodes $x_0, x_q \in \phi$ where $e_i = (x_i, x_{i+1})$ denotes an edge in the multigraph is said to be a *causal path* if the weights of the edges e_i are *strictly increasing* with i.

This means that the edge e_{i-1} was formed before e_i for $0 < i < q$. For the rest of the paper, we always mean causal path when speaking about a path. We observe that the random graph $g(k)$ is a snapshot of the Poisson network

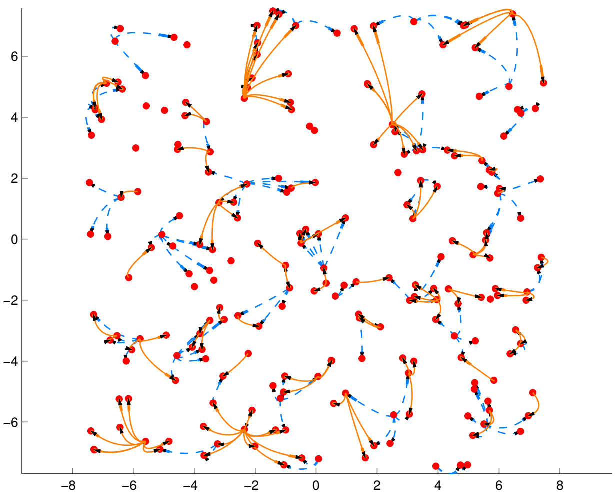

Fig. 2 Illustration of $G(0,1)$, $p = 0.2$, $\beta = 1.2$. Dashed line represent edges in $g(0)$ (edges with weight 0) and solid lines represent edges in $g(1)$ (edges with weight 1).

at time instant k. The random graph process $G(0, m)$ captures the entire connectivity history up to time m. In the graph $G(0, m)$ there is a notion of time and causality, i.e., packets can propagate only along a causal path.

3 Properties of the Snapshot Graph $g(k)$

In this section, we will analyze the properties of the random graph $g(k)$. We first observe that the graphs $g(k)$ are identically distributed for all k. So for this section we will drop the time index unless otherwise indicated. It can be shown that g is a planar Euclidean graph *even with straight lines* as edges [10, Lemma 2]. In Figure 1, realizations of g are shown for $p = 0.2$ and $p = 0.3$. We first characterize the distribution of the in-degree of a receiver node and the out-degree of a transmit node.

3.1 Node degree distributions

Let $N_t(x)$ denote the number of receivers a transmitter located at x can connect to, i.e., the out-degree of a transmitting node. Similarly, let $N_r(x)$ denote the number of transmitters that can connect to a receiver at x, i.e., the in-degree of a receiving node. We first calculate the average out-degree of a transmitting node.

Proposition 1 $\mathbb{E}\left[N_t(x)\right] = \frac{1-p}{p}\beta^{-2}\left(1 - \exp\left(-\lambda p \pi \beta^2 \eta^2\right)\right)$.

Proof By stationarity of ϕ, we have $N_t(x) \stackrel{d}{=} N_t(o)$ where $\stackrel{d}{=}$ stands for equality in distribution. So it is sufficient to consider the out-degree of a transmitter placed at the origin, which is given by $\sum_{x\in\phi_r}\mathbf{1}(o\to x,\phi_t,\eta)$. Note that this process is not an independent thinning of ϕ_r, hence the out-degree is not Poisson. So the average degree is

$$\begin{aligned}
\mathbb{E}[N_t(o)] &= \mathbb{E}\left[\sum_{x\in\phi_r}\mathbf{1}(o\to x,\phi_t,\eta)\right] \\
&\overset{(a)}{=} \lambda(1-p)\int_{\mathbb{R}^2}\mathbb{E}_{\phi_t}\left[\mathbf{1}(o\to x,\phi_t,\eta)\right]\mathrm{d}x \\
&\overset{(b)}{=} \lambda(1-p)\int_{B(o,\eta)}\exp\left(-\lambda p\pi\beta^2\|x\|^2\right)\mathrm{d}x \\
&= \frac{1-p}{p}\beta^{-2}\left(1-\exp(-\lambda p\pi\beta^2\eta^2)\right),
\end{aligned}$$

where (a) follows from Campbell's theorem [22] and the independence of ϕ_r and ϕ_t. Equality (b) follows from the fact that $\mathbf{1}(o\to x,\phi_t)$ is equal to one if and only if the ball $B(x,\beta\|x\|)$ does not contain any interferers.

The average out-degree in the interference-limited case is obtained by the limit $\lim_{\eta\to\infty}\mathbb{E}\left[N_t(x)\right]$ and is $\frac{1-p}{p}\beta^{-2}$. Similarly the average out-degree in the noise-limited case is $\lim_{\beta\to 0}\mathbb{E}\left[N_t(x)\right]$ and is equal to $\lambda(1-p)\pi\eta^2$.

Proposition 2 *The probability distribution of N_t is given by*

$$\mathbb{P}\left(N_t=m\right)=\sum_{k=m}^{\infty}\binom{k}{m}(-1)^{k+m}\left(\frac{1-p}{p}\right)^k\frac{V_k}{k!}, \tag{3}$$

where

$$V_k=\int_{B(o,\sqrt{\lambda p}\eta)}\cdots\int_{B(o,\sqrt{\lambda p}\eta)}\exp\left(-\mathrm{area}\left(\bigcup_{i=1}^{k}B(x_i,\beta\|x_i\|)\right)\right)\mathrm{d}x_1\cdots\mathrm{d}x_k$$

and $V_0=1$. $\mathrm{area}(A)$ *is the area of* $A\subset\mathbb{R}^2$.

Proof We provide the complete characterization of N_t using the Laplace transform, given by

$$\begin{aligned}
\mathcal{L}_{N_t}(s) &= \mathbb{E}\left[\exp\left(-sN_t\right)\right] \\
&= \mathbb{E}\left[\exp\left(-s\sum_{x\in\phi_r}\mathbf{1}(o\to x,\phi_t,\eta)\right)\right] \\
&\overset{(a)}{=} \mathbb{E}_{\phi_t}\exp\left[-\lambda(1-p)\int_{\mathbb{R}^2}1-\exp(-s\mathbf{1}(o\to x,\phi_t,\eta))\mathrm{d}x\right] \\
&= \mathbb{E}_{\phi_t}\exp\left[-\lambda(1-p)(1-e^{-s})\int_{\mathbb{R}^2}\mathbf{1}(o\to x,\phi_t,\eta)\mathrm{d}x\right] \qquad (4)\\
&= \mathbb{E}_{\phi_t}\exp\left[-\lambda(1-p)(1-e^{-s})\int_{B(o,\eta)}\mathbf{1}(o\to x,\phi_t,\infty)\mathrm{d}x\right], \qquad (5)
\end{aligned}$$

where (a) follows from the probability generating functional of a PPP. Let ν denote a two dimensional PPP of density 1. Using the scale-invariance property

$$\mathbf{1}(o \to x, \phi_t, \infty) \stackrel{d}{=} \mathbf{1}(o \to x\sqrt{\lambda p}, \nu, \infty).$$

we can write

$$\mathcal{L}_{N_t}(s) = \mathbb{E}_\nu \exp\left[-\frac{1-p}{p}(1-e^{-s})\int_{B(o,\sqrt{\lambda p}\eta)} \mathbf{1}(o \to x, \nu, \infty)\mathrm{d}x\right]. \quad (6)$$

Letting $a = \frac{1-p}{p}(1-e^{-s})$ and expanding the outer exponential we obtain

$$\mathcal{L}_{N_t}(s) = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \underbrace{\mathbb{E}_\nu\left(\int_{B(o,\sqrt{\lambda p}\eta)} \mathbf{1}(o \to x, \nu, \infty)\mathrm{d}x\right)^k}_{L_k}.$$

The expected value L_k can be written as

$$\begin{aligned} L_k &= \int\limits_{B(o,\sqrt{\lambda p}\eta)} \cdots \int\limits_{B(o,\sqrt{\lambda p}\eta)} \mathbb{E}_\nu\left(\mathbf{1}(o \to x_1, \nu, \infty) \cdots \mathbf{1}(o \to x_k, \nu, \infty)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k \\ &= V_k\,. \end{aligned}$$

The result follows from a comparison of coefficients after replacing e^{-s} with z.

A lower bound on $\mathcal{L}_{N_t}(s)$ from (6) is

$$\begin{aligned} \mathcal{L}_{N_t}(s) &\stackrel{(a)}{\geq} \exp\left[-\frac{1-p}{p}(1-e^{-s})\int_{B(o,\sqrt{\lambda p}\eta)} \mathbb{E}_\nu \mathbf{1}(o \to x, \nu, \infty)\mathrm{d}x\right] \\ &\stackrel{(b)}{=} \exp\left[-\frac{1-p}{p\beta^2}(1-e^{-s})(1-e^{-\pi\beta^2\lambda p\eta^2})\right] \end{aligned}$$

where (a) follows from Jensen's inequality and (b) follows since $\mathbb{E}_\nu \mathbf{1}(o \to x, \nu, \infty) = \exp(-\beta^2\pi\|x\|^2)$. This is the Laplace transform of a Poisson random variable with mean $\frac{1-p}{p\beta^2}(1-e^{-\pi\beta^2\lambda p\eta^2})$, which implies the following lower bound on the probability of a transmit node being isolated:

$$\mathbb{P}(N_t = 0) \geq \exp\left(-\frac{1-p}{p\beta^2}(1-e^{-\pi\beta^2\lambda p\eta^2})\right) = \exp(-\mathbb{E}[N_t(o)]).$$

We next evaluate the in-degree distribution of a receiving node. Since the point process is stationary, the distribution of $N_r(x)$ is the same for all receivers x.

Proposition 3 *The average in-degree $\mathbb{E}[N_r(x)]$ of a node in g is $\beta^{-2}(1-e^{-\pi\beta^2\lambda p\eta^2})$. For $\beta > 1$, N_r is a Bernoulli random variable.*

Proof We have $N_r(x) \stackrel{d}{=} N_r(o)$ and thus

$$\begin{aligned}\mathbb{E}[N_r(o)] &= \mathbb{E}\left[\sum_{y\in\phi} \mathbf{1}_{\phi_t}(y)\mathbf{1}(y \to o, \phi_t, \eta)\right] \\ &= \lambda p \int_{\mathbb{R}^2} \mathbb{E}_{\phi_t}\left[\mathbf{1}(y \to o, \phi_t, \eta)\right] \mathrm{d}y \\ &= \lambda p \int_{B(o,\eta)} \exp\left(-\lambda p \pi \beta^2 \|y\|\right) \mathrm{d}y \\ &= \beta^{-2}(1 - e^{-\pi\beta^2\lambda p \eta^2}). \qquad (7)\end{aligned}$$

If $\beta > 1$, at most one transmitter can connect to any receiver, so N_r is Bernoulli, with the mean given in (7).

As a sanity check, we can confirm that the average in- and out-degrees are equal, i.e., $p\mathbb{E}[N_t(o)] = (1-p)\mathbb{E}[N_r(o)]$. Observe that these are spatio-temporal averages, not time averages.

3.2 Average time for single-hop connectivity

A node may require multiple attempts (time slots) before it is able to connect to any other node. In this subsection we will consider the time it takes for a node to (opportunistically) connect to some other node. We add a virtual node at the origin and define the number of time slots required to connect to any node,

$$T_O = \underset{k}{\operatorname{argmin}} \left[\mathbf{1}(o \in \phi_t(k)) \prod_{x\in\phi_r(k)} (1 - \mathbf{1}(o \to x, \phi_t(k), \eta))\right].$$

Lemma 1 *The average single-hop connection time in a Poisson network is infinite:*

$$\mathbb{E}T_O = \infty.$$

Proof In the point process ϕ the probability that the ball $B(o,\eta)$ is empty is equal to $\exp(-\lambda\pi\eta^2)$. Hence a typical transmitter at the origin cannot connect to any node with probability $\exp(-\lambda\pi\eta^2)$ regardless of the number of attempts. Hence $\mathbb{E}T_O = \infty$.

From the above lemma we observe that the presence of noise which implies a finite connectivity radius makes the average single-hop connectivity time infinite. In a Poisson network this happens because the nearest-neighbor distance is Rayleigh [12] and there exists a positive fraction of nodes with large nearest-neighbor distance. We now consider an interference-limited network, i.e., neglect the finite connectivity radius assumption. Let $\tilde{T}_O$ denote the opportunistic connectivity time with the interference limited assumption. Let $\tilde{T}_N$

denote the time required for a connection to form between the origin and its nearest neighbor. We then have

$$\tilde{T}_O \leq \tilde{T}_N.$$

Lemma 2 *The average time for nearest-neighbor connectivity is*

$$\mathbb{E}\tilde{T}_N = \begin{cases} (p(1-p) - p^2\nu(\beta))^{-1}, & p < \frac{1}{1+\nu(\beta)} \\ \infty, & \textit{otherwise.} \end{cases}$$

where

$$\nu(\beta) = \begin{cases} \beta^2 - \pi^{-1}\left\{\beta^2\cos^{-1}\frac{\beta}{2} + \cos^{-1}\left(1 - \frac{\beta^2}{2}\right) - \frac{\beta}{2}\sqrt{4-\beta^2}\right\}, & \beta < 2 \\ \beta^2 - 1 & , \ \beta > 2. \end{cases}$$

Proof Let z denote the nearest neighbor of the origin o. We first condition on the fact that the node at the origin transmits and the node at z listens. We have

$$\mathbf{1}(o \to z, \phi_t(k)) = \left[\prod_{x \in \phi \cap B(o,\|z\|)^c} (1 - \mathbf{1}(x \in B(z, \beta\|z\|))\mathbf{1}(x \in \phi_t(k)))\right]$$

The probability that $\tilde{T}_N > k$ is

$$\mathbb{P}(\tilde{T}_N > k) = \mathbb{E}\prod_{m=1}^{k}(1 - \mathbf{1}(o \to z, \phi_t(m))). \tag{8}$$

Let $N(o)$ denote the nearest neighbor of the origin o. Conditioning on the point process we have

$$\mathbb{P}(\tilde{T}_N > k \mid \phi, N(o) = z) = \left[1 - \prod_{x \in \phi \cap B(o,\|z\|)^c} (1 - \mathbf{1}(x \in B(z, \beta\|z\|))p)\right]^k \tag{9}$$

and thus

$$\begin{aligned} \mathbb{E}[\tilde{T}_N \mid N(o) = z] &= \mathbb{E}\sum_{k=0}^{\infty}\mathbb{P}(\tilde{T}_N > k \mid \phi) \\ &= \mathbb{E}\left[\prod_{x \in \phi \cap B(o,\|z\|)^c}(1 - \mathbf{1}(x \in B(z, \beta\|z\|))p)\right]^{-1} \\ &= \exp\left(-\lambda\int_{B(o,\|z\|)^c} 1 - \frac{1}{1 - \mathbf{1}(x \in B(z, \beta\|z\|))p}\mathrm{d}x\right) \\ &= \exp\left(\frac{p}{1-p}\lambda\pi\|z\|^2\nu(\beta)\right). \end{aligned} \tag{10}$$

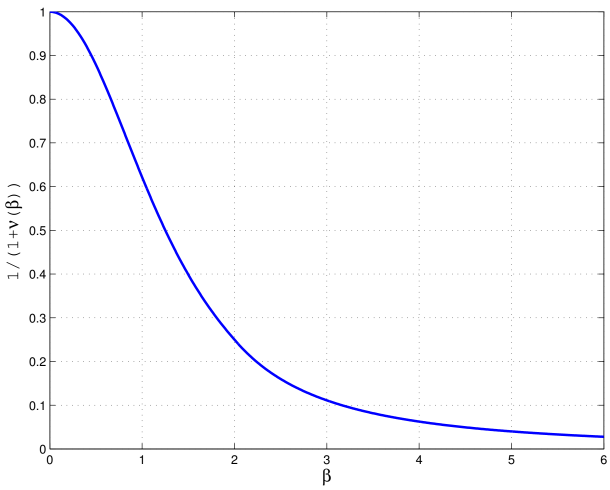

Fig. 3 The ALOHA parameter p above which the average time for nearest-neighbor connectivity $\mathbb{E}\tilde{T}_N$ is infinite as a function of β.

Averaging with respect to the nearest-neighbor distribution yields

$$\mathbb{E}\tilde{T}_N = 2\pi\lambda \int_0^\infty z \exp(-\lambda\pi z^2) \exp\left(\frac{p}{1-p}\lambda\pi z^2 \nu(\beta)\right) \mathrm{d}z \tag{11}$$

$$= \frac{1}{1 - p(1-p)^{-1}\nu(\beta)}, \quad p < \frac{1}{1+\nu(\beta)}. \tag{12}$$

Removing the conditioning on the node at o transmitting and the nearest neighbor listening, the result follows.

From the above lemma we observe that there exists a cutoff value for the ALOHA contention parameter above which $\mathbb{E}\tilde{T}_O = \infty$. See Figure 4. We also observe that the minimum value of $\mathbb{E}\tilde{T}_N$ occurs at $p = 0.5/(1 + \nu(\beta))$ and is equal to $4(1 + \nu(\beta))$. A similar cut-off phenomena was also observed in the physical model [3].

We now provide a lower bound to the average time required for opportunistic communication for $\beta > 1$.

Lemma 3 *The average time for opportunistic communication is lower bounded by:*
$1 < \beta < 2$:

$$\mathbb{E}\tilde{T}_O > \frac{(\beta-1)^2[2 + p + (\beta-1)^2]}{p(1-p^2)}$$

$\beta > 2$:

$$\mathbb{E}\tilde{T}_O > \begin{cases} (p - p^2(\beta-1)^2)^{-1}, & p < (\beta-1)^{-2} \\ \infty, & \textit{otherwise.} \end{cases}$$

Proof We observe that

$$\mathbf{1}(o \to x, \phi_t(k)) \leq \mathbf{1}(\phi_t(k) \cap B(o, (\beta-1)\|x\|) = \{o\}).$$

So the event for opportunistic success probability is upper bounded as

$$1 - \prod_{x \in \phi_r(k)} [1 - \mathbf{1}(o \to x, \phi_t(k))] \leq 1 - \prod_{x \in \phi_r(k)} [1 - \mathbf{1}(\phi_t(k) \cap B(o, (\beta-1)\|x\|) = \{o\})]. \tag{13}$$

Case 1: $1 < \beta < 2$.
Let $z \in \phi_r$ be the nearest receiver to the origin. We then have

$$B(o, (\beta-1)\|z\|) \subset B(o, (\beta-1)\|x\|) \quad \forall x \in \phi_r \setminus \{z\}.$$

Hence the success probability at time instant k is bounded by

$$\mathbb{P}(\text{success} \mid \phi) \leq \mathbb{P}(\phi_t(k) \cap B(o, (\beta-1)\|z\|) = \{o\}),$$

where z is the nearest node of $\phi_r(k)$ to the origin. Let ξ denote the nearest point of the point process ϕ. Then the right hand side of the above equation is equal to the probability that there is at least one receiver among the nodes in the annulus A centered around the origin and radii ξ and $\xi/(\beta-1)$. Let m denote the number of nodes of ϕ in A. We then have

$$\mathbb{P}(\phi_t(k) \cap B(o, (\beta-1)\|z\|) = \{o\} \mid \phi) = 1 - p^{m+1}.$$

Hence

$$\mathbb{P}(\tilde{T}_O > n \mid \phi) = p^{(m+1)n}.$$

So we have

$$\mathbb{E}\tilde{T}_O > \mathbb{E}\left[\frac{1}{1-p^{m+1}}\right]$$

and thus

$$\begin{aligned}
\mathbb{E}\tilde{T}_O &> \mathbb{E}\left[\frac{1}{1-p} \mid m=0\right] + \mathbb{E}\left[\frac{1}{1-p^{m+1}} \mid m>1\right] \\
&= \frac{1}{(1-p)(A(\beta)+1)} + \sum_{n=0}^{\infty} p^n \mathbb{E}[p^{nm} \mid m>0] \\
&= \frac{1}{(1-p)(A(\beta)+1)} + 2A(\beta) \sum_{n=0}^{\infty} \frac{p^{2n}}{(A(\beta)+1)(A(\beta)(1-p^n)+1)} \\
&> \frac{1}{(1-p)(A(\beta)+1)} + \frac{A(\beta)}{(A(\beta)+1)^2(1-p^2)},
\end{aligned}$$

where $A(\beta) = (\beta-1)^{-2} - 1$. Multiplying with the average time for the origin at o to be a transmitter, we have the result.
Case 2: $\beta > 2$. For $\beta > 2$, we observe that the right hand side of (13) is

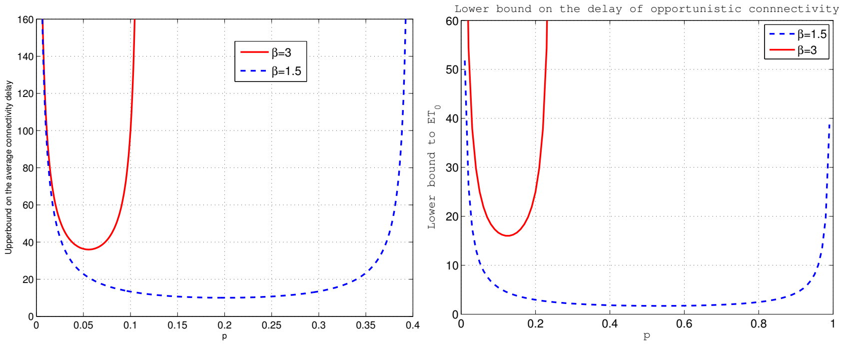

Fig. 4 The lower and upper bounds for $\mathbb{E}\tilde{T}_O$ as a function of p for different values of β. The upper bound corresponds to the average connectivity delay for the nearest-neighbor connectivity $\mathbb{E}\tilde{T}_N$.

equal to 1 if and only if the closest point of ϕ to the origin ξ is a receiver and $B(o,(\beta-1)\xi)$ is devoid of any transmitters. So we have

$$\mathbb{P}(\text{Success}) < (1-p)^{m+1},$$

where m is the number of points of ϕ in the annulus of radii $\|\xi\|$ and $(1-\beta)\|\xi\|$. Hence we have

$$\begin{aligned}\mathbb{E}\tilde{T}_O &> \mathbb{E}(1-p)^{-m-1}\\ &= (1-p)^{-1}\mathbb{E}\exp(\lambda\pi((\beta-1)^2-1)\xi^2 p(1-p)^{-1})\\ &= (1-p)^{-1}2\pi\lambda\int_0^\infty x\exp(\lambda\pi((\beta-1)^2-1)x^2p(1-p)^{-1}-\pi\lambda x^2)\mathrm{d}x.\end{aligned}$$

If $p<(\beta-1)^{-2}$ the last integral converges. Removing the conditioning on the origin being a transmitter we obtain the result.

4 The Time Evolution Graph $G(0,n)$

In the previous section we analyzed the snapshot connectivity graph formed at a particular time instant. In this section we will consider the superposition of these snapshot graphs and study how the connectivity evolves over time. Recall that the time evolution graph $G(0,n)$ is defined in (2).

4.1 Asymptotic analysis of $G(0,n)$

We first define the connection time between two nodes. For $x,y\in\phi$, we denote the *path formation time* between x and y as

$$T(x,y)=\min\{k\colon\ G(0,k)\text{ has a path from }x\text{ to }y\}.$$

For general $x, y \in \mathbb{R}^2$, define $T(x,y) = T(x^*, y^*)$ where x^* (resp. y^*) is the point in ϕ closest to x (resp. y), with some fixed deterministic rule for breaking ties (there are no ties almost surely). Since the point process is isotropic, it is sufficient for most cases to consider destinations along a given direction. For notational convenience we define for $y \in \mathbb{R}$, $T(x,y) = T(x,(y,0))$.

This path formation time is the minimum time required for a packet to propagate from a source x to its destination y in a Poisson network. In this section we show that this propagation delay increases linearly with the source-destination distance. Similar to $T(x,y)$ we define

$$T_n(x,y) = \min_{k>n} \{k - n\colon\ G(n,k) \text{ has a path from } x \text{ to } y\}.$$

The evolution of the graph $G(0,n)$ is similar to the growth of an epidemic on the plane, and one can relate the spread of information on the graph $G(0,n)$ to the theory of Markovian contact processes [21] which was used to analyze the growth of epidemics. We now provide bounds on the path formation time between two points.

In the following arguments we rely on the spatial subadditivity of $T(o,x)$ to analyze the asymptotic properties. Subadditivity of random variables is a powerful tool which is often used to prove results in percolation and geometric graph theory. The problem of finding the minimum-delay path is similar to the problem of first-passage percolation. From the definition of $T(o,y)$, we observe that

$$T(o,y) \le T(o,x) + T_{T(o,x)}(x,y). \tag{14}$$

We also have that $T_{T(o,n)}(x,y) \stackrel{d}{=} T(x,y)$ from the way the graph process is defined. Observe that (14) resembles the triangle inequality (especially if $T_{T(o,y)}(x,y)$ was $T(x,y)$) and thus provides a pseudo-metric, which holds in FPP problems and is the reason that the shortest paths in FPP are called geodesics. In the next two lemmata we show that the average time for a path to form between two nodes scales linearly with the distance between them.

Lemma 4 *The time constant defined by*

$$\mu = \lim_{x\to\infty} \frac{\mathbb{E}T(o,x)}{x}$$

exists.

Proof From (14), we have

$$T(o,y+x) \le T(o,y) + T_{T(o,y)}(y,y+x). \tag{15}$$

From the definition of the graph, the edge set E_k does not depend on $E_i,\ i<k$. Hence $T_{T(o,y)}(y,y+x)$ has the same distribution as $T(y,y+x)$. Also from the invariance of the point process ϕ, we have $T(y,y+x) \stackrel{d}{=} T(o,x)$. Taking expectations of (15), we obtain

$$\mathbb{E}T(o,y+x) \le \mathbb{E}T(o,y) + \mathbb{E}T(o,x),$$

and the result follows from the basic properties of subadditive functions.

Consistent with the FPP terminology we will call μ the time constant of the process.

Lemma 5 *The time constant for the disc model is infinite,*

$$\mu = \infty.$$

Proof Follows from Lemma 1.

The time constant is infinite because of noise. Because of the finite connectivity radius a positive fraction of the nodes will not be able to connect to any other node and hence the time constant is infinite. But if $\eta > \sqrt{1.435/\lambda}$ [4] the disc graph with radius η and node set ϕ percolates. Hence there is a giant connected component that corresponds to the disc graph formed by just considering the noise and not the interference. We denote this giant connected component by Ψ_η.

4.2 Finiteness and positivity of the time constant μ

We now prove that the any two nodes in this giant component can communicate in a time that scales linearly with the distance in between. Similar to $G(0,n)$ we define $G(0,n,\eta)$ as the dynamic graph on Ψ_η. We can similarly define for $x, y \in \Psi_\eta$.

$$T(x,y,\eta) = \min\{k : \quad G(0,k,\eta) \text{ has a path from } x \text{ to } y\},$$

and for $x, y \in \mathbb{R}^2$, $T(x,y,\eta) = T(x^*, y^*, \eta)$ where x^* and y^* are the points in Ψ_η closest to x and y. The following Lemma has been proven in [18].

Lemma 6 *For $x, y \in \mathbb{R}^2$ and $\|x-y\| < \infty$, $\|x^* - y^*\| < \infty$ almost surely.*

We also have the following lemma from [18] which deals with the lengths of the shortest path in terms of the number of hops.

Lemma 7 *For $x, y \in \Psi_\eta$, let $L(x,y)$ denote the length (in terms of number of hops) of the shortest path of the disc graph. If $\|x-y\| < \infty$, then $L(x,y) < \infty$.*

We now prove that the time constant is finite and positive on the giant connected component.

Lemma 8 *For any two nodes in Ψ_η, the average path formation time scales linearly with the distance, i.e.,*

$$0 < \mu < \infty,$$

if $0 < p < 1$.

Proof Upper bound: Let n denote the point $(n, 0)$. By subadditivity and homogenity we have

$$\mathbb{E}T(o, n, \eta) \leq n\mathbb{E}T(o, 1, \eta),$$

and hence it is sufficient to show that $\mathbb{E}T(o, 1, \eta) < \infty$ to prove $\mu < \infty$. By Lemmata 6 and 7 we have $L(o^*, 1^*) < \infty$ almost surely. Hence the shortest path that connects 0^* and 1^* in the disc graph has a finite number of edges. Denote the edges by $e_i, 1 \leq i \leq L(o^*, 1^*)$ and its corresponding Euclidean length by $|e_i|$. By the protocol model $|e_i| < \eta$. Let T_i denote the average time for a direct connection to form on the edge e_i. Since the transmitting set of the giant component at time instant k is a subset of $\phi_t(k)$, the number of interfering nodes (nodes that affect the formation of edge e_i) is smaller. Hence the average time obtained in (10) with $z = \eta$ upper-bounds T_i. Hence we have

$$T_i \leq \exp\left(\frac{p}{1-p}\lambda\pi\eta^2\nu(\beta)\right).$$

So

$$\mathbb{E}T(o, 1, \eta) < \sum_{i=1}^{L(o^*,1^*)} T_i < L(o^*, 1^*) \exp\left(\frac{p}{1-p}\lambda\pi\eta^2\nu(\beta)\right),$$

which is finite when $p < 1$, and hence $\mu < \infty$.
Lower bound: By the protocol model any path between o and n should have at least n/η hops and hence the average time is always greater than n/η and hence $\mu > 0$.

Hence the information propagation time on the giant component scales linearly with distance. The fraction of nodes in the giant component increases as the maximum connectivity distance η increases, and hence the set of nodes for which $\mu < \infty$ increases with increasing η.

5 Simulation Results

In this section we illustrate the results using simulation results. For the purpose of simulation we consider a PPP of unit density in the square $[-50, 50]^2$. For most of the simulations, we use $\beta = 1.2$, and we average over 200 independent realizations of the point process. In Figure 5, $\mathbb{E}T(o, x)$ is plotted with respect to x for different values of p. The time constant μ is plotted as a function of p in Figure 6. We make the following observations:

1. The time constant increases with the ALOHA parameter p.
2. In Figure 5, we observe that $\mathbb{E}T(o, x) \approx \mu(p)x + C(p)$, where $C(p)$ is a decreasing function of p and $\mu(p)$ is increasing. For smaller values of p, the time taken for a node to become a transmitter is large, but the probability of a successful transmission is also high because of the low density of transmitters. This results in a large $C(p)$ and smaller $\mu(p)$ for small p.

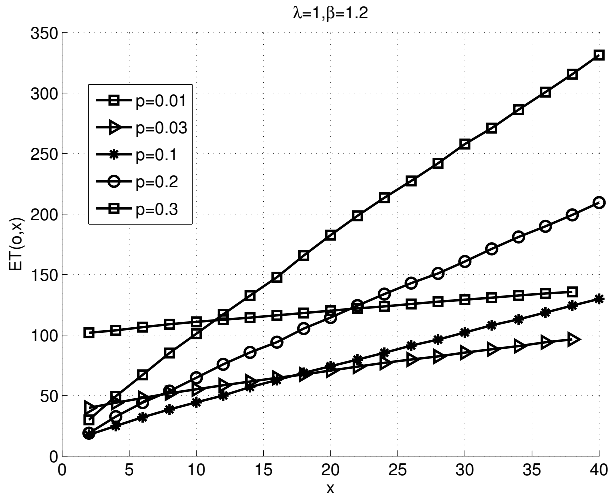

Fig. 5 $\mathbb{E}T(o,x)$ as a function of x, for $\beta = 1.2$. We first observe the linear scaling of $\mathbb{E}T(o,x)$ with the distance x and that the slope increases with p. Also for small values of x we observe that $\mathbb{E}T(o,x) \approx p^{-1}$ since for small x the path delay time is dominated by the MAC contention time. For small values of p, once the source is a transmitter, long edges form due to the low interference.

3. Figure 5 also implies that the presence of interfering transmitters causes the delay to increase when the packet has to be transmitted over longer distances. So when the packet transmission distance is large, it is beneficial to decrease the density of contending transmitters.
4. For each x, there is an optimal p which minimizes the delay, and the optimum p is a decreasing function of x.

For two nodes located at o and x and $\|x\|$ large, there will in general be many paths between o and x which form by time $\mu\|x\|$. From such an ensemble of delay-optimal paths, we will consider paths which have the minimum number of hops and call them *fastest paths.* In Figure 7, we show the average number of hops in these paths. We observe that for a given p, the average hop length decreases as the source-destination distance x increases. This shows that for larger source-destination distance, it is beneficial to use shorter hops since they are more reliable and form faster than longer hops. Also from Figure 6, we observe that for larger x, it is beneficial to be less aggressive in terms of spatial reuse and use a smaller p.

6 Conclusions

Connectivity in a wireless network is dynamic and directed because of the MAC scheduler and the half-duplex radios. Since these properties are not captured in static graph models that are usually used, we have introduced a dynamic connectivity graph and applied it to analyze properties of Poisson

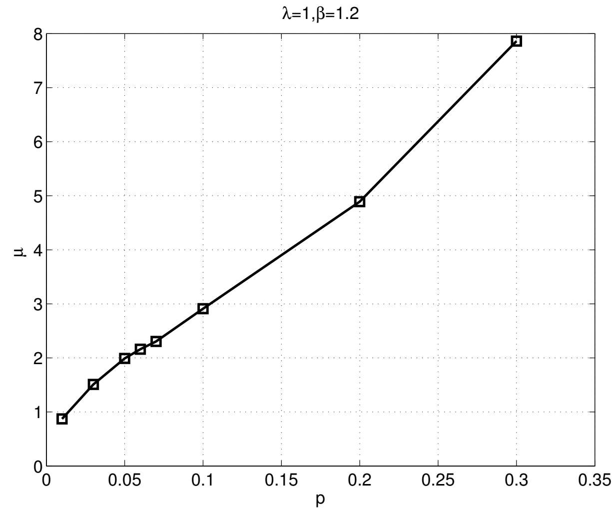

Fig. 6 The time constant μ as a function of p, for $\beta = 1.2$

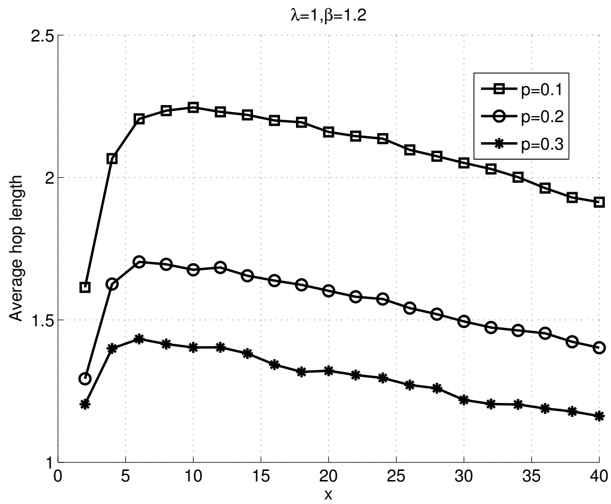

Fig. 7 Average hop length in the fastest path versus the source-destination distance.

network. We have shown that the time taken for a causal path to form between a source and a destination on this dynamic digraph scales linearly with the source-destination distance for a large fraction of nodes. The fraction of nodes for which the time-constant is finite increases with increasing power. So we can state the following: *Networks are inherently noise-limited (or power-limited) as given sufficient time, the MAC protocol can induce enough randomness or diversity to deal with the interference.* By simulations we showed that it is beneficial to use higher values of the ALOHA contention parameter for smaller source-destination distances and lower values for large distances, and that the average hop length of the fastest path first increases rapidly but then decreases slowly as a function of the source-destination distance. These observations provide some insight how to choose the hop length for efficient routing in ad hoc networks.

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