

A STOCHASTIC GEOMETRY APPROACH TO THE INTERFERENCE AND
OUTAGE CHARACTERIZATION OF LARGE WIRELESS NETWORKS

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Abstract

by

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This thesis characterizes the relationship between the geometry of a wireless network and its performance. The geometry of the network is largely influenced by the wireless node locations and the large-scale path loss and in part by fading and the transmit power levels. The performance of a wireless link is governed by interference and signal power which in turn depend on the node locations. Interference characterization is important in understanding the performance of a wireless system. But unfortunately the distribution of the interference is known only for few spatial distributions of node locations. More specifically, interference was characterized when the nodes were either distributed as a Poisson point process or as a lattice process. But in reality, the wireless nodes may neither be so random nor so regular but somewhere in between. In this thesis the location of nodes is modeled as a general stationary and isotropic point process, and we use stochastic geometry to analyze the interference and outage.

We prove that the interference distribution depends critically on the path-loss model under consideration. When the path-loss is unbounded at the origin, *i.e.*, $\ell(x) = \|x\|^{-\alpha}$, interference has a heavy-tailed distribution with parameter $2/\alpha$. When the path-loss is bounded, *i.e.*, $\ell(x) = (1 + \|x\|^\alpha)^{-1}$ the interference distribution depends on the fading statistics. We prove that the interference has an

exponential distribution when the fading is exponential and has a heavy-tailed distribution when the fading is heavy-tailed. This proves that Gaussian modeling of interference is not appropriate. We also provide the temporal and spatial correlation of interference when ALOHA is used for MAC scheduling.

Wireless nodes may cluster because of physical constraints or because of MAC scheduling. For example, soldiers (with radios) cluster on the battle field or sensor networks are clustered for energy reduction. But it is not clear if clustering of transmitters is beneficial compared to randomizing the transmissions from the perspective of link outages. We derive the outage probability of a Poisson clustered network by obtaining its conditional probability generating functional.

It is difficult for the base station in a cellular network to connect to mobile stations on the cell boundary because of the distance and the inter-cell interference. It has therefore been proposed for the base station to communicate with the mobile at the cell use using multiple hops. We use stochastic geometry to analyze the outage probability in a two-hop cellular system. We provide the asymptotic gain of a two-hop system over direct transmission, for three different relay selection schemes. The major emphasis of the thesis is the inclusion of the spatial statistics of the node locations in the performance analysis of a wireless network. To that end we concentrate on effect of the spatial distribution of nodes on the interference distribution. We provide results on the PDF, correlation and the tail behavior of the interference. One of the main contributions of the thesis is the methodology and the tools of analysis that we develop. The thesis concentrates on developing spatial analysis techniques that have wide applicability rather than concentrating on very specific details of a communication system.

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CHAPTER 1

INTRODUCTION

Information in a wireless communication system is modulated and transmitted by an electro-magnetic wave. In the three dimensional space, wave energy dissipates across the space with distance and in practice the dissipation follows an inverse power law $r^{-\alpha}$, $\alpha > 0$ where r is the distance from the source. Even from this simple model of signal propagation, it is easy to observe that as the distance between the source-destination pair increases, the observable signal energy decreases. Addition of thermal noise at the receiver requires the received signal energy greater than a threshold (decided by the digital rate of transmission) for the extraction of digital information. Hence the transmitter has to boost the signal or limit the transmission distance for information transmission. When one considers a single source-destination pair, the source can boost the signal power to compensate for the distance, and noise becomes a limiting factor (given that the source has enough power). Even in this case, observe that the distance translates into power and hence the performance of the system *implicitly* depends on the geometry, in this case the source-destination distance.

Since the medium is wireless, in addition to the desired signal, there is interference from other wireless transmitters. From the simple power law dissipation, it can be easily seen that the desired signal will get drowned in interference if the interferer is closer than the source (assuming both transmit with equal power). So

the relative distances of the source from the destination and the interferer from the destination becomes critical in recovering the transmitted information, *i.e.*, the geometry of the network plays a role in the performance. In most cases, the power of the interfering signal cannot be controlled by the source and hence in this case, boosting up the signal power cannot do the trick. This problem becomes more significant when there are multiple sources and destinations and there is no coordination among them. This problem can be tackled partially by allocating system resources to coordinate the different transmitters, *i.e.*, schedule the transmitters and decide the transmit powers so as to reduce interference.

In the present wireless systems, generally a central controller schedules the nodes so as to reduce the interference, for example base stations in the cellular networks. On the other end of the spectrum are the distributed medium access control (MAC) protocols by which a source self schedules. ALOHA and CSMA are two classes of MAC protocols that are extensively used in the wireless systems. But even after scheduling, interference cannot be entirely avoided in the network, unless one is willing to drastically under-utilize the system resources. So even after scheduling the system performance depends on the interference which in turn depend on the geometry of the network, *i.e.*, node locations. But unfortunately the effect of space on a wireless system, is not well understood as compared to the dimensions of frequency and time.

One of the major difficulty in incorporating the node locations in the performance analysis of a communicating system is that, most often node locations cannot be engineered. So one has to resort to various models of spatial locations that would facilitate the analysis. A common and analytically convenient assumption for the node distribution in wireless networks is the homogeneous (or

stationary) Poisson point process (PPP) of intensity λ , where the number of nodes in a certain area of size A is Poisson with parameter λA , and the numbers of nodes in two disjoint areas are independent random variables.

In the assumption of a PPP network, the node locations are being modeled randomly. One can instead model the node locations in a deterministic manner, for example $[x_1, \dots, x_n]$ instead of the n nodes being uniformly distributed. Determining the system performance for any deterministic node placement is more difficult, and the network geometry depends on the particular values of $[x_1, \dots, x_n]$. Also from a practical point of view it is more easier and more flexible to model the radio locations in a statistical manner.

An alternative approach to node location modeling is to abstract the communication channel between the users. This is the approach taken in multi-user information theory in which the channel is modeled as a general probabilistic function. Although this approach would lead to the analysis of a large class of channels the problems are two-fold:

- The problems posed by such abstraction are very difficult to solve.
- Once the capacity of such a probabilistic function is found, the stochastic modeling of node locations is again required to understand and apply the results.

So an elegant way to understand the performance of a communication network is to start with a probabilistic model for the node locations and use the tools from wireless communication and information theory to study the performance (1). This approach of using PPP to model the spatial locations of radios is used extensively in the analysis of large wireless ad hoc networks which constitute of wireless nodes that operate in a distributed fashion. This fully distributed nature

of the network makes the geometry of the network very prominent in its analysis. The use of the PPP model simplified the analysis and provided insight into the operation of the network in the form of scaling laws.

For sensor networks, this assumption of PPP is usually justified by claiming that sensor nodes may be dropped from aircraft in large numbers; for mobile ad hoc networks, it may be argued that terminals move independently from each other. While this may be the case for certain networks, it is much more likely that the node distribution is not "completely spatially random" (CSR), *i.e.*, that nodes are either clustered or more regularly distributed. Moreover, even if the complete set of nodes constitutes a PPP, the subset of *active* nodes (*e.g.*, transmitters in a given time-slot or sentries in a sensor network), may not be homogeneously Poisson. Certainly, it is preferable that simultaneous transmitters in an ad hoc network or sentries in a sensor network form more regular processes to maximize spatial reuse or coverage respectively. On the other hand, many protocols have been suggested that are based on clustered processes. This motivates the need to extend the rich set of results available for PPPs to other node distributions.

The major emphasis of the thesis is the inclusion of the spatial statistics of the node locations in the performance analysis of a wireless network. To that end we concentrate on effect of the spatial distribution of nodes on the interference distribution. We provide results on the PDF, correlation and the tail behavior of the interference. One of the main contributions of the thesis is the methodology and the tools of analysis that we obtain. The thesis concentrates on developing spatial analysis techniques that can be used to analyze various communication systems. We focus on bare bone problems rather than very specific details of a communication system.

1.1 Definitions and Notation

In this section, we state the general system model, state the assumptions and set the notation.

Time: We assume that the time is discretized. The duration of each time slot is fixed and in general we assume that the slot is big enough to transmit a packet (a sizeable chunk of information). We also assume that all the nodes are perfectly synchronized.

Fading: We assume i.i.d unit mean fading between any two pair of nodes and time. We use $\mathbf{h}_{xy}[k]$ to denote the square of the fading, *i.e.*, the power coefficient between nodes located at x and y at time k . We also assume that the fading does not change for the time slot k , *i.e.*, we assume quasi-static fading. When there is no confusion about the time instant under consideration, we shall drop k and use \mathbf{h}_{xy} to denote the power fading coefficient. Some of the most common models for fading used are the Rayleigh and Nakagami- m fading. In these cases, the PDF of the power fading denoted by f_h is given by:

1. Rayleigh:

$$f_h(y) = \exp(-y).$$

2. Nakagami- m :

$$f_h(y) = \frac{m^m}{\Gamma(m)} y^{m-1} e^{-my}.$$

Observe that $\mathbb{E}[h] = 1$.

Path-loss model: The path-loss model is denoted by $\ell(x) : \mathbb{R}^2 \setminus \{o\} \rightarrow \mathbb{R}^+$ and satisfies the following conditions.

1. $\ell(x)$ is a continuous, positive, non-increasing function of $\|x\|$ and

$$\int_{\mathbb{R}^2 \setminus B(o, \epsilon)} \ell(x) dx < \infty, \quad \forall \epsilon > 0.$$

- 2.

$$\lim_{\|x\| \rightarrow \infty} \frac{\ell(x)}{\ell(x-y)} = 1, \quad \forall y \in \mathbb{R}^2 \tag{1.1}$$

$\ell(x)$ is usually taken to be a power law in the form

1. Singular path-loss model: $\|x\|^{-\alpha}$.
2. Non-singular path-loss model: $(1 + \|x\|^\alpha)^{-1}$.

$\min\{1, \|x\|^{-\alpha}\}$ is also an example of a non-singular path-loss function.

Remarks:

1. To satisfy Condition 1, we require $\alpha > 2$ in all the above models.
2. In practice, the decay of the power as $r^{-\alpha}$ can be observed only when $r \gg \lambda_w$ where λ_w is the wavelength of the electromagnetic wave used. This phenomena is generally referred as the near and far field effect. In practice a receiver is in a transmitters far field if the distance is larger than $2D^2/\lambda_w$, where D is the largest physical dimension of the transmitting antenna. The exponent α is greater than 2 because of power dissipation in the medium. In free space α would be equal to 2. In the near field the physics becomes more difficult because of the coupling between the transmitter and receiver.

Node Locations:

- The node locations are modeled as a point process Φ on the plane.

- For each node $x \in \Phi$, its destination is in a random direction located at $r(x)$ with $\|x - r(x)\| = R$.

For a brief introduction to the theory of point processes please refer to Chapter 6 and (2) for more details.

Definition 1 (Interference). *The interference at location $z \in \mathbb{R}^2$ when Ψ is the interfering set is given by*

$$I_{\Psi}(z, k) = \sum_{x \in \Psi} h_{xz}[k] \ell(x - z). \quad (1.2)$$

When there is no confusion, Ψ , k and z will be dropped from $I_{\Phi}(z, k)$.

In this thesis, interference is always treated as noise. From an information theoretic point of view this is not entirely correct since more sophisticated signaling schemes can be used to increase the rate of the system. But in practice most of the receivers are designed by considering interference as noise. In the thesis we also use the fact that Gaussian signaling is used and hence the capacity of a point to point link with interference is

$$\log_2(1 + \text{SINR}).$$

This assumption leads to the following outage based model of connectivity.

Definition 2 (SINR model). *A transmitting node at x can connect to a receiver y if the received SINR is greater than θ . More precisely x can connect to y at time instant k if*

$$\text{SINR}(x, y, \Psi) = \frac{h_{xy}[k] \ell(x - y)}{\sigma^2 + I_{\Psi}(y, k)} > \theta, \quad (1.3)$$

where Ψ is the interfering set and σ^2 denotes the power of the additive noise at the receiver.

We assume that σ^2 is constant throughout the network. θ depends on the modulation and the rate of transmission. If one uses Gaussian code book for transmission $\theta = 2^a - 1$, where a is the rate of transmission. This method of connectivity is typically referred to as the outage model.

Definition 3 (Local connectivity). *We define the local connectivity function as follows*

$$\mathbf{1}(x \rightarrow y \mid \Psi) = \begin{cases} 1 & \text{if } \text{SINR}(x, y, \Psi) \geq \theta, \\ 0 & \text{Otherwise.} \end{cases} \quad (1.4)$$

We now define the success probability between a source and destination talking the statistics of the transmit process. Let $B_n \subset \mathbb{R}^2$ denote an increasing sequence of convex sets with $B_{n-1} \subset B_n$ and $\lim_{n \rightarrow \infty} |B_n| = \infty$.

Definition 4 (Average spatial success probability). *Let Φ be the transmitting set. The average spatial success probability is defined as*

$$P_s = \lim_{n \rightarrow \infty} \frac{1}{\Lambda(B_n)} \mathbb{E} \left[\sum_{x \in \Phi \cap B_n} \mathbf{1}(x \rightarrow r(x) \mid \Phi \setminus \{x\}) \right], \quad (1.5)$$

for some convex set $B \subset \mathbb{R}^2$. Here $\Lambda(B_n)$ denotes the average number of points of Φ in the set B_n . The expectation is with respect to the point process, fading and the random direction.

In the above definition of the success probability we are averaging over space (different source-destination pairs) as well different realizations of the point process. Alternatively one may define the link success probability between a transmitter x and its receiver $r(x)$.

Definition 5 (Link success probability). *The success probability of an individual link is defined as*

$$P_{sl} = \mathbb{P}^{!x}(\text{SINR}(x, r(x), \Phi) > \theta) \quad (1.6)$$

Here $\mathbb{P}^{!x}$ denotes the reduced conditional probability that there is a point of the transmit point process Φ at x . In the definition of P_{sl} the success probability is averaged over different realizations of the point process but not over space. We now remove the averaging over different realizations and define the ergodic success probability.

Definition 6 (Ergodic success probability).

$$P_{se} = \lim_{n \rightarrow \infty} \frac{1}{\Lambda(B_n)} \left[\sum_{x \in \Phi \cap B_n} \mathbf{1}(x \rightarrow r(x) \mid \Phi \setminus \{x\}) \right]. \quad (1.7)$$

When the underlying transmitter set Φ is stationary and isotropic, the success probability simplifies as follows:

Theorem 1. *When the transmitters are distributed as a stationary and isotropic point process of density λ ,*

$$P_s = P_{sl} = \mathbb{P}^{!o}(\text{SINR}(o, R, \Phi) > \theta). \quad (1.8)$$

Proof. By the Campbell Mecke theorem (3), we have

$$P_s = \lim_{n \rightarrow \infty} \frac{\lambda^{-1}}{|B_n|} \int_{B_n} \mathbb{E}^{!x} \mathbf{1}(x \rightarrow r(x) \mid \Phi) dx.$$

Expanding in terms of the interference we have

$$\begin{aligned} \mathbf{P}_s &= \lim_{n \rightarrow \infty} \frac{\lambda^{-1}}{|B_n|} \int_{B_n} \mathbb{P}^{!x} \left(\frac{\mathbf{h}_{xr(x)} \ell(R)}{\sigma^2 + I_\Phi(r(x))} > \theta \right) dx \\ &\stackrel{(a)}{=} \lim_{n \rightarrow \infty} \frac{\lambda^{-1}}{|B_n|} \int_{B_n} \mathbb{P}^{!o} \left(\frac{\mathbf{h}_{oR} \ell(R)}{\sigma^2 + I_\Phi(x - r(x))} > \theta \right) dx \end{aligned}$$

(a) follows from the fact that the fading is i.i.d over space and $\mathbb{P}^{!x}(Y) = \mathbb{P}^{!o}(Y_x)$. Since the Palm measure is isotropic for a motion-invariant process, the integrand does not depend on x and hence we have

$$\mathbf{P}_s = \mathbb{P}^{!o} \left(\frac{\mathbf{h}_{oR} \ell(R)}{\sigma^2 + I_\Phi(R)} > \theta \right).$$

$\mathbf{P}_s = \mathbf{P}_{sl}$ follows from the properties of the Palm distribution of a stationary point process. \square

Theorem 2. *When the underlying transmitter set Φ is stationary, isotropic and ergodic,*

$$\mathbf{P}_s = \mathbf{P}_{sl} = \mathbf{P}_{se} = \mathbb{P}^{!o}(\text{SINR}(o, R, \Phi) > \theta).$$

Proof. The result follows from Theorem 1 and (2, Prop. 13.4.1). \square

In this thesis we concentrate only on transmitters distributed as stationary and isotropic point processes and hence use \mathbf{P}_s unless otherwise indicated. We now provide a general formula for the success probability when the fading is Nakagami-m distributed.

Theorem 3. *The Probability of success when the transmitters form a stationary*

point process Φ , $\sigma^2 = 0$ and the fading is Nakagami- m is equal to

$$P_s = \sum_{k=0}^{m-1} \frac{(-1)^k}{k!} \frac{d^k}{ds^k} \mathcal{L}_{I_\Phi(z)}^{lo}(s) \exp(-\sigma^2 s) \Big|_{s=\theta m/\ell(R)} \quad (1.9)$$

where $\mathcal{L}_{I_\Phi(z)}^{lo}(s)$ denotes the Laplace transform of the interference with respect to the reduced Palm measure of Φ .

Proof. From (1.8) the success probability is given by

$$\begin{aligned} P_s &= \mathbb{P}^{lo}(\text{SINR}(o, R, \Phi) > \theta) \\ &= \mathbb{P}^{lo}\left(\frac{h_{oR}\ell(R)}{\sigma^2 + I_\Phi(R)} > \theta\right) \\ &= \left(\frac{\theta}{\ell(R)}\right)^m \frac{m^m}{\Gamma(m)} \int_0^\infty y^{m-1} \exp\left(-\frac{\theta m}{\ell(R)} y\right) \mathbb{P}^{lo}(I_\Phi(R) + \sigma^2 < y) dy \end{aligned}$$

Integrating by parts we have

$$\begin{aligned} P_s &= \frac{1}{\Gamma(m)} \int_0^\infty \Gamma\left(m, \frac{\theta m}{\ell(R)} y\right) d\mathbb{P}^{lo}(I_\Phi(z) + \sigma^2 < y) \\ &\stackrel{(a)}{=} \sum_{k=0}^{m-1} \frac{1}{k!} \int_0^\infty y^k \exp\left(-\frac{\theta m}{\ell(R)} y\right) d\mathbb{P}^{lo}(I_\Phi(z) + \sigma^2 < y) \end{aligned}$$

where (a) follows from the series expansion of the incomplete Gamma function.

The result follows from the properties of the Laplace transform. \square

The following Corollary is obtained by setting $m = 1$.

Corollary 7. *For Rayleigh fading, the success probability is equal to*

$$P_s = \mathcal{L}_{I_\Phi(z)}^{lo}\left(\frac{\theta}{\ell(R)}\right) \exp\left(-\frac{\sigma^2 \theta}{\ell(R)}\right). \quad (1.10)$$

For other fading distributions a more abstract formula can be obtained for the success probability which involves the conditional Laplace transform of the interference. The following result is valid only for $\ell(x) = \|x\|^{-\alpha}$ and $\ell(x) = (1 + \|x\|^\alpha)^{-1}$.

Proposition 1. *Let $\hat{F}_c(\xi)$ denote the Fourier transform of the complementary cumulative distribution function (CCDF) $F_c(x)$ of the fading. We then have*

$$P_s = \int_{-\infty}^{\infty} \hat{F}_c(\xi) \exp\left(j\xi\theta \frac{\sigma^2}{\ell(R)}\right) \mathcal{L}_{I_\Phi(z)}^{\text{!o}}\left(\frac{j\xi\theta}{\ell(R)}\right) d\xi.$$

Proof. The probability of success is

$$P_s = \mathbb{P}^{\text{!o}}\left(\frac{h_{oR}\ell(R)}{\sigma^2 + I_\Phi(R)} > \theta\right)$$

which is equal to

$$\begin{aligned} P_s &= \mathbb{E}^{\text{!o}} F_c\left(\theta \frac{\sigma^2 + I_\Phi(z)}{\ell(R)}\right) \\ &= \mathbb{E}^{\text{!o}} \int_{-\infty}^{\infty} \hat{F}_c(\xi) \exp\left(j\xi\theta \frac{\sigma^2 + I_\Phi(z)}{\ell(R)}\right) d\xi \\ &= \int_{-\infty}^{\infty} \hat{F}_c(\xi) \exp\left(j\xi\theta \frac{\sigma^2}{\ell(R)}\right) \mathcal{L}_{I_\Phi(z)}^{\text{!o}}\left(\frac{j\xi\theta}{\ell(R)}\right) d\xi. \end{aligned}$$

□

From Theorem 3 and Proposition 1, we observe that the conditional Laplace transform of the interference is sufficient to evaluate the success probability. In the next section, we describe the prior work related to the interference and outage characterization in wireless networks pertinent to Chapters 2, 3 and 4.

1.2 Related Work

There exists a significant body of literature for networks with Poisson distributed nodes. In (4), the characteristic function of the interference was obtained when there is no fading and the nodes are Poisson distributed. They also provide the probability distribution function of the interference as an infinite series. Mathar et al., in (5), analyze the interference when the interference contribution by a transmitter located at x to a receiver located at the origin is exponentially distributed with parameter $\|x\|^2$. Using this model they derive the density function of the interference when the nodes are arranged as a one dimensional lattice. Also the Laplace transform of the interference is obtained when the nodes are Poisson distributed.

It is known that the interference in a planar network of nodes can be modeled as a shot noise process. Let $\{x_j\}$ be a point process in \mathbb{R} . Let $\{\beta_j(\cdot)\}$ be a sequence of independent and identically distributed random functions on \mathbb{R}^d , independent of $\{x_j\}$. Then a generalized shot noise process can be defined as (6)

$$Y(x) = \sum_j \beta_j(x - x_j)$$

If $\beta_j(\cdot)$ is the path loss model with fading, $Y(x)$ is the interference at location x if all nodes x_j are transmitting. The shot noise process is a very well studied process for noise modeling. It was first introduced by Schottky in the study of fluctuations in the anode current of a thermionic diode and it was studied in detail by Rice (7; 8). Daley in 1971 defined multi-dimensional shot noise and examined its existence when the points $\{x_j\}$ are Poisson distributed in \mathbb{R}^d . The existence of generalized shot-noise process, for any point process was studied by Westcott

in (6). Westcott also provides the Laplace transform of the shot-noise when the points $\{x_j\}$ are distributed as a Poisson cluster process. Normal convergence of the multidimensional shot-noise process is shown by Heinrich and Schmidt (9). They also show that when the points $\{x_j\}$ form a Poisson point process of intensity λ , the rate of convergence to a normal distribution is $\sqrt{\lambda}$.

In (10), Ilow and Hatzinakos model the interference as a shot noise process and show that the interference is a symmetric α -stable process (11) when the nodes are Poisson distributed on the plane. They also show that channel randomness affects the dispersion of the distribution, while the path-loss exponent affects the exponent of the process. The throughput and outage in the presence of interference are analyzed in (12; 13; 14). In (12), the shot-noise process is analyzed using stochastic geometry when the nodes are distributed as Poisson and the fading is Rayleigh. In (15) upper and lower bounds are obtained under general fading and Poisson arrangement of nodes.

Even in the case of the PPP, the interference distribution is not known for all fading distributions and all channel attenuation models. Only the characteristic function or the Laplace transform of the interference can be obtained in most of the cases. The Laplace transform can be used to evaluate the outage probabilities under Rayleigh fading characteristics (12; 16). In the analysis of outage probability, the *conditional* Laplace transform is required, *i.e.*, the Laplace transform given that there is a point of the process located at the origin. For the PPP, the conditional Laplace transform is equal to the unconditional Laplace transform.

(17) introduces the notion of *transmission capacity*, which is a measure of the area spectral efficiency of the successful transmissions resulting from the optimal contention density as a function of the link distance. Transmission capacity is

defined as the product of the maximum density of successful transmissions and their data rate, given an outage constraint. Weber et al. provide bounds for the transmission capacity under different models of fading, when the node location are Poisson distributed. Transmission capacity is used as a metric to evaluate the merit of power control in a Poisson ad hoc network in (18). In (19; 20) the optimal number of sub-bands into which the bandwidth can be divided so as to maximize the transmission capacity is obtained.

1.3 Contribution and Organization of the Thesis

In Chapter 2, we derive the statistical properties of the interference when the transmitting nodes are distributed as a stationary and motion-invariant point process. We obtain:

- Upper and lower bounds on the CCDF of interference which are asymptotically tight.
- We prove that interference is heavy tailed when the path-loss model is $\ell(x) = \|x\|^{-\alpha}$ irrespective of the fading.
- Interference is dictated by the fading when the path-loss model is bounded.
- Gaussian modeling of interference (power) is a bad assumption.

In Chapter 3, the spatial and temporal correlation properties of the interference are obtained for a PPP set of transmitters with ALOHA as the MAC protocol.

In Chapter 4, we focus on Poisson clustered transmitters and obtain expressions for the outage probability:

- The conditional probability generating functional of a PCP is obtained and used in the derivation of P_s .

- We provide conditions for which clustering the transmitters is more beneficial than randomizing the transmissions.
- We prove that the transmission capacity of a Poisson clustered network is equal to the transmission capacity of PPP network with the same density.

In Chapter 5, we analyze three relay selection schemes in a cellular architecture. We provide a detailed analysis of the success probability by taking the spatial statistics into account. We also provide exact computable asymptotic results in the high SNR regime.

Chapter 6 provides a basic introduction to the theory of point processes.

CHAPTER 2

INTERFERENCE IN GENERAL MOTION-INVARIANT NETWORKS.

2.1 System Model

The transmitters are modeled as a motion invariant (stationary and isotropic) point process Φ of intensity λ on the plane. Examples of such process include the stationary PPP and the PCP. Another example of such a process is the following: *Shifted lattice process*: The point process Φ is equal to the randomly translated and rotated integer lattice. More precisely

$$\Phi = \sqrt{\lambda}Z^2 e^{j\beta} + U,$$

where β is uniformly distributed in $[0, 2\pi]$ and U is uniformly distributed in $[0, \sqrt{\lambda}]$. Every transmitter is assumed to transmit with unit power. From (1.8) we observe that the average success probability can be calculated by placing a typical transmitter at the origin. So in this chapter we derive the properties of the interference by placing a typical transmitter at the origin. Also in this chapter we shall exclusively concentrate on the properties of the interference at a single location z and hence all the fading coefficients are with respect to this location. The interference observed at $z \in \mathbb{R}^2$ is equal to

$$I_{\Phi}(z) = \sum_{x \in \Phi \setminus \{o\}} h_{xz} \ell(x - z). \quad (2.1)$$

In this chapter we provide the characterization of the (complementary) CDF of $I_\Phi(z)$ given that there is a transmitting node at the origin. So all probabilities are conditioned on the event that there is a transmitting node at the origin, *i.e.*, Palm probabilities (2; 3; 21). Let $\tilde{\mathcal{G}}[v]$ denote the conditional probability generating functional (CPGFL) of the point process Φ , *i.e.*,

$$\tilde{\mathcal{G}}[v] = \mathbb{E}^{!o} \left[\prod_{\mathbf{x} \in \Phi} v(\mathbf{x}) \right] \quad (2.2)$$

where $v : \mathbb{R}^2 \rightarrow [0, \infty)$ is a well behaved function (2).

2.2 Conditional Laplace Transform of the Interference: $\mathcal{L}_{I_\Phi(z)}^{!o}(s)$

We first start by deriving the first and the second conditional moments of the interference.

Lemma 1. *The mean of the interference is given by*

$$\mathbb{E}^{!o}[I_\Phi(z)] = \frac{1}{\lambda} \int_{\mathbb{R}^2} \ell(x-z) \rho^{(2)}(x) dx,$$

where $\rho^{(2)}(z)$ is the second order product density.

Proof. The mean of the interference is equal to

$$\begin{aligned} \mathbb{E}^{!o}[I_\Phi(z)] &= \mathbb{E}^{!o} \sum_{\mathbf{x} \in \Phi} \mathbf{h}_x \ell(\mathbf{x} - z) \\ &\stackrel{(a)}{=} \mathbb{E}[\mathbf{h}] \lambda \int_{\mathbb{R}^2} \ell(x-z) \mathcal{K}_2(dx) \\ &\stackrel{(b)}{=} \frac{\mathbb{E}[\mathbf{h}]}{\lambda} \int_{\mathbb{R}^2} \ell(x-z) \rho^{(2)}(x) dx. \end{aligned}$$

(a) follows from the definition of the n -th factorial measure and (b) from the

definition of $\rho^{(2)}(x)$. □

The average interference is finite for the non-singular path-loss model. For $\ell(x) = \|x\|^{-\alpha}$, the average interference is finite if and only if $\rho^{(2)}(x)$ is zero (or $\rho^{(2)}(x) = o(x^\alpha)$, $x \rightarrow \infty$) on a small neighborhood of z . From the definition of $\rho^{(2)}(x)$ this implies that there should not be any interfering transmitter close to the receiver located at z . For a PPP and PCP $\rho^{(2)}(z) \neq 0$ for $z > 0$ and hence the average interference is infinity. For the shifted lattice process, $\rho^{(2)}(x) = 0$ for $\|x\| < \sqrt{\lambda}$ and hence the average interference is finite for $z < \sqrt{\lambda}$.

Lemma 2. *The second moment of the interference is equal to*

$$\begin{aligned} \mathbb{E}^{\text{!o}}[I_{\Phi}^2(z)] &= \frac{\mathbb{E}[\mathbf{h}^2]}{\lambda} \int_{\mathbb{R}^2} \ell^2(x-z) \rho^{(2)}(x) dx \\ &\quad + \frac{1}{\lambda} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ell(x_1-z) \ell(x_2-z) \rho^{(3)}(x_1, x_2) dx_1 dx_2. \end{aligned}$$

Proof. We have

$$\begin{aligned} \mathbb{E}^{\text{!o}}[I_{\Phi}^2(z)] &= \mathbb{E}^{\text{!o}} \left[\sum_{x \in \Phi} \mathbf{h}_x \ell(x-z) \right]^2 \\ &= \mathbb{E}^{\text{!o}} \left[\sum_{x \in \Phi} \mathbf{h}_x^2 \ell^2(x-z) \right] + \mathbb{E}^{\text{!o}} \left[\sum_{x_1, x_2 \in \Phi}^{\neq} \mathbf{h}_{x_1} \mathbf{h}_{x_2} \ell(x_1-z) \ell(x_2-z) \right] \\ &\stackrel{(a)}{=} \mathbb{E}[\mathbf{h}^2] \mathbb{E}^{\text{!o}} \left[\sum_{x \in \Phi} \ell^2(x-z) \right] + \mathbb{E}[\mathbf{h}]^2 \mathbb{E}^{\text{!o}} \left[\sum_{x_1, x_2 \in \Phi}^{\neq} \ell(x_1-z) \ell(x_2-z) \right]. \end{aligned}$$

(a) follows from the independence of the fading. \sum_{x_1, x_2}^{\neq} is the sum over all tuples

(x_1, x_2) such that $x_1 \neq x_2$. We have,

$$\begin{aligned} \mathbb{E}^{!o} \left[\sum_{x \in \Phi} \ell^2(x - z) \right] &\stackrel{(a)}{=} \lambda \int_{\mathbb{R}^2} \ell^2(x - z) \mathcal{K}_2(dx) \\ &\stackrel{(b)}{=} \frac{1}{\lambda} \int_{\mathbb{R}^2} \ell^2(x - z) \rho^{(2)}(x) dx. \end{aligned}$$

Similarly we have

$$\begin{aligned} \mathbb{E}^{!o} \left[\sum_{x_1, x_2 \in \Phi}^{\neq} \ell(x_1 - z) \ell(x_2 - z) \right] &\stackrel{(c)}{=} \lambda^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ell(x_1 - z) \ell(x_2 - z) \mathcal{K}_3(dx_1 \times dx_2) \\ &\stackrel{(d)}{=} \frac{1}{\lambda} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ell(x_1 - z) \ell(x_2 - z) \rho^{(3)}(x_1, x_2) dx_1 dx_2. \end{aligned}$$

where (a), (c) follow from (6.7) and (b), (d) follow from (6.8). \square

We observe that the interference moments and hence the distribution depends on the location z for a general point process. The distribution of the $I_\Phi(z)$ does not depend on the direction of z because of the isotropy of the Palm distribution for stationary processes. We now derive the conditional Laplace transform of the interference in terms of the CPGFL.

Theorem 4. *The conditional Laplace transform of the interference is given by*

$$\mathcal{L}_{I_\Phi(z)}^{!o}(s) = \tilde{\mathcal{G}}[\mathcal{L}_h(sl(\cdot - z))] \tag{2.3}$$

Proof. The Conditional Laplace transform of the interference is

$$\begin{aligned}
\mathcal{L}_{I_{\Phi}(z)}^{\text{lo}}(s) &= \mathbb{E}^{\text{lo}} \exp\left(-s \sum_{x \in \Phi} h_x \ell(x - z)\right) \\
&\stackrel{(a)}{=} \mathbb{E}^{\text{lo}} \prod_{x \in \Phi} \mathbb{E} \exp(-s h_x \ell(x - z)) \\
&= \mathbb{E}^{\text{lo}} \prod_{x \in \Phi} \mathcal{L}_h(s \ell(x - z)) \\
&\stackrel{(b)}{=} \tilde{\mathcal{G}}[\mathcal{L}_h(s \ell(\cdot - z))]
\end{aligned}$$

where (a) follows from the independence of the fading and (b) follows from the definition of the CPGFL. \square

From the above expression we observe that the interference distribution can be evaluated if the CPGFL of the transmit point process Φ is known. But unfortunately the CPGFL is known only for a few point process.

2.2.1 Interference in Poisson Networks.

When Φ is PPP, we have $\rho^{(2)}(x) = \lambda^2$ and $\rho^{(3)}(x) = \lambda^3$ and hence the average interference is

$$\mathbb{E}^{\text{lo}}[I_{\Phi}(z)] = \lambda \int_{\mathbb{R}^2} \ell(x) dx.$$

The second moment of the interference is equal to

$$\mathbb{E}^{\text{lo}}[I_{\Phi}^2(z)] = \mathbb{E}[h^2] \lambda \int_{\mathbb{R}^2} \ell^2(x) dx + \lambda^2 \left(\int_{\mathbb{R}^2} \ell(x) dx \right)^2.$$

Hence the variance of the interference is equal to $\mathbb{E}[h^2] \lambda \int_{\mathbb{R}^2} \ell^2(x) dx$. By Slivnyak's theorem we have $\mathbb{P}^{\text{lo}} = \mathbb{P}$ and hence the CPGFL is equal to the PGFL. Hence from

(6.1) and Theorem 4 the Laplace transform of the interference is equal to

$$\begin{aligned}\mathcal{L}_{I_{\Phi}(z)}^{!o}(s) &= \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \mathcal{L}_h(s\ell(x-z))dx\right) \\ &= \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \mathcal{L}_h(s\ell(x))dx\right).\end{aligned}$$

Observe that the interference distribution does not depend on the location z . This is because of the stationary of the reduced Palm measure of the PPP. When $\ell(x) = \|x\|^{-\alpha}$, by a change of variables we obtain

$$\mathcal{L}_{I_{\Phi}(z)}^{!o}(s) = \exp\left(-\lambda s^{\delta} \int_{\mathbb{R}^2} 1 - \mathcal{L}_h(\|x\|^{-\alpha})dx\right), \quad (2.4)$$

where $\delta = 2/\alpha$. From the Laplace transform of the interference, we observe that the interference is a stable distribution with parameter δ when the node distribution is PPP and the path-loss model singular. In this case integer moments of the interference do not exist. In the singular case the Laplace transform can be simplified further:

$$\begin{aligned}\mathcal{L}_{I_{\Phi}(z)}^{!o}(s) &= \exp\left(-\lambda s^{\delta} \int_{\mathbb{R}^2} \int_0^{\infty} \left[1 - e^{-\|x\|^{-\alpha}y} f(y)\right] dy dx\right) \\ &\stackrel{(a)}{=} \exp\left(-\lambda \mathbb{E}[h^{\delta}] s^{\delta} \int_{\mathbb{R}^2} 1 - e^{-\|x\|^{-\alpha}} dx\right) \\ &= \exp\left(-\pi \lambda \mathbb{E}[h^{\delta}] \Gamma(1-\delta) s^{\delta}\right).\end{aligned}$$

where (a) follows from the substitution $xy^{-1/\alpha} \rightarrow x$. From the above expression we observe that the distribution depends only on the δ -th moment of the fading. This is in contrast with the non-singular channel model in which the fading distribution plays a more dominant role.

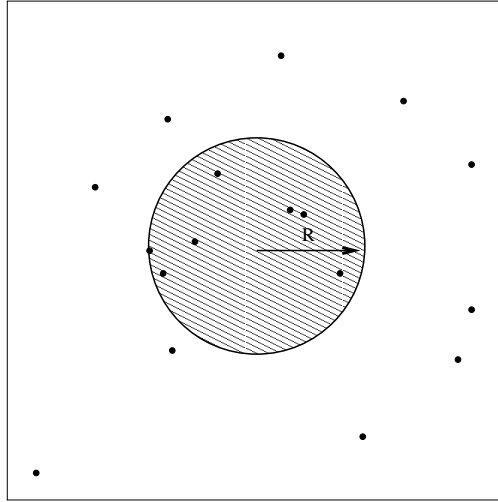


Figure 2.1. The hashed disc of radius $R = y^{-1/\alpha}$ and corresponds to the set of transmitters (near set) which individually contribute at least y to the interference I .

2.3 Bounds on the Interference Distribution

In the previous section, we have derived the Laplace transform of the interference and although in theory, the Laplace transform provides the complete description of the interference, it would be more beneficial to have the CDF or the PDF of the interference. In this section we provide bounds on the CCDF of the interference when the transmitting nodes are homogeneously distributed on the plane.

The basic idea behind the proof is easy to understand when fading is absent, $\ell(x) = \|x\|^{-\alpha}$ and for a PPP. In this case the interference at the origin is given by $I = \sum_{x \in \Phi} l(x)$. We can then divide the transmitting set into two subsets, the near set and the far set. The near set consists of all the nodes that individually contribute atleast y to I and the far set is the complement of the near set. See

Figure 2.1. Since there is no fading it is easy to see that the near set consists of the nodes in $b(o, y^{-1/\alpha})$. We can lower bound $\mathbb{P}(I > y)$ by neglecting the contribution of the far set and this will be a tight bound if $\alpha - 2$ is not too small since most of the contribution to the interference is from the near set. A lower bound for $\mathbb{P}(I < y)$ follows by first observing that the event $I < y$ requires the near set to be empty and secondly that the contribution of the far set can be replaced by its average using the Markov inequality (the average interference caused by the far set is finite for $y > 0$). When there is fading, there is an effective reordering of the points (22) but nevertheless the near and the far sets can be defined in a similar fashion.

Theorem 5. *When the transmitters are distributed as a stationary point process Φ , the CCDF $\bar{F}_I(y)$ of the interference at location z , conditioned on a transmitter present at the origin but not included in the interference is lower bounded by $\bar{F}_I^l(y)$ and upper bounded by $\bar{F}_I^u(y)$, where*

$$\bar{F}_I^l(y) = 1 - \tilde{\mathcal{G}} \left[F_h \left(\frac{y}{\ell(\cdot - z)} \right) \right] \quad (2.5)$$

$$\bar{F}_I^u(y) = 1 - (1 - \varphi(y)) \tilde{\mathcal{G}} \left[F_h \left(\frac{y}{\ell(\cdot - z)} \right) \right] \quad (2.6)$$

where $F_h(x)$ denotes the CDF of the fading coefficient h , and

$$\varphi(y) = \frac{1}{y\lambda} \int_{\mathbb{R}^2} \ell(x - z) \rho^{(2)}(x) \int_0^{y/\ell(x-z)} \nu dF_h(\nu) dx. \quad (2.7)$$

If $\mathbb{E}^{!o}[I_\Phi^p] < \infty$, we can also use a loose $\varphi(y) = \mathbb{E}^{!o}[I_\Phi^p] y^{-p}$, $p \geq 1$.

Proof. The basic idea is to partition the transmitter set Φ into two subsets Φ_y

and Φ_y^c where,

$$\begin{aligned}\Phi_y &= \{x \in \Phi, h_x \ell(x-z) > y\} \quad (\text{near set}), \\ \Phi_y^c &= \{x \in \Phi, h_x \ell(x-z) \leq y\} \quad (\text{far set}).\end{aligned}$$

Φ_y consists of those transmitters whose contribution to the interference exceeds y . We have $I_\Phi(z) = I_{\Phi_y}(z) + I_{\Phi_y^c}(z)$, where $I_{\Phi_y}(z)$ corresponds to the interference due to the transmitter set Φ_y and $I_{\Phi_y^c}(z)$ corresponds to the interference due to the transmitter set Φ_y^c . Hence we have

$$\begin{aligned}\bar{F}_I(y) &= \mathbb{P}(I_{\Phi_y}(z) + I_{\Phi_y^c}(z) \geq y) \\ &\geq \mathbb{P}(I_{\Phi_y}(z) \geq y) \\ &= 1 - \mathbb{P}(I_{\Phi_y}(z) < y) \\ &= 1 - \mathbb{P}(\Phi_y = \emptyset).\end{aligned}\tag{2.8}$$

We can evaluate the probability $\mathbb{P}(\Phi_y = \emptyset)$ that Φ_y is empty using the conditional Laplace functional as follows:

$$\begin{aligned}\mathbb{P}(\Phi_y = \emptyset) &= \mathbb{E}^{\text{lo}} \prod_{x \in \Phi} \mathbf{1}_{h_x \ell(x-z) \leq y} \\ &\stackrel{(a)}{=} \mathbb{E}^{\text{lo}} \prod_{x \in \Phi} \mathbb{E}_{h_x} (\mathbf{1}_{h_x \ell(x-z) \leq y}) \\ &= \mathbb{E}^{\text{lo}} \prod_{x \in \Phi} F_h \left(\frac{y}{\ell(x-z)} \right) \\ &= \tilde{\mathcal{G}} \left[F_h \left(\frac{y}{\ell(\cdot - z)} \right) \right],\end{aligned}\tag{2.9}$$

where (a) follows from the independence of h_x . To obtain the upper bound

$$\begin{aligned}
\bar{F}_I(y) &= \mathbb{P}(I_\Phi > y \mid I_{\Phi_y} > y) \bar{F}_I^l(y) + \mathbb{P}(I_\Phi > y \mid I_{\Phi_y} \leq y) (1 - \bar{F}_I^l(y)) \\
&\stackrel{(a)}{=} 1 - \tilde{\mathcal{G}} \left[F_h \left(\frac{y}{\ell(\cdot - z)} \right) \right] + \mathbb{P}(I_\Phi > y \mid I_{\Phi_y} \leq y) \tilde{\mathcal{G}} \left[F_h \left(\frac{y}{\ell(\cdot - z)} \right) \right] \\
&= 1 - (1 - \mathbb{P}(I_\Phi > y \mid I_{\Phi_y} \leq y)) \tilde{\mathcal{G}} \left[F_h \left(\frac{y}{\ell(\cdot - z)} \right) \right] \tag{2.10}
\end{aligned}$$

where (a) follows from the lower bound we have established. To evaluate $\mathbb{P}(I_\Phi > y \mid I_{\Phi_y} \leq y)$ we use the Markov inequality. We have

$$\begin{aligned}
\mathbb{P}(I_\Phi > y \mid I_{\Phi_y} \leq y) &= \mathbb{P}(I_\Phi > y \mid \Phi_y = \emptyset) \\
&\stackrel{(a)}{\leq} \frac{\mathbb{E}^{!o}(I_\Phi \mid \Phi_y = \emptyset)}{y} \\
&= \frac{1}{y} \mathbb{E}^{!o} \sum_{x \in \Phi} h_x \ell(x - z) \mathbf{1}_{h_x \ell(x - z) \leq y} \\
&= \frac{1}{y} \mathbb{E}^{!o} \sum_{x \in \Phi} \ell(x - z) \int_0^{y/\ell(x - z)} \nu dF_h(\nu) \\
&\stackrel{(b)}{=} \frac{1}{y\lambda} \int_{\mathbb{R}^2} \ell(x - z) \int_0^{y/\ell(x - z)} \nu dF_h(\nu) \rho^{(2)}(x) dx.
\end{aligned}$$

(a) follows from the Markov inequality, and (b) follows from a procedure similar to the calculation of the mean interference from the previous section.

□

When Φ is a PPP, we have $\tilde{\mathcal{G}}[v] = \exp(-\lambda \int 1 - v(x) dx)$ (6.1). For Rayleigh fading and $\ell(x) = \|x\|^{-\alpha}$, the lower bound is equal to

$$\bar{F}_I^l(y) = 1 - \exp(-\pi \lambda y^{-\delta} \Gamma(1 + \delta)),$$

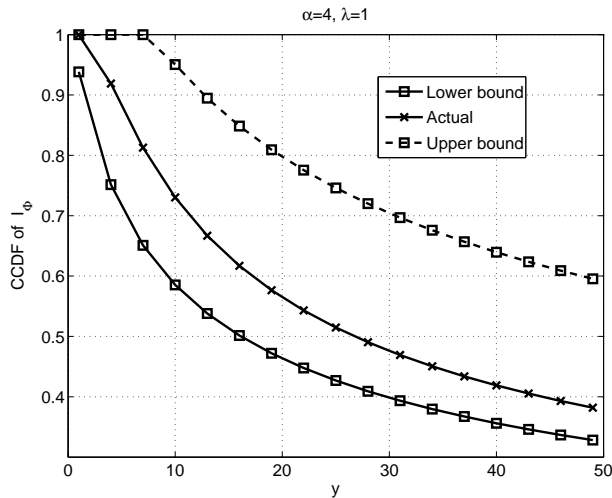


Figure 2.2. The CCDF of the interference is plotted when Φ is a PPP and $\ell(x) = \|x\|^{-4}$. In this case the interference is a Levy stable distribution.

and the upper bound is equal to

$$\bar{F}_I^u(y) = 1 - \left(1 - \frac{2\pi\lambda\Gamma(1+\delta)}{\alpha-2}y^{-\delta}\right) \exp(-\pi\lambda y^{-\delta}\Gamma(1+\delta)).$$

From Figure 2.2, we observe that the lower bound is closer to the actual CCDF than the upper bound. In the above derivation, the upper bound may be loose because a simple Markov inequality is used to bound $\mathbb{P}(I_\Phi > y \mid I_{\Phi_y} \leq y)$, and a better bound may be obtained by using the Chernoff bound. The upper bound diverges as $\alpha \downarrow 2$ since the average interference contribution from the far set diverges. From the upper and the lower bound we can observe that the interference in a PPP network is heavy tailed with parameter δ for the singular path-loss model.

In the next Lemma we prove that the upper and the lower bounds are asymp-

totically tight when $\ell(x) = \|x\|^{-\alpha}$. We will use $g_1(x) \sim g_2(x)$ to denote

$$\lim_{x \rightarrow \infty} g_1(x)/g_2(x) = 1.$$

Lemma 3. *When $\ell(x) = \|x\|^{-\alpha}$, $\alpha > 2$, $z \in \mathbb{R}^2$,*

$$\varphi(y) \begin{cases} \sim \frac{2\pi\rho^{(2)}(z)\mathbb{E}[h^\delta]}{\lambda(\alpha-2)}y^{-\delta} & \rho^{(2)}(z) \neq 0 \\ = o(y^{-\delta}) & \rho^{(2)}(z) = 0. \end{cases}$$

Proof. When $g(x) = \|x\|^{-\alpha}$, $\alpha > 2$, we have

$$\begin{aligned} \varphi(y) &= \frac{1}{y\lambda} \int_{\mathbb{R}^2} \|x-z\|^{-\alpha} \rho^{(2)}(x) \int_0^{\|x-z\|^\alpha} \nu dF_h(\nu) dx \\ &\stackrel{(a)}{=} \frac{1}{y\lambda} \int_0^\infty \nu dF_h(\nu) \int_{\mathbb{R}^2} \|x\|^{-\alpha} \mathbf{1}(\|x\|^\alpha > \nu y^{-1}) \rho^{(2)}(x+z) dx \\ &\stackrel{(b)}{=} \frac{y^{-2/\alpha}}{\lambda} \int_0^\infty \nu dF_h(\nu) \int_{\mathbb{R}^2} \|x\|^{-\alpha} \mathbf{1}(\|x\|^\alpha > \nu) \rho^{(2)}\left(\frac{x}{y^{1/\alpha}} + z\right) dx \end{aligned}$$

where (a) follows from the substitution $x \rightarrow x+z$ and interchanging the integrals.

(b) follows by the substitution $y^{1/\alpha}x \rightarrow x$. So by the dominated convergence theorem when $\rho^{(2)}(z) \neq 0$, we have

$$\begin{aligned} \lim_{y \rightarrow 0} \frac{\varphi(y)}{y^{2/\alpha}} &= \frac{\rho^{(2)}(z)}{\lambda} \int_0^\infty \nu dF_h(\nu) \int_{\mathbb{R}^2} \|x\|^{-\alpha} \mathbf{1}_{\|x\|^\alpha > \nu} dx \\ &= \frac{2\pi\rho^{(2)}(z)\mathbb{E}[h^{2/\alpha}]}{\lambda(\alpha-2)}. \end{aligned}$$

□

2.4 Asymptotic Behavior of the Interference Distribution

In the previous section, bounds on the CCDF were provided, and they depend on the conditional PGFL. But the PGFL (let alone the conditional PGFL) is not known, except for a few point processes. In this section we take an alternate approach and evaluate the tail of the CDF. We show that the CCDF of the interference depends critically on the path-loss model $\ell(x)$.

2.4.1 Singular Path-Loss Function

For a real-valued function $f(x)$, the behavior of the function for large x can be evaluated by the value of the Laplace transform at $s = 0$. The generalization of this idea is expressed by the following Tauberian theorem.

Theorem 6 (Tauberian theorem (23; 24)). *Let R_0 represent the set of functions which satisfy the property*

$$\frac{\eta(\lambda x)}{\eta(x)} \rightarrow 1, \quad (x \rightarrow \infty).$$

For $0 \leq \beta \leq 1$, $\eta \in R_0$, the following are equivalent:

$$1 - \int_{-\infty}^{\infty} e^{-sx} f(x) dx \sim s^\beta \eta\left(\frac{1}{s}\right), \quad (s \downarrow 0)$$

$$1 - \int_0^x f(y) dy \sim \frac{\eta(x)}{x^\beta \Gamma(1 - \beta)}, \quad (x \rightarrow \infty).$$

In this section we will derive the tail behavior of the interference using the conditional Laplace transform of the interference. We first prove that the interference is heavy-tailed when the path-loss model is singular. The basic idea of the proof is as follows: From the previous chapter, we know that the conditional

Laplace transform of the interference is given by $\tilde{\mathcal{G}}[\mathcal{L}_h(s\ell(\cdot - z))]$. We have

$$\tilde{\mathcal{G}}[\mathcal{L}_h(s\ell(\cdot - z))] = \tilde{\mathcal{G}}[1 - v(s, x - z)],$$

where $v(s, x) = (1 - \mathcal{L}_h(s\ell(x)))$. We observe that $v(s, x) \rightarrow 0$ as $s \rightarrow 0$. Since $v(s, x)$ is small, we show that the following approximation holds true when s is small.

$$\begin{aligned} \tilde{\mathcal{G}}[1 - v(s, x)] &\approx 1 - \mathbb{E}^{1o} \sum_{x \in \Phi} v(s, x - z) \\ &= 1 - \int_{\mathbb{R}^2} v(s, x - z) \rho^{(2)}(x) dx. \end{aligned}$$

It is then shown that $\int_{\mathbb{R}^2} v(s, x - z) \rho^{(2)}(x) dx = \Theta(s^{2/\alpha})$ when $\ell(x) = \|x\|^{-\alpha}$ and the Tauberian theorem is used to prove that the tail of the interference is heavy.

Theorem 7. *Let $z \in \mathbb{R}^2$ such that $\rho^{(2)}(z) \neq 0$ and $\rho^{(2)}(z)$ continuous around a small neighborhood of z . If $\ell(x) = \|x\|^{-\alpha}$, $\alpha > 2$, factorial moments of Φ exist then $I_\Phi(z)$ is heavy-tailed with parameter δ . More precisely*

$$\mathbb{P}(I_\Phi(z) \geq y) \sim \frac{\pi \rho^{(2)}(z)}{\lambda} \mathbb{E}[h^\delta] y^{-\delta}, \quad y \rightarrow \infty.$$

Proof. The idea is use the scaling of Laplace transform of $I_\Phi(z)$ at $s = 0$ and derive the properties of the tail properties of $I_\Phi(z)$ using the Tauberian Theorem 6.

Let $\ell_r(x) = \ell(x) \mathbf{1}_{b(o, r)}(x)$, be a truncated version of $\ell(x)$. Observe that $\ell_r(x) \rightarrow \ell(x)$ uniformly as $r \rightarrow \infty$. We now prove that $\mathbb{E}_h(e^{-sh\ell_r(x-z)})$ belongs to the class of functions over which the conditional PGFL is continuous.

Bounded support: Observe that $1 - \mathbb{E}_h(e^{-sh\ell_r(x-z)})$ has a bounded support.

Moment convergence: We now prove that

$$\int_{\mathbb{R}^2} |\mathbb{E}_h(e^{-sh\ell_r(x-z)}) - \mathbb{E}_h(e^{-sh\ell(x-z)})| \rho^{(2)}(x) dx \rightarrow 0 \quad (2.11)$$

The left side of the above equation is

$$\begin{aligned} & \int_{\mathbb{R}^2} \left| \int_0^\infty [\exp(-s\ell_r(x-z)t) - \exp(-s\ell(x-z)t)] f(t) dt \right| \rho^{(2)}(x) dx \\ &= \int_{\|x-z\|>r} \int_0^\infty [1 - \exp(-s\ell(x-z)t)] f(t) dt \rho^{(2)}(x) dx \\ &\stackrel{(a)}{\leq} s \int_{\|x-z\|>r} \|x-z\|^{-\alpha} \rho^{(2)}(x) dx \int_0^\infty t f(t) dt \\ &\stackrel{(b)}{=} \frac{\mathbb{E}[h]s}{r^{\alpha-2}} \int_{\|x\|>1} \|x\|^{-\alpha} \rho^{(2)}(rx+z) dx \\ &\stackrel{(c)}{=} \rightarrow 0, \text{ when } r \rightarrow \infty. \end{aligned} \quad (2.12)$$

(a) follows from the inequality $1 - \exp(-x) \leq x$, (b) follows from the substitution $r^{-1}(x-z) \rightarrow x$ and (c) follows since $\rho^{(2)}(rx+z) \rightarrow \lambda^2$ and $\int_{\|x\|>1} \|x\|^{-\alpha} dx < \infty$.

From (2.11) and from (2, Problem 9.4.5), we have

$$\mathcal{L}_{I_\Phi(z)}(s) = \lim_{r \rightarrow \infty} \mathbb{E}^{!o} \left[\prod_{x \in \Phi} \mathbb{E}_h(e^{-sh\ell_r(x-z)}) \right]. \quad (2.13)$$

Hence we can work with $\ell_r(x)$ and take the limit $r \rightarrow \infty$ at the end. Define

$$k_r(s, x) = \mathbb{E}_h(e^{-sh\ell_r(x-z)}).$$

Since $\ell_r(x) = \ell(x) \mathbf{1}_{b(o,r)}(x)$ is defined on a compact subset, we have that $k_r(s, x) \neq 1$ only on a compact subset B_r of \mathbb{R}^2 . Since Φ is a simple and finite point process,

$\Phi(B_r) < \infty$ with probability one. So we have from (25, p. 458),

$$\begin{aligned} 0 &\leq \prod_{x \in \Phi} [1 - (1 - k_r(s, x))] - \left[1 - \sum_{x \in \Phi} (1 - k_r(s, x)) \right] \\ &\leq \sum_{x_1 \neq x_2 \in \Phi} (1 - k_r(s, x_1))(1 - k_r(s, x_2)). \end{aligned} \quad (2.14)$$

First taking expectation and then taking the limit, we have from (2.13),

$$\lim_{r \rightarrow \infty} \mathbb{E}^{lo} \left[\prod_{x \in \Phi} [1 - (1 - k_r(s, x))] \right] = \mathcal{L}_{I_\Phi(z)}(s).$$

We also have that

$$\begin{aligned} \eta(s) &= \lim_{r \rightarrow \infty} \mathbb{E}^{lo} \left[\sum_{x \in \Phi} (1 - k_r(s, x)) \right] \\ &\stackrel{(a)}{=} \lim_{r \rightarrow \infty} \lambda^{-1} \int_{\mathbb{R}^2} (1 - k_r(s, x)) \rho^{(2)}(x) dx \\ &= \lambda^{-1} \int_{\mathbb{R}^2} (1 - k(s, x)) \rho^{(2)}(x) dx, \end{aligned}$$

where (a) follows from the definition of $\rho^{(2)}(x)$ and $k(s, x) = \lim_{r \rightarrow \infty} k_r(s, x)$. Similarly we have

$$\begin{aligned} \beta(s) &= \lim_{r \rightarrow \infty} \mathbb{E}^{lo} \left[\sum_{x_1 \neq x_2 \in \Phi} (1 - k_r(s, x_1))(1 - k_r(s, x_2)) \right] \\ &\stackrel{(a)}{=} \lambda^{-2} \int \int (1 - k(s, x_1))(1 - k(s, x_2)) \rho^{(3)}(x_1, x_2) dx_1 dx_2 \end{aligned}$$

where (a) follows from the definition of $\rho^{(3)}(x)$. We now prove that $\lim_{s \rightarrow 0} \eta(s) s^{-\delta} >$

0 and $\lim_{s \rightarrow 0} \beta(s)s^{-\delta} = 0$. We have

$$\begin{aligned}
\eta(s) &= \lambda^{-1} \int_{\mathbb{R}^2} [1 - \mathbb{E}_h(e^{-sh\ell(x-z)})] \rho^{(2)}(x) dx \\
&= \lambda^{-1} \int_0^\infty \int_{\mathbb{R}^2} [1 - \exp(-s\|x\|^{-\alpha}t)] \rho^{(2)}(x+z) dx f(t) dt \\
&= \lambda^{-1} s^\delta \int_0^\infty \int_{\mathbb{R}^2} [1 - \exp(-\|x\|^{-\alpha}t)] \rho^{(2)}(xs^\delta + z) dx f(t) dt.
\end{aligned}$$

We therefore have

$$\begin{aligned}
\lim_{s \rightarrow 0} \frac{\eta(s)}{s^\delta} &= \lambda^{-1} \lim_{s \rightarrow 0} \int_0^\infty \int_{\mathbb{R}^2} [1 - \exp(-\|x\|^{-\alpha}t)] \cdot \\
&\quad \rho^{(2)}(xs^\delta + z) dx f(t) dt \\
&\stackrel{(a)}{=} \lambda^{-1} \rho^{(2)}(z) \int_0^\infty \int_{\mathbb{R}^2} [1 - \exp(-\|x\|^{-\alpha}t)] dx f(t) dt \\
&= \lambda^{-1} \rho^{(2)}(z) \mathbb{E}[h^\delta] \pi \Gamma(1 - \delta). \tag{2.15}
\end{aligned}$$

Here (a) follows from the dominated convergence theorem. We now prove that $\beta(s)$ divided by s^δ tends to zero. Consider

$$\begin{aligned}
\beta(s) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (1 - k(s, x_1))(1 - k(s, x_2)) \rho^{(3)}(x_1, x_2) dx_1 dx_2 \\
&= s^{4/\alpha} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\int_0^\infty 1 - \exp(-\|x_1\|^{-\alpha}t) f(t) dt \right) \cdot \\
&\quad \left(\int_0^\infty 1 - \exp(-\|x_2\|^{-\alpha}t) f(t) dt \right) \cdot \\
&\quad \rho^{(3)}(s^{1/\alpha}x_1 + z, s^{1/\alpha}x_2 + z) dx_1 dx_2.
\end{aligned}$$

We also observe that the integral is a finite integral because of our assumption $\rho^{(3)}(x, y) < \log(\|x\|) \log(\|y\|)$. So we have

$$\lim_{s \rightarrow 0} \frac{\beta(s)}{s^\delta} = 0. \tag{2.16}$$

From (2.14), we have

$$0 \leq \mathcal{L}_{I_{\Phi}(z)}(s) - (1 - \eta(s)) \leq \beta(s).$$

Dividing both sides by s^δ , taking the limit $s \rightarrow 0$ and using (2.15) and (2.16),

$$\lim_{s \rightarrow 0} \frac{1 - \mathcal{L}_{I_{\Phi}(z)}(s)}{s^\delta} = \frac{\pi \rho^{(2)}(z)}{\lambda} \mathbb{E}[h^\delta] \Gamma(1 - \delta).$$

So we have

$$1 - \mathcal{L}_{I_{\Phi}(z)}(s) \sim \frac{\pi \rho^{(2)}(z)}{\lambda} \mathbb{E}[h^\delta] \Gamma(1 - \delta) s^\delta.$$

So using the Tauberian Theorem 6,

$$\mathbb{P}(I_{\Phi}(z) \geq y) \sim \frac{\pi \rho^{(2)}(z)}{\lambda} \mathbb{E}[h^\delta] y^{-\delta}.$$

□

From the above theorem we observe that interference at z is always a heavy tailed distribution with parameter δ when the path-loss model is singular and $\rho^{(2)}(z) \neq 0$. This is because the receiver is not a part of the process and a transmitter can be arbitrarily close to it, which causes the interference to blow up.

2.4.2 Non-Singular Path-Loss Function

We now investigate the interference distribution when the path-loss is non-singular. In this case, the existence of an interferer close to a receiver will not alter the magnitude of the interference significantly. Instead the fading becomes

an important factor, and the distribution tail depends mainly on the tail of the fading. We will use the following theorem that connects the convergence region of the Laplace transform of a random variable and the decay of its CCDF.

Theorem 8 (Nakagawa). *Let X be a non-negative random variable, and $F(x) = \mathbb{P}(X \leq x)$ be the probability distribution function of X . Let*

$$\mathcal{L}_X(s) = \int_0^\infty e^{-sx} dF(x), \quad s = \sigma + j\tau \in \mathbb{C} \quad (2.17)$$

be the Laplace-Stieltjes transform of $F(x)$ and σ_0 be the abscissa of convergence of $\mathcal{L}_X(s)$. We assume $-\infty < \sigma_0 < 0$. If $s = \sigma_0$ is a pole of $\mathcal{L}_X(s)$, then we have

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \mathbb{P}(X > x) = \sigma_0. \quad (2.18)$$

The above theorem is useful in dealing with fading distributions whose tail decays exponentially. Since the conditional Laplace transform of the interference is known, it can be used in conjunction with the above theorem to prove the exponential decay of the interference tail when the fading is exponential. When the fading distribution is heavy-tailed, we use the upper and the lower bounds on the CCDF provided in Theorem 5 to prove that interference is also heavy tailed.

Theorem 9. *Let $\ell(x) = 1/(1 + \|x\|^\alpha)$. Then*

1. *If the fading has at-most an exponential tail, i.e., $\bar{F}_h(x) < \exp(-ax)$ for large x , then the interference tail is also exponential. Formally: if $\exists a > 0$ s.t. $\bar{F}_h(x) = \Theta(e^{-ax})$, $x \rightarrow \infty$ implies $\bar{F}_I(x) = \Theta(e^{-ax})$.*
2. *If the fading has a polynomial or heavy tail the interference is heavy tailed.*

$$\bar{F}_h(x) \sim x^{-\alpha} \Rightarrow \bar{F}_I(x) \sim x^{-\alpha}.$$

Proof. Case 1):*Exponential fading:* We will first show that the conditional Laplace transform of the interference converges for $s < \sigma$, $\sigma < 0$ and diverges for $s > \sigma$. We have

$$\mathcal{L}_{I_\Phi(z)}(s) = \mathbb{E}^{\text{lo}} \prod_{\mathbf{x} \in \Phi} k(s, \mathbf{x})$$

where $k(s, x) = \mathcal{L}_h(s\ell(x-z))$. From the above equation we observe that $\mathcal{L}_{I_\Phi(z)}(s)$ is finite if and only if

$$\eta(s) = \mathbb{E}^{\text{lo}} \sum_{\mathbf{x} \in \Phi} |\log k(s, \mathbf{x})| < \infty.$$

We now show that the abscissa of convergence σ of $\mathcal{L}_{I_\Phi(z)}(s)$ is strictly less than zero, *i.e.*, $\eta(s) < \infty$ for some $s < 0$.

Let $\beta(s, x, h) = \exp\left(\frac{-sh}{(1+\|x-z\|^\alpha)}\right)$. We have

$$\begin{aligned} \eta(s) &= \mathbb{E}^{\text{lo}} \sum_{\mathbf{x} \in \Phi} |\log k(s, \mathbf{x})| \\ &= \int_{\mathbb{R}^2} |\log (\mathbb{E}_h[\beta(s, x, h)])| \rho^{(2)}(x) dx. \end{aligned}$$

When $s > 0$, it is trivial to see that $\eta(s) < \infty$. The rest of the proof can be better understood by considering $s < 0$. Since $\bar{F}_h(x) \sim \exp(-ax)$, $x \rightarrow \infty$ it can be assumed without any loss of generality that the PDF of the fading is equal to $f(x) = a \exp(-ax)$, $x > R$, for some large R . We have

$$\begin{aligned} k(s, x) &= \int_0^R \beta(s, x, t) dF(t) \\ &\quad + a \int_R^\infty \exp\left(-t \left[a + \frac{s}{(1+\|x-z\|^\alpha)} \right]\right) dt \end{aligned} \quad (2.19)$$

The first term is always finite. Considering the second term, we observe that the

term in the exponent will be positive for all x when $s > -a$. Hence the integral converges when $s > -a$ and $k(s, x)$ is well defined (especially in the neighborhood of z)¹.

Also observe that $k(s, x) > 1$ for $s \in (-a, 0)$. Let $b \in (-a, 0)$. We now prove that $\eta(b) < \infty$. Since $k(b, x) > 1$, we have $\log(k(b, x)) \leq k(b, x) - 1$. Hence

$$\begin{aligned} \eta(s) &\leq \int_{\mathbb{R}^2} [k(b, x) - 1] \rho^{(2)}(x) dx \\ &= \int_{b(o, \kappa)} [k(b, x) - 1] \rho^{(2)}(x) dx + \int_{b(o, \kappa)^c} [k(b, x) - 1] \rho^{(2)}(x) dx \end{aligned}$$

for $\kappa \in (0, \infty)$. Since $b > -a$ we observe that $k(b, x)$ is a well behaved function, *i.e.*, bounded and smooth, and hence the first term in the above equation is bounded for any $\kappa < \infty$. So if we prove that the second term is finite, then $\eta(b) < \infty$. For large $\|x\|$ we have $\rho^{(2)}(x) \rightarrow \lambda^2$. Choose κ such that for all $\|x\| > \kappa$, $\rho^{(2)}(x)$ is very close to λ^2 . Hence $\rho^{(2)}(x)$ is essentially a constant and proving

$$\int_{b(o, \kappa)^c} (k(b, x) - 1) dx < \infty$$

will be sufficient. We have

$$\begin{aligned} \int_{b(o, \kappa)^c} (k(b, x) - 1) dx &= \underbrace{\int_{b(o, \kappa)^c} \int_0^R [\beta(-|b|, x, t) - 1] f(t) dt dx}_A \\ &\quad + \underbrace{\int_{b(o, \kappa)^c} \int_R^\infty [\beta(-|b|, x, t) - 1] f(t) dt dx}_B. \end{aligned}$$

¹Observe the importance of 1 in the denominator of the second term. If the one wasn't present, then $\forall s < 0$, $k(s, x)$ would be become undefined on an open neighborhood of z .

Considering the first term A, we can increase κ such that

$$\beta(-|b|, x, t) = \exp\left(\frac{|b|t}{(1 + \|x - z\|^\alpha)}\right) \approx 1 + \frac{|b|t}{(1 + \|x - z\|^\alpha)}. \quad (2.20)$$

This can be done since $t < R$. Hence we have

$$A = \int_{b(o, \kappa)^c} \int_0^R \frac{|b|t}{(1 + \|x - z\|^\alpha)} f(t) dt dx < \infty.$$

Considering the second integral B, substituting for the PDF of the fading and after some algebraic manipulation, we have

$$\begin{aligned} B &= e^{-aR} \int_{b(o, \kappa)^c} \frac{a(1 + \|x - z\|^\alpha) (\beta(-|b|, x, R) - 1)}{a(1 + \|x - z\|^\alpha) - |b|} dx \\ &\quad + e^{-aR} \int_{b(o, \kappa)^c} \frac{|b|}{a(1 + \|x - z\|^\alpha) - |b|} dx. \end{aligned}$$

Since κ is large, using the approximation (2.20), we observe that $B < \infty$. So we have shown that $\eta(b) < \infty$ for all $b \in (-a, \infty)$. We also observe that $\eta(s) = \infty$ for $s < -a$. So the abscissa is equal to $-a < 0$. Using Theorem 3 in (26), we have that the tail falls exponentially with parameter a .

Case 2: $\bar{F}_h \sim h^{-a}$ is a heavy tailed distribution: In this case $k(s, x) = \infty$ for all $s < 0$, and hence (26, Thm. 3) cannot be applied. We will use Theorem 5 that provides upper and lower bounds for the CCDF of the interference. We first evaluate $\tilde{\mathcal{G}} \left[F_h \left(\frac{y}{\ell(\cdot, -z)} \right) \right]$ for large y :

$$\begin{aligned} \tilde{\mathcal{G}} \left[F_h \left(\frac{y}{\ell(\cdot, -z)} \right) \right] &= \tilde{\mathcal{G}} [1 - [1 - F_h(y(1 + \|x - z\|^\alpha))]] \\ &\stackrel{(a)}{\approx} \tilde{\mathcal{G}} (1 - [y(1 + \|x - z\|^\alpha)]^{-a}) \\ &\stackrel{(b)}{\approx} 1 - y^{-a} \int_{\mathbb{R}^2} [(1 + \|x - z\|^\alpha)]^{-a} \rho^{(2)}(x) dx \end{aligned}$$

where (a) follows by the continuity of $\tilde{\mathcal{G}}$ and large y . (b) follows from an argument similar to Theorem 7. We also have from Theorem 5, that $\varphi(y) = y^{-a}\mathbb{E}^{\text{lo}}[I_{\Phi}(z)^a]$. Here $\mathbb{E}^{\text{lo}}[I_{\Phi}(z)^a] < \infty$ because of the bounded nature of $\ell(x)$ and its fast decaying tail. So from the upper bound in Theorem 5, we have

$$\mathbb{P}(I_{\Phi}(z) > y) < y^{-a} \left[\int_{\mathbb{R}^2} [(1 + \|x - z\|^\alpha)]^{-\alpha} \rho^{(2)}(x) dx + \mathbb{E}^{\text{lo}}[I_{\Phi}(z)^a] \right],$$

and from the lower bound

$$P(I_{\Phi}(z) > y) > y^{-a} \int_{\mathbb{R}^2} [(1 + \|x - z\|^\alpha)]^{-\alpha} \rho^{(2)}(x) dx.$$

So the tail decays like y^{-a} . □

We now show that the distribution of interference decays exponentially fast at origin. The basic idea is that there is some contribution from some point of the process, however small it is.

Theorem 10. *The CDF of the interference decays faster than any polynomial at the origin, i.e., $\forall n \in \mathbb{N}$,*

$$\mathbb{P}(I_{\Phi}(z) < y) = o(y^n)$$

as $y \rightarrow 0$.

Proof. Find a such that $F_h(x) = o(x^a)$. Choose $k \in \mathbb{N}$ such that $k > n/a$. From Theorem 5, we have

$$\begin{aligned} \mathbb{P}(I_{\Phi}(z) < y) &< \tilde{\mathcal{G}} \left(F_h \left(\frac{y}{\ell(\cdot - z)} \right) \right) \\ &= \mathbb{E}^{\text{lo}} \left[\prod_{x \in \Phi} F_h \left(\frac{y}{\ell(x - z)} \right) \mid \Phi \text{ has at least } k \text{ points} \right]. \end{aligned}$$

Multiply both sides by y^{-n} and take the limit $y \rightarrow 0$. On the right hand side, the limit can be moved inside the expectation by the dominated convergence theorem. Since there are at least k points almost surely on the plane (because Φ is stationary), we have that the limit on the right goes to zero by our choice of k . \square

2.5 Examples and Simulation Results

In this section we give examples of the interference for different point processes and fading distributions. We specifically concentrate on three different point processes: the PPP, the Thomas cluster process and the shifted lattice process.

Poisson point process:

$$1) \ell(x) = \|x\|^{-\alpha}$$

From Section 2.2.1, the Laplace transform of the interference is given by, $\mathcal{L}_{I_\Phi}(s) = \exp(-\lambda\pi s^\delta \mathbb{E}[h^\delta] \Gamma(1 - \delta))$. Observe that the Laplace transform is independent of z and hence the distribution is independent of the location z . Also $\mathcal{L}_{I_\Phi}(s)$ is the Laplace transform of a stable random variable with parameter δ and hence heavy tailed.

$$2) \ell(x) = (1 + \|x\|^\alpha)^{-1}$$

We first consider the case exponential fading *i.e.*, $f(x) = \mu \exp(-\mu x)$. In this case,

$$\mathcal{L}_{I_\Phi}(s) = \exp\left(-\lambda\pi^2\delta \csc(\pi\delta) \frac{s}{(\mu + s)^{1-\delta}}\right).$$

Observe that $\mathcal{L}_{I_\Phi}(s)$ is well defined for $s > -\mu$. Let $\nu_{I_\Phi}(x)$ denote the PDF of interference, we then have by the final value theorem, $\lim_{x \rightarrow \infty} e^{\mu_1 x} \nu_{I_\Phi}(x) = \lim_{s \rightarrow 0} \mathcal{L}_{I_\Phi}(s - \mu_1) < \infty$ for all $\mu_1 < \mu$. So the PDF is a combination of many

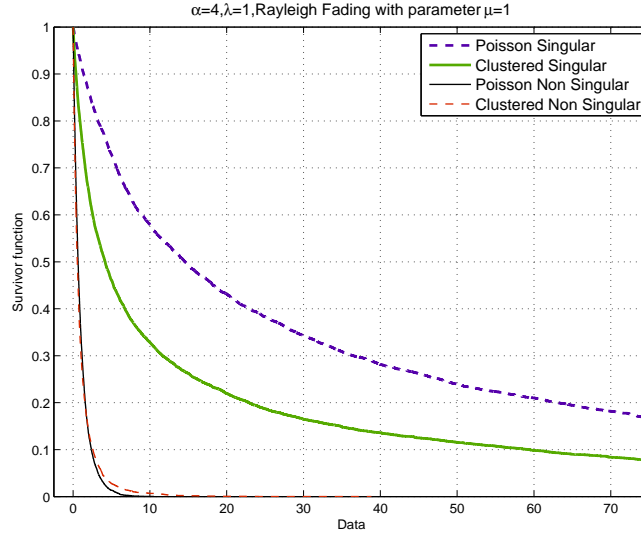


Figure 2.3. CCDF of the interference for Rayleigh fading and path loss $\alpha = 4, z = (3, 0)$

decaying exponentials. In Figure 2.3, we plot the CCDF of Poisson interference with Rayleigh fading. We observe the heavy tailed distribution for $\ell(x) = \|x\|^{-\alpha}$ and the exponential decay when $\ell(x) = (1 + \|x\|^\alpha)^{-1}$.

Thomas cluster processes: We proved in (27, Lemma 3) that the interference has a heavy tailed distribution for $\ell(x) = \|x\|^{-\alpha}$ with parameter δ .

Shifted lattice process: The interference results for this process are verified by simulation. In Figure 2.4, we plot the CCDF for different values of z and with Rayleigh fading. We observe that the tail properties depend heavily on z . When $\|z\| < 1$, we have that $\rho^{(2)}(z) = 0$ (actually the associated measure $\mathcal{K}_2(A) = 0, \forall A \subset b(o, 1)$). So here the effective path loss model is bounded and hence the interference tail follows that of the fading. When $\|z\| > 1$, there is a positive probability that a transmitting node can be arbitrarily close to z and hence the interference follows a heavy tail distribution.

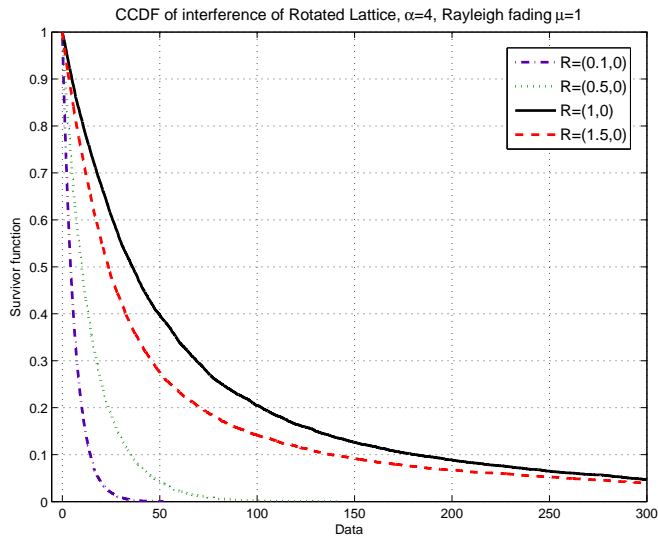


Figure 2.4. CCDF of the interference for exponential power fading and path loss $\alpha = 4$ when the point process is the lattice process.

Approximation of the distribution of interference: From the previous theorems, we have the following observations:

1. The CDF $F_I(y)$ of the interference decays faster than any polynomial as $y \downarrow 0$.
2. When $\ell(x) = (1 + \|x\|^\alpha)^{-1}$, $\alpha > 2$, the mean interference is finite, and the CCDF tail decays like that of the fading distribution.
3. When $\ell(x) = \|x\|^{-\alpha}$, the mean diverges, and the CDF has a heavy tail.

Observation 1 eliminates the use of Gaussian distribution to model the interference except when the mean $\mu = \mathbb{E}[I_\Phi]$ is very large (but finite), so that $\exp(-\mu^2/2\sigma^2)$ is small. We choose three probability distributions which have these properties. The gamma distribution, and the inverse Gaussian distribution.

1. Gamma distribution: $f(x) = x^{k-1} \exp(-x/a)/\Gamma(k)a^k$. Mean : ka , Variance: ka^2 .

2. Inverse Gaussian :

$$f(x) = \left[\frac{\nu}{2\pi x^3} \right]^{1/2} \exp \left(-\frac{\kappa(x - \nu)^2}{2\nu^2 x} \right)$$

Mean: ν , Variance: ν^3/κ .

3. Inverse Gamma :

$$f(x) = \beta^a x^{-a-1} \exp(-\beta/x) \Gamma(a)^{-1}$$

Mean: $\beta/(a - 1)$, Variance: $\beta^2/((a - 1)^2(a - 2))$.

Observe that in the inverse Gaussian distribution, the mean and variance can be chosen independently of each other. Observe that the gamma distribution only has a $(k - 1)$ th order of decay at the origin and has an exponential tail. On the other hand, the inverse Gaussian distribution has an exponential decay at origin and a slightly super exponential tail. In Figure 2.5, we have plotted² the PDF of the interference using Monte-Carlo simulation when the underlying node distribution is PPP and the fading is Rayleigh with a non-singular path loss model. We observe that the normal fit performs the worst. Both the gamma and inverse Gaussian give us a good fit. Also the inverse gamma PDF is a bad fit since it has a fourth-order decaying tail, while the fading is exponentially decaying. In finite networks, where the number of nodes is finite and fixed and the nodes are distributed on a bounded subset of the Euclidean plane, the interference does not

²We have used a square of size 40×40 for simulation and averaged over 200000 instances.

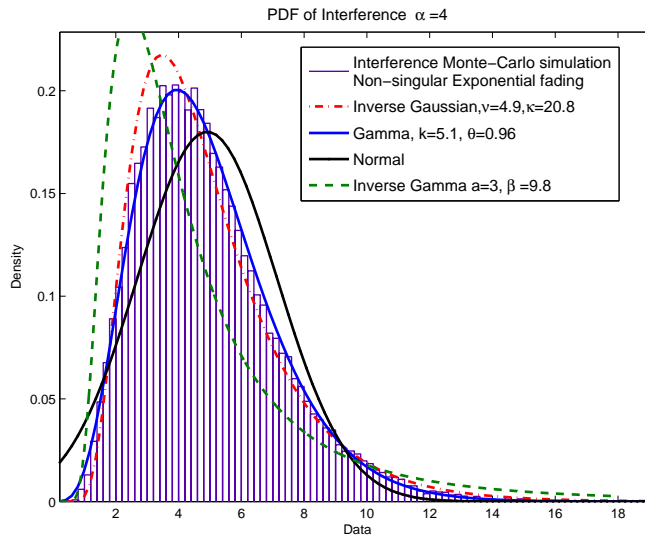


Figure 2.5. PDF of Interference and the corresponding fits for $\ell(x) = (1 + \|x\|^4)^{-1}$ and the square of fading as exponential with parameter 1.

decay infinitely fast at the origin, but only goes to zero like y^{na} where n is the number of transmitters and a the decay of the fading at origin.

2.6 Chapter Summary

The statistical properties of interference caused by transmitters that are arranged as a general motion-invariant process are analyzed. Bounds of the CCDF have been provided that are asymptotically tight. We make the following observations:

1. The distribution of the interference is greatly influenced by the path-loss model.
2. If one assumes a singular path-loss, *i.e.*, $\ell(x) = \|x\|^{-\alpha}$ the interference is a heavy tailed distribution irrespective of the fading. This heavy-tailed

nature of the interference stems mainly from the contribution of the nearest interferer to the receiver.

3. With a bounded path-loss model, the interference distribution is dictated by fading. For example, when the fading is exponential the tail of the interference also decays exponentially.
4. Modeling interference by a Gaussian distribution is not appropriate and we have shown by simulations and reasoned that distributions like gamma, inverse gamma or the inverse Gaussian provide a better fit.

CHAPTER 3

SPATIAL AND TEMPORAL CORRELATION OF THE INTERFERENCE IN ALOHA AD HOC NETWORKS.

Interference in a wireless ad hoc network is a spatial phenomenon which depends on the set of transmitters, the path loss, and the fading. The presence of common randomness in the locations of the interferers induces temporal and spatial correlations in the interference, even for ALOHA. These correlations affect the retransmission strategies and the routing. In the literature, these correlations are generally neglected for the purpose of analytical tractability and because these correlations do not change the scaling behavior of an ad hoc wireless network. For example, in (12) and (28), the spatial correlations are neglected for the purpose of routing. Also extending results like the transmission capacity (17) from a single-hop to a multi-hop scenario requires taking the spatio-temporal correlations into account.

From Chapter 2, we observe that the interference distribution in a Poisson network does not depend on the spatial location. This is because of the stationarity of the Palm measure of the PPP. Even though the interference distribution is identical on the entire plane, the interference is not independent across the plane. This is because the interference is caused by the common randomness, *i.e.*, the point process Φ . Since each node in a wireless network uses a MAC protocol to decide whether to transmit or receive, the transmitting set Φ_k changes with time

k , and since $\Phi_k \subset \Phi$, the interference becomes correlated over time because of the common randomness Φ . In this section we consider ALOHA as the MAC protocol in which each node transmits with probability p and receives with probability $1 - p$ independently of other nodes. We observe that ALOHA as a MAC protocol introduces no correlations since the transmitters are independently chosen. Nevertheless the presence of the common randomness Φ causes interference to be temporally correlated. The temporal and the spatial correlation of the interference makes the outages correlated temporally (important for retransmissions) and spatially correlated (important for routing).

In this chapter we quantify the temporal and spatial correlation of the interference in a wireless ad hoc network whose nodes are distributed as a Poisson point process on the plane when ALOHA is used as the multiple-access scheme.

3.1 System Model

We model the location of the nodes (radios) as a Poisson point process (PPP) $\Phi = \{x_1, x_2, \dots\} \subset \mathbb{R}^2$ of density λ . We assume that all the nodes transmit with unit power and that the fading is spatially and temporally independent with unit mean. The (power) fading coefficient between two pairs of nodes x and y at time instant n is denoted by $h_{xy}[n]$. The large scale path loss function $\ell(x)$ is assumed to follow an additional constraint of integrability:

$$\int_0^\infty x\ell(x)dx < \infty. \quad (3.1)$$

For example, a valid path loss model is given by

$$\ell_\epsilon(x) = \frac{1}{\epsilon + \|x\|^\alpha}, \quad \epsilon \in (0, \infty), \quad \alpha > 2. \quad (3.2)$$

We can model the standard singular path loss model $\ell(x) = \|x\|^{-\alpha}$ by considering the limit $\lim_{\epsilon \rightarrow 0} \ell_\epsilon(x)$. The interference at time instant k and (spatial) location z is given by

$$I_k(z) = \sum_{x \in \Phi} \mathbf{1}(x \in \Phi_k) h_{xz}[k] \ell(x - z). \quad (3.3)$$

where Φ_k denotes the transmitting set at time k . We assume that the MAC protocol used is ALOHA where each node decides to transmit independently with probability p in each slot.

3.2 Spatio-Temporal Correlation of Interference

In a wireless system the transmitting set changes at every time slot because of the MAC scheduler. Since the transmitting sets at different time slots are chosen from Φ (a common source of randomness), the interference exhibits temporal and spatial correlation. Since ALOHA chooses the transmitting sets identically across time, $I_k(u)$ is identically distributed for all k . Since nodes transmit independently of each other in ALOHA, the transmitting set $\Phi_k \subset \Phi$ is also spatially stationary, and hence $I_k(u) \stackrel{d}{=} I_k(o)$ where $\stackrel{d}{=}$ denotes equality in distribution and o denotes the origin in \mathbb{R}^2 . Hence we have

$$\begin{aligned} \mathbb{E}I_k(u) &= \mathbb{E}I_k(o) \\ &\stackrel{(a)}{=} \mathbb{E} \sum_{x \in \Phi} \mathbf{1}(x \in \phi_k) h_{xo}[k] \ell(x) \\ &\stackrel{(b)}{=} p\lambda \int_{\mathbb{R}^2} \ell(x) dx, \end{aligned} \quad (3.4)$$

where (a) follows from Campbell's theorem (3) and (b) follows since $\mathbb{E}[h] = 1$. The second moment of the interference is given by

$$\begin{aligned}
\mathbb{E}[I_k(o)^2] &= \mathbb{E} \left[\left(\sum_{\mathbf{x} \in \Phi_k} \mathbf{h}_{xo}[k] \ell(\mathbf{x}) \right)^2 \right] \\
&= \mathbb{E} \sum_{\mathbf{x} \in \Phi_k} \mathbf{h}_{xo}[k]^2 \ell^2(\mathbf{x}) + \mathbb{E} \sum_{\substack{\mathbf{x} \neq \mathbf{y} \\ \mathbf{x}, \mathbf{y} \in \Phi_k}} \mathbf{h}_{xo}[k] \mathbf{h}_{yo}[k] \ell(\mathbf{x}) \ell(\mathbf{y}) \\
&\stackrel{(a)}{=} p \mathbb{E}[h^2] \lambda \int_{\mathbb{R}^2} \ell^2(x) dx + p^2 \mathbb{E}[h]^2 \lambda^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ell(x) \ell(y) dx dy. \quad (3.5)
\end{aligned}$$

where (a) follows from the independence of $\mathbf{h}_{xo}[k]$ and $\mathbf{h}_{yo}[k]$ and the second-order product density formula of the Poisson point process (3). When the fading follows a Nakagami- m^1 distribution and the path loss model is given by $\ell_\epsilon(x)$, the variance of the interference follows from (3.4) and (3.5) and is given by

$$\text{Var} [I_k(o)] = \frac{2\pi^2(\alpha - 2)p\lambda}{\epsilon^{2-2/\alpha} \alpha^2 \sin(2\pi/\alpha)} \frac{m + 1}{m},$$

and the mean product of $I_k(u)$ and $I_l(v)$ at times k and l , $k \neq l$ is given by

$$\begin{aligned}
\mathbb{E}[I_k(u)I_l(v)] &= \mathbb{E} \left[\sum_{\mathbf{x} \in \Phi_k} \mathbf{h}_{xu}[k] \ell(\mathbf{x} - u) \sum_{\mathbf{y} \in \Phi_l} \mathbf{h}_{yv}[l] \ell(\mathbf{y} - v) \right] \\
&= p^2 \mathbb{E}[h]^2 \lambda \int_{\mathbb{R}^2} \ell(x - u) \ell(x - v) dx \\
&\quad + \mathbb{E} \sum_{\substack{\mathbf{x} \neq \mathbf{y} \\ \mathbf{x}, \mathbf{y} \in \Phi}} \mathbf{1}(\mathbf{x} \in \Phi_k) \mathbf{1}(\mathbf{y} \in \Phi_l) \mathbf{h}_{xu}[k] \mathbf{h}_{yv}[l] \ell(\mathbf{x}) \ell(\mathbf{y}).
\end{aligned}$$

¹The distribution is given by

$$F(x) = 1 - \frac{\Gamma_{\text{ic}}(m, mx)}{\Gamma(m)},$$

where Γ_{ic} denotes the incomplete gamma function.

By Campbell's theorem and the second order product density of a PPP, we have

$$\begin{aligned}
\mathbb{E}[I_k(u)I_l(v)] &= p^2\mathbb{E}[h]^2\lambda \int_{\mathbb{R}^2} \ell(x-u)\ell(x-v)dx \\
&\quad + \lambda^2 p^2 \mathbb{E}[h]^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ell(x)\ell(y)dx dy \\
&= p^2\lambda \int_{\mathbb{R}^2} \ell(x-u)\ell(x-v)dx + \lambda^2 p^2 \left(\int_{\mathbb{R}^2} \ell(x)dx \right)^2. \quad (3.6)
\end{aligned}$$

Lemma 4. *The spatio-temporal correlation coefficient of the interferences $I_k(u)$ and $I_l(v)$, $k \neq l$, when the path loss function $\ell(x)$ satisfies (3.1) is given by*

$$\zeta(u, v) = \frac{p \int_{\mathbb{R}^2} \ell(x)\ell(x - \|u - v\|)dx}{\mathbb{E}[h^2] \int_{\mathbb{R}^2} \ell^2(x)dx}. \quad (3.7)$$

Proof. Since $I_k(u)$ and $I_l(v)$ are identically distributed, we have

$$\zeta(u, v) = \frac{\mathbb{E}[I_k(u)I_l(v)] - \mathbb{E}[I_k(u)]^2}{\mathbb{E}[I_k(u)^2] - \mathbb{E}[I_k(u)]^2}.$$

Since $I_k(u) \stackrel{d}{=} I_k(o)$ and by substituting for the above quantities we have,

$$\begin{aligned}
\zeta(u, v) &= \frac{p \int_{\mathbb{R}^2} \ell(x-u)\ell(x-v)dx}{\mathbb{E}[h^2] \int_{\mathbb{R}^2} \ell^2(x)dx} \\
&\stackrel{(a)}{=} \frac{p \int_{\mathbb{R}^2} \ell(x)\ell(x - \|u - v\|)dx}{\mathbb{E}[h^2] \int_{\mathbb{R}^2} \ell^2(x)dx}, \quad (3.8)
\end{aligned}$$

where (a) follows by using the substitution $y = x - u$ and the fact that $\ell(x)$ depends only on $\|x\|$. \square

We have the following result about the temporal correlation by setting $\|u - v\| = 0$.

Corollary 8. *The temporal correlation coefficient with ALOHA as the MAC pro-*

total is given by

$$\zeta_t = \frac{p}{\mathbb{E}[h^2]}. \quad (3.9)$$

When the fading is Nakagami- m , the correlation coefficient is $\zeta_t = \frac{pm}{m+1}$. In particular, for $m = 1$ (Rayleigh fading), the temporal correlation coefficient is $p/2$ and for $m \rightarrow \infty$ (no fading), the temporal correlation coefficient is p .

We first observe that the correlation increases with increasing m , i.e., fading decreases correlation which is intuitive. Further, the correlation coefficient does not depend on λ , since, unlike the ALOHA parameter p which introduces additional randomness between time slots, it just uniformly scales the interference.

Observe that in the above derivation, $\int_{\mathbb{R}^2} \ell^2(x) dx$ is not defined when $\ell(x) = \|x\|^{-\alpha}$, but we can use $g_\epsilon(x)$ and take $\epsilon \rightarrow 0$. We now find the correlation for the singular path-loss model as a limit of $\ell_\epsilon(x)$.

Corollary 9. *Let the path loss model be given by $\ell_\epsilon(x) = 1/(\epsilon + \|x\|^\alpha)$. We then have*

$$\lim_{\epsilon \rightarrow 0} \zeta(u, v) = 0, \quad u \neq v.$$

Proof. We have

$$\begin{aligned} \zeta(u, v) &= \lim_{\epsilon \rightarrow 0} \frac{p \int_{\mathbb{R}^2} \ell_\epsilon(x-u) \ell_\epsilon(x-v) dx}{\mathbb{E}[h^2] \int_{\mathbb{R}^2} \ell_\epsilon^2(x) dx} \\ &\stackrel{(a)}{=} \lim_{\epsilon \rightarrow 0} \frac{p \int_{\mathbb{R}^2} \frac{1}{1+\|x-u\epsilon^{-1/\alpha}\|^\alpha} \frac{1}{1+\|x-v\epsilon^{-1/\alpha}\|^\alpha} dx}{\mathbb{E}[h^2] \int_{\mathbb{R}^2} \left(\frac{1}{1+\|x\|^\alpha} \right)^2 dx} \\ &= 0, \end{aligned}$$

where (a) follows from change of variables. □

The correlation coefficient being 0 is an artifact of the singular path loss model.

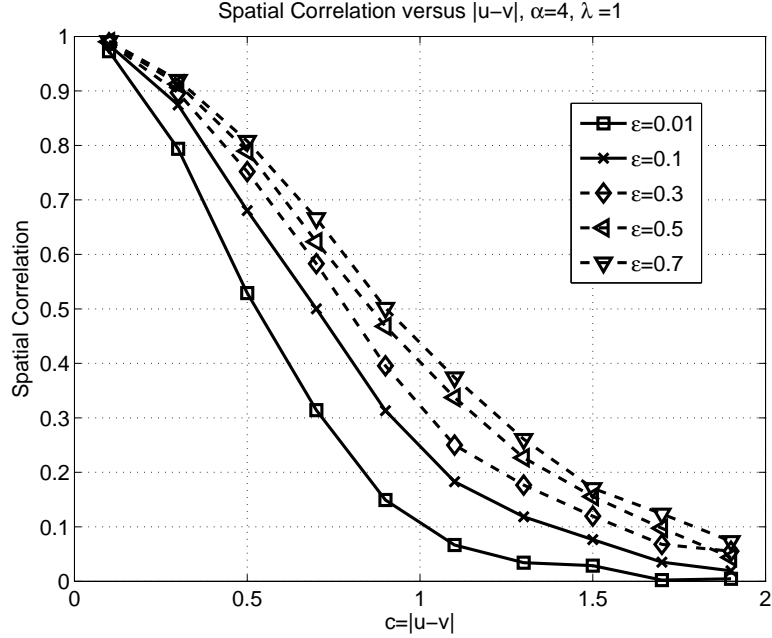


Figure 3.1. Spatial correlation $\zeta(u, v)/p$ versus $\|u - v\|$, when the path-loss model is given by $\ell_\epsilon(x)$, $\lambda = 1$ and $\alpha = 4$. We observe that $\zeta_s(u, v) \rightarrow 0$, $u \neq v$, for $\epsilon \rightarrow 0$.

When the path loss is $\|x\|^{-\alpha}$, the nearest transmitter is the main contributor to the interference. So for $u \neq v$, the interference as viewed by u is dominated by transmitters in a disc $B(u, \delta)$, $\delta > 0$ of radius δ centered at u and for v dominated by transmitters in $B(v, \delta)$ for small δ . The transmitters locations being independent in $B(v, \delta)$ and $B(u, \delta)$ for a PPP, makes the correlation-coefficient go to zero. A more powerful metric like mutual information would be better able to capture the dependence of interference for the singular path loss model. In Figure 3.1, the spatial correlation is plotted as a function of $\|u - v\|$ for different ϵ .

3.3 Temporal Correlation of Link Outages

In the standard analysis of retransmissions in a wireless ad hoc system, the link failures are assumed to be uncorrelated across time. But this is not so, since the interference is temporally correlated. We now provide the conditional probability of link formation assuming a successful transmission.

We assume that a transmitter at the origin has a destination located at $z \in \mathbb{R}^2$. Let A_k denote the event that the origin is able to connect to its destination z at time instant k , i.e.,

$$\text{SIR} = \frac{\mathbf{h}_{oz}[k]\ell(z)}{I_k(z)} > \theta.$$

For simplicity we shall assume the fading is Rayleigh (similar methods can be used for Nakagami- m). We now provide the joint probability of success $\mathbb{P}(A_k, A_l)$, $k \neq l$. We have

$$\begin{aligned} \mathbb{P}(A_k, A_l) &= \mathbb{P}(\mathbf{h}_{oz}[k] > \theta_z I_k(z), \mathbf{h}_{oz}[l] > \theta_z I_l(z)) \\ &\stackrel{(a)}{=} \mathbb{E}[\exp(-\theta_z I_k(z)) \exp(-\theta_z I_l(z))] \\ &= \mathbb{E}[\exp(-\theta_z \sum_{x \in \Phi} \ell(x) [\mathbf{1}(x \in \Phi_k) \mathbf{h}_{xz}[k] + \mathbf{1}(x \in \Phi_l) \mathbf{h}_{xz}[l]])] \\ &\stackrel{(b)}{=} \mathbb{E} \left[\prod_{x \in \Phi} \left(\frac{p}{1 + \theta_z \ell(x)} + 1 - p \right)^2 \right] \\ &\stackrel{(c)}{=} \exp \left(-\lambda \int_{\mathbb{R}^2} 1 - \left(\frac{p}{1 + \theta_z \ell(x)} + 1 - p \right)^2 dx \right), \end{aligned} \tag{3.10}$$

where $\theta_z = \theta/\ell(z)$. (a) follows from the independence of $\mathbf{h}_{oz}[k]$ and $\mathbf{h}_{oz}[l]$, $k \neq l$, (b) follows by taking the average with respect to $\mathbf{h}_{xz}[k]$, $\mathbf{h}_{xz}[l]$ and the ALOHA, (c)

follows from the probability generating functional of the PPP. Similarly we have

$$\mathbb{P}(A_l) = \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \left(\frac{p}{1 + \theta_z \ell(x)} + 1 - p\right) dx\right).$$

So the ratio of conditional and the unconditional probability is given by

$$\begin{aligned} \frac{\mathbb{P}(A_k|A_l)}{\mathbb{P}(A_l)} &= \frac{\mathbb{P}(A_k, A_l)}{\mathbb{P}(A_l)^2} \\ &= \exp\left(\lambda p^2 \int_{\mathbb{R}^2} \left(\frac{\theta_z \ell(x)}{1 + \theta_z \ell(x)}\right)^2 dx\right) \\ &> 1. \end{aligned} \tag{3.11}$$

When $\ell(x) = \|x\|^{-\alpha}$, we have

$$\frac{\mathbb{P}(A_k|A_l)}{\mathbb{P}(A_l)} = \exp\left(2\lambda\theta^{2/\alpha}\|z\|^2 p^2 \pi^2 \frac{\alpha - 2}{\alpha^2} \csc\left(\frac{2\pi}{\alpha}\right)\right). \tag{3.12}$$

In Figure we plot the conditional and the unconditional link success probabilities.

We make the following observations:

1. From (3.11), we observe that the link formation is correlated a cross time.
2. If a transmission succeeds at a time instant m , there is a higher probability that a transmission succeeds at a time instant n .
3. From (3.11), we also have $\mathbb{P}(A_k^c|A_l^c) > \mathbb{P}(A_l^c)$. So a link in outage is always more likely to be in outage and hence the retransmission strategy should reduce the rate of transmission or change the density of transmitters rather than retransmit "blindly".
4. We observe that $\frac{\mathbb{P}(A_k|A_l)}{\mathbb{P}(A_l)}$ always increases with θ, p . The increase in p is because of the larger transmit set due to which the probability of the same

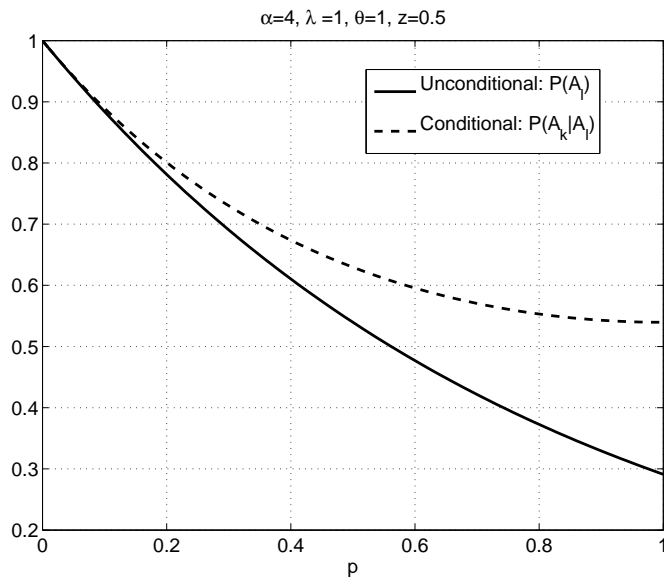


Figure 3.2. $\mathbb{P}(A_k|A_l)$ and $\mathbb{P}(A_l)$ versus the ALOHA parameter p . $\lambda = 1$, $\ell(x) = \|x\|^{-4}$, $z = 0.5$, $\theta = 1$.

sub-set of nodes transmitting at different times increases, thereby causing more correlation. When θ is large, the outage is a result of the interfering transmissions caused by a larger number of nodes. Hence by a similar reasoning as above, the correlation increases.

5. $\frac{\mathbb{P}(A_k|A_l)}{\mathbb{P}(A_l)}$ increases with λ when $\ell(x) = \|x\|^{-\alpha}$ since the link distance z is not scaled with λ . Indeed, by normalizing the link distance by $1/\sqrt{\lambda}$, we observe that (3.12) does not depend on λ .

3.4 Chapter Summary

We obtained the spatial and temporal correlations of interference in an ALOHA wireless network when the nodes are Poisson distributed. We can see that the path-loss model is critical in the spatial correlation and metrics like KL-distance

are required to understand the spatial dependence of the interference when the path-loss model is singular. We observe that that the link outages are temporally correlated and this information should be taken into account when analyzing ad hoc network performance and designing retransmission strategies.

CHAPTER 4

INTERFERENCE AND OUTAGE IN CLUSTERED NETWORKS.

4.1 System Model

The location of the transmitting nodes is modeled as a stationary and isotropic Poisson cluster process Φ on \mathbb{R}^2 . See Section 6.1.2 for more details. We further focus on more specific models for the representative cluster, namely Matern cluster processes and Thomas cluster processes. In these processes the number of points in the representative cluster is Poisson distributed with mean \bar{c} . For the Matern cluster process each point is uniformly distributed in a ball of radius σ around the origin. So the density function $\zeta(x)$ is given by

$$\zeta(x) = \begin{cases} \frac{1}{\pi\sigma^2}, & \|x\| \leq \sigma \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

In the Thomas cluster process each point is scattered using a symmetric normal distribution with variance σ^2 around the origin. So the density function $f(x)$ is given by

$$\zeta(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right).$$

A Thomas cluster process is illustrated in Fig.4.1. When the number of points in the representative cluster is Poisson with mean \bar{c} , as in the case of Matern

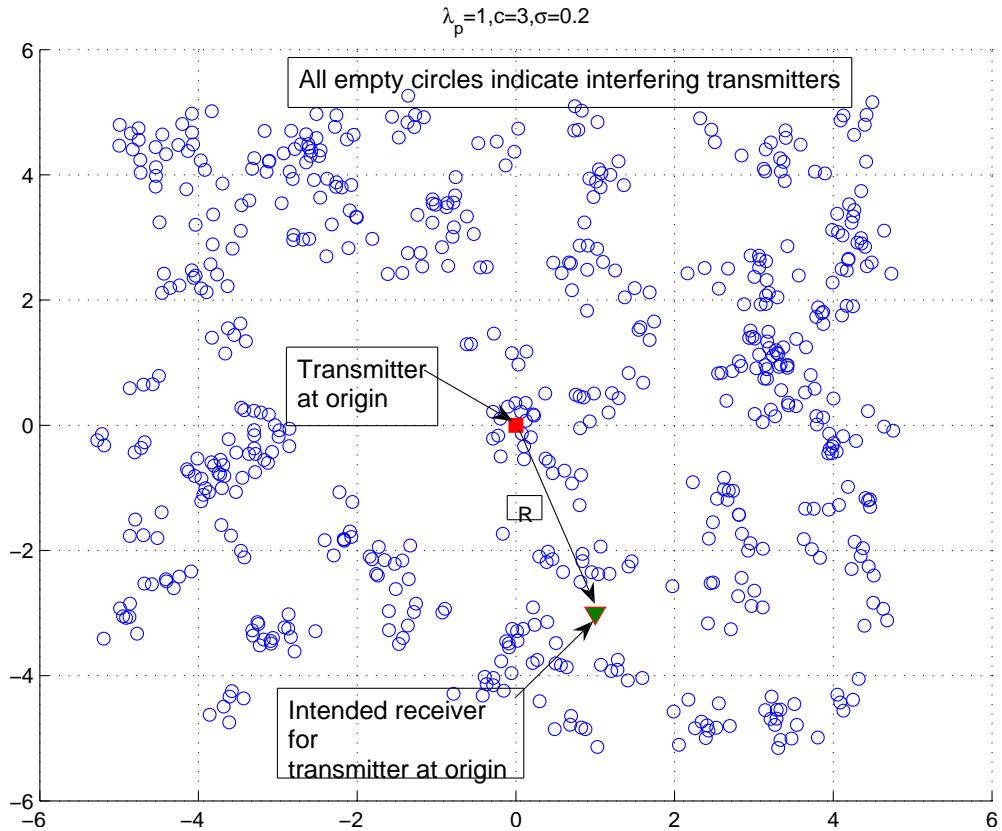


Figure 4.1. Illustration of transmitters and receivers. Cluster density is 1. Transmitter density in each cluster is 3. The spread of each cluster is Gaussian with standard deviation $\sigma = 0.25$. Observe that the intended receiver for the transmitter at the origin is not a part of the cluster process. The transmitter at the origin is a part of the cluster located around the origin.

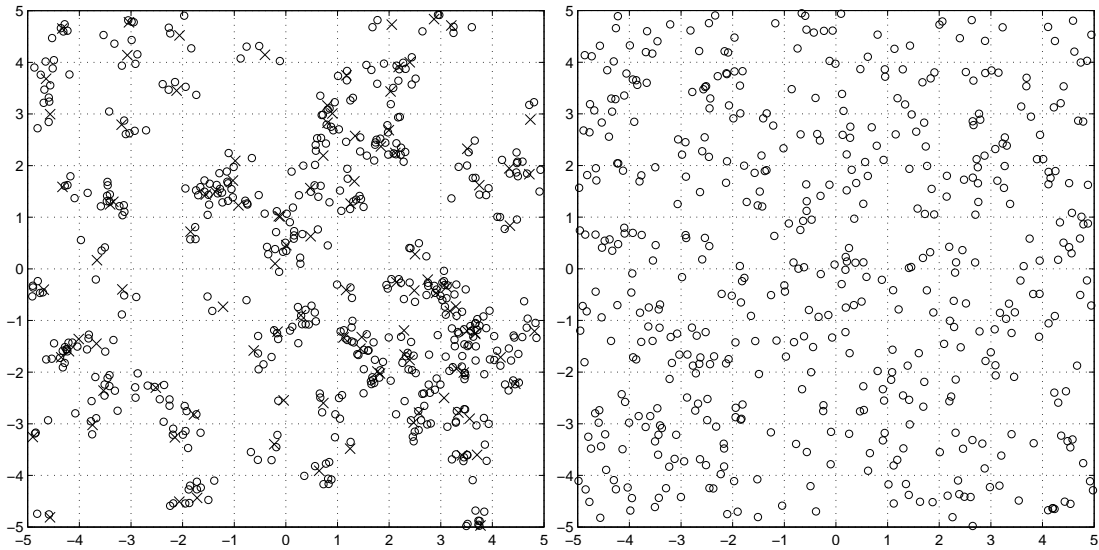


Figure 4.2. (Left) Thomas cluster process with parameters $\lambda_p = 1$, $\bar{c} = 5$ and $\sigma = 0.2$. The crosses indicate the parent points. (Right) PPP with the same intensity $\lambda = 5$ for comparison.

and Thomas cluster processes, the moment generating function of the number of points in the cluster is ,

$$M(z) = \exp(-\bar{c}(1 - z)).$$

The second order product density of the Matern and the Thomas cluster processes (3, Sec. 5.3) is

$$\rho^{(2)}(z) = \lambda_p \bar{c}^2 [(\zeta * \zeta)(z) + \lambda_p]. \quad (4.2)$$

See Section 6.1.2 and (3, Sec. 5.3) for a detailed description of clustered point processes.

4.2 Interference in Poisson Clustered Process (PCP)

In this section, we will derive the statistical characteristics of the interference when the transmitters are distributed as a PCP. As shall be evident in the next section, we will derive the interference properties conditioning on a transmitter being at the origin (the desired transmitter). Differently from a PPP, placing an additional transmitter at the origin (even though it does not contribute to the interference) makes the distribution of the interference non-stationary, *i.e.*, the interference $I_{\Phi}(z)$ distribution depends on z . From Lemma 1 the average interference (conditioned on the event that there is a point of the process at the origin is given by) is equal to

$$\mathbb{E}^{!o}[I_{\Phi}(z)] = \frac{\mathbb{E}[h]}{\lambda} \int_{\mathbb{R}^2} \ell(x-z) \rho^{(2)}(x) dx \quad (4.3)$$

Example: Thomas Cluster Process. In this case, from (3, Pg. 160)

$$\frac{\rho^{(2)}(x)}{\lambda^2} = 1 + \frac{1}{4\pi\lambda_p\sigma^2} \exp\left(\frac{-\|x\|^2}{4\sigma^2}\right)$$

where $\lambda = \lambda_p \bar{c}$. We obtain

$$\mathbb{E}^{!o}[I_{\Phi}(z)] = \mathbb{E}I_{\text{Poi}(\lambda)} + \frac{\bar{c}\mathbb{E}[h]}{4\pi\sigma^2} \int_{\mathbb{R}^2} \ell(x-z) \exp\left(\frac{-\|x\|^2}{4\sigma^2}\right) dx \quad (4.4)$$

Where $\mathbb{E}I_{\text{Poi}(\lambda)}$ is the average interference seen by a receiver located at z , when the nodes are distributed as a PPP with intensity λ . $\mathbb{E}I_{\text{Poi}(\lambda)}$ is finite only when $\ell(x)$ is bounded at the origin. The above expression also shows that the mean interference is indeed larger than for the PPP. We also observe that $\mathbb{E}^{!o}[I_{\Phi}(z)] \rightarrow \mathbb{E}I_{\text{Poi}(z)}$ when $\|z\| \rightarrow \infty$, which is intuitive since the effect of the

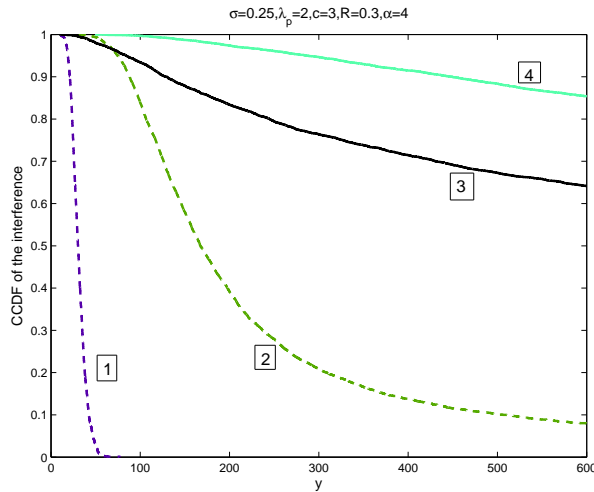


Figure 4.3. $\lambda_p = 2$, $\bar{c} = 3$, $\sigma = 0.25$, $\alpha = 4$, $\|z\| = 0.3$: Comparison of the interference CCDF for different path loss models and different fading.

cluster at the origin reduces as $\|z\|$ increases. From Chapter 2, we observe that the interference is heavy-tailed when $\ell(x) = \|x\|^{-\alpha}$ and the interference follows the fading for a bounded distribution. In Figure 4.3 curves #1 and #2 correspond to $\ell(x) = (1 + \|x\|^\alpha)^{-1}$. Curve #1 corresponds to Rayleigh fading and exhibits an exponential decay. Curve #2 for which h is distributed as generalized Pareto¹ with parameters $k = 1, \theta = 0, \sigma_p = 1$ (a hypothetical power fading distribution which exhibits power law decay) exhibits a power law decay. Curves #3 (generalized Pareto) and #4 (Rayleigh) correspond to $\ell(x) = \|x\|^{-\alpha}$ and exhibit a heavy tail for both fading distributions.

¹The PDF of the generalized Pareto distribution is $f(x) = \sigma^{-1} (1 + k \frac{x-\theta}{\sigma})^{-1-1/k}$

4.2.1 Conditional Probability Generating Functional of a PCP

Theorem 11. *Let $0 \leq v(x) \leq 1$. The conditional generating functional of the Thomas and Matern clustered processes is*

$$\tilde{\mathcal{G}}[v] = \mathcal{G}[v] \int_{\mathbb{R}^2} G_c[v(\cdot - y)]\zeta(y)dy,$$

where

$$G_c[v] = \exp\left(-\bar{c}\left[1 - \int_{\mathbb{R}^2} v(x)\zeta(x)dx\right]\right)$$

is the generating functional of the representative cluster for a Matern (or Thomas) process.

Proof. Let $Y_x = Y + x$. From Theorem 16, we have

$$\tilde{\Omega}^{!o}(Y) = \frac{1}{\bar{c}}\mathbb{E}\left(\sum_{x \in N_o} \mathbf{1}_{Y_x}(N_o \setminus \{x\})\right) \quad (4.5)$$

Let $\Omega(\cdot)$ denote the probability distribution of the representative cluster. Using the Campbell-Mecke theorem (3, Pg. 119), we get

$$\begin{aligned} \tilde{\Omega}^{!o}(Y) &= \frac{1}{\bar{c}} \int_{\mathbb{R}^2} \int_{\mathbf{N}} \mathbf{1}_{Y_x}(N_o) \Omega^{!x}(dN_o) \bar{c} \zeta(x) dx \\ &= \int_{\mathbb{R}^2} \int_{\mathbf{N}} \mathbf{1}_{Y_x}(N_o) \Omega^{!x}(dN_o) \zeta(x) dx \end{aligned} \quad (4.6)$$

Since the representative cluster has a Poisson distribution of points, by Slivnyak's theorem we have $\Omega_x^!(\cdot) = \Omega(\cdot)$. Hence

$$\begin{aligned} \tilde{\Omega}^{!o}(Y) &= \int_{\mathbb{R}^2} \int_{\mathbf{N}} \mathbf{1}_{Y_x}(N_o) \Omega(dN_o) \zeta(x) dx \\ &= \int_{\mathbb{R}^2} \Omega(Y_x) \zeta(x) dx \end{aligned} \quad (4.7)$$

For notational convenience let ψ denote N_o and let $\psi_y = \psi + y$. Let \mathbf{N} denote the set of finite and simple sequences on \mathbb{R}^2 (see Chapter 6). Using Theorem 15, we have

$$\begin{aligned}
\tilde{\mathcal{G}}[v] &= \int_{\mathbf{N}} \int_{\mathbf{N}} \prod_{x \in \phi \cup \psi} v(x) \mathbf{P}(\mathrm{d}\phi) \tilde{\Omega}^{!o}(\mathrm{d}\psi) \\
&= \int_{\mathbf{N}} \prod_{x \in \Phi} v(x) \mathbf{P}(\mathrm{d}\phi) \int_{\mathbf{N}} \prod_{x \in \psi} v(x) \tilde{\Omega}^{!o}(\mathrm{d}\psi) \\
&= \mathcal{G}[v] \int_{\mathbf{N}} \prod_{x \in \psi} v(x) \tilde{\Omega}^{!o}(\mathrm{d}\psi) \tag{4.8}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{G}}[v] &\stackrel{(a)}{=} \mathcal{G}(v) \int_{\mathbf{N}} \prod_{x \in \psi} v(x) \int_{\mathbb{R}^2} \Omega(\mathrm{d}\psi_y) \zeta(y) \mathrm{d}y \\
&= \mathcal{G}[v] \int_{\mathbb{R}^2} \int_{\mathbf{N}} \prod_{x \in \psi} v(x) \Omega(\mathrm{d}\psi_y) \zeta(y) \mathrm{d}y \\
&= \mathcal{G}[v] \int_{\mathbb{R}^2} \int_{\mathbf{N}} \prod_{x \in \psi} v(x - y) \Omega(\mathrm{d}\psi) \zeta(y) \mathrm{d}y \\
&\stackrel{(b)}{=} \mathcal{G}[v] \int_{\mathbb{R}^2} G_c[v(\cdot - y)] \zeta(y) \mathrm{d}y
\end{aligned}$$

(a) follows from (4.7), and (b) follows from the definition of $\mathcal{G}(\cdot)$. □

The above result can be interpreted as follows: The Palm measure of the clustered process is the independent superposition of the original process and a cluster at the origin which is randomly translated by an amount drawn from the density $\zeta(y)$. Hence the resultant conditional PGFL is the product of the PGFL of the clustered process and the PGFL of the representative cluster shifted by y

and averaged by the density $\zeta(y)$. So from the above lemma, we have

$$\begin{aligned} \mathcal{G}[v] &= \exp\left(-\lambda_p \int_{\mathbb{R}^2} 1 - M\left[\int_{\mathbb{R}^2} v(x+y)\zeta(y)dy\right] dx\right) \\ &\quad \times \int_{\mathbb{R}^2} M\left[\int_{\mathbb{R}^2} v(x-y)f(x)dx\right] \zeta(y)dy \end{aligned} \quad (4.9)$$

Since $\zeta(x) = \zeta(-x)$, then $\int_{\mathbb{R}^2} v(x+y)f(y)dy = \int_{\mathbb{R}^2} v(x-y)\zeta(y)dy = v * \zeta$, so

$$\mathcal{G}[v] = \exp\left(-\lambda_p \int_{\mathbb{R}^2} 1 - M[(v * \zeta)(x)]dx\right) \int_{\mathbb{R}^2} M[(v * \zeta)(y)]\zeta(y)dy \quad (4.10)$$

Likelihood and nearest neighbor functions of the Poisson cluster process, which involve similar calculations with Palm distributions are provided in (29). One can obtain the nearest-neighbor distribution functions of Thomas or Matern cluster process as $D(r) = \mathcal{G}(1_{B(o,r)^c}(\cdot))$. In some other cases the number of points per cluster may be fixed rather than Poisson.

We now derive the conditional Laplace transform in a Poisson cluster process, when the number of points in each cluster $\bar{c} \in \mathbb{N}$ is fixed. We also assume that each point is independently distributed with density $f(x)$. In this case the moment generating function of the number of points in the representative cluster is given by

$$M(z) = z^{\bar{c}}$$

As in the previous definition, we have

$$\begin{aligned}
\tilde{\mathcal{G}}[v] &= G[v] \int_{\mathbf{N}} \prod_{x \in \psi} v(x) \tilde{\Omega}^{!o}(\mathrm{d}\psi) \\
&= G[v] \int_{\mathbf{N}} \prod_{x \in \psi} v(x) \int_{\mathbb{R}^2} \Omega^{!y}(\mathrm{d}\psi_y) \zeta(y) \mathrm{d}y \\
&= G[v] \int_{\mathbb{R}^2} \int_{\mathbf{N}} \prod_{x \in \psi} v(x) \Omega^{!y}(\mathrm{d}\psi_y) \zeta(y) \mathrm{d}y \\
&\stackrel{(a)}{=} G[v] \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} v(x-y) \zeta(x) \right)^{\bar{c}-1} \zeta(y) \mathrm{d}y \tag{4.11}
\end{aligned}$$

where (a) follows from the fact that the points are independently distributed and we are not counting the point at the origin. In this case $G[v]$ is given by

$$G[v] = \exp \left\{ -\lambda_p \int_{\mathbb{R}^2} 1 - \left(\int_{\mathbb{R}^2} v(x+y) \zeta(y) \mathrm{d}y \right)^{\bar{c}} \mathrm{d}x \right\}.$$

4.3 Outage Analysis

Let the desired transmitter be located at the origin and the receiver at location z at distance $R = \|z\|$ from the transmitter. With a slight abuse of notation we shall be using R to denote the point $(R, 0)$. From Theorem 1, the probability of success for this pair is given by

$$\mathbb{P}_s = \mathbb{P}^{!o} \left(\frac{\mathbf{h}_{oz} \ell(z)}{\sigma^2 + I_{\Phi}(z)} \geq \theta \right) \tag{4.12}$$

We now assume Rayleigh fading, *i.e.*, the received power is exponentially distributed with mean μ . Then by Corollary 7, we have

$$\mathbb{P}_s = \mathcal{L}_{I_{\Phi}(z)}^{!o}(\theta/\ell(z)) \exp(-\sigma^2 \theta/\ell(z)), \tag{4.13}$$

Lemma 5. *[Success probability] The probability of successful transmission between the transmitter at the origin and the receiver located at $z \in \mathbb{R}^2$ with Rayleigh fading and $\sigma^2 = 0$ (no noise), is given by*

$$P_s = \underbrace{\exp \left\{ - \lambda_p \int_{\mathbb{R}^2} \left[1 - \exp(-\bar{c}\beta(z, y)) \right] dy \right\}}_{T_1} \underbrace{\int_{\mathbb{R}^2} \exp(-\bar{c}\beta(z, y)) \zeta(y) dy}_{T_2}$$

where

$$\beta(z, y) = \int_{\mathbb{R}^2} \frac{\ell(x - y - z)}{\frac{\ell(z)}{\theta} + \ell(x - y - z)} \zeta(x) dx \quad (4.14)$$

Proof. From (4.13) and Theorem 11. □

The success probability, when the number of nodes in each cluster is fixed, can be calculated by using (4.11) and (4.13).

Remarks:

Observe that the interference can be written as a sum of two independent terms, one being the interference caused at the receiver by the transmitter's own cluster and the other being the interference caused by other clusters.

$$I_{\Phi}(z) = I_{\Phi \setminus \text{Tx-cluster}}(z) + I_{\text{Tx-cluster}}(z) \quad (4.15)$$

Since we are considering a Poisson cluster process, these two terms are independent. The contribution of the interference in the success probability is $\mathcal{L}_{I_{\Phi}(z)}^{\circ}(\theta/\ell(z))$. Since the Laplace transform of the sum of independent random variables is the product of the individual Laplace transforms, we have the product of two terms in Theorem 5. The term T_1 in (4.12) captures the interference without the cluster at the origin (*i.e.*, without conditioning); it is independent of the position z since the original cluster process is stationary which can be verified by change of variables

$y_1 = y + z$. The second term T_2 is the contribution of the transmitter's cluster; it is identical for all z with $\|z\| = R$ since ζ and ℓ are isotropic (*i.e.*, do not depend on the direction). So the success probability itself is the same for all z at distance R . This is because the Palm distribution is always isotropic when the original distribution is motion-invariant (3, Eq. 4.4.8).

From the above argument we observe that $\mathbb{P}(\text{success})$ depends only on $\|z\| = R$ and not on the angle of z . So the success probability should be interpreted as an average over the circle $\|z\| = R$, *i.e.*, the receiver may be uniformly located anywhere on the circle of radius R around the origin. We now derive closed-form upper and lower bounds on $\mathbb{P}(\text{success})$.

Lemma 6 (Lower bound).

$$P_s \geq P_p(\lambda)P_p(\bar{c}\hat{\zeta}^*) \quad (4.16)$$

where $P_p(\lambda)$ denotes the success probability when Φ is a PPP, $\hat{\zeta}^* = \sup_{y \in \mathbb{R}^2} (\zeta * \zeta)(y)$, and $\lambda = \lambda_p \bar{c}$.

Proof. The first factor in (4.12), T_1 can be lower bounded by the success probability in the standard PPP $P_p(\lambda)$, and the second factor can be lower bounded by $P_p(\bar{c}\hat{\zeta}^*)$. From (4.12) and the fact that $1 - \exp(-\delta x) \leq \delta x, \delta \geq 0$, we have

$$P_s \geq \underbrace{\exp\left(-\lambda_p \bar{c} \int_{\mathbb{R}^2} \beta(R, y) dy\right)}_{\text{Term1}} \underbrace{\int_{\mathbb{R}^2} \exp(-\bar{c}\beta(R, y)) \zeta(y) dy}_{\text{Term2}}$$

$$\begin{aligned}
\text{Term1} &= \exp\left(-\lambda \int_{\mathbb{R}^2} \beta(R, y) dy\right) \\
&\stackrel{(a)}{=} \exp\left(-\lambda \int_{\mathbb{R}^2} \frac{\ell(y)}{\frac{\ell(R)}{\theta} + \ell(y)} dy\right) \\
&= P_p(\lambda)
\end{aligned} \tag{4.17}$$

(a) follows from change of variables, interchanging integrals and using $\int \zeta(x) dx = 1$.

$$\text{Term2} = \int_{\mathbb{R}^2} \exp(-\bar{c}\beta(R, y)) \zeta(y) dy$$

Since $\exp(-x)$ is convex and $\zeta(x) > 0$, $\int \zeta(x) dx = 1$, Using Jensen's inequality ($\mathbb{E}\zeta(x) \geq \zeta(\mathbb{E}(x))$) we have,

$$\text{Term2} \geq \exp\left(-\bar{c} \int_{\mathbb{R}^2} \beta(R, y) \zeta(y) dy\right)$$

Changing variables and using $\zeta(x) = \zeta(-x)$, we get,

$$\begin{aligned}
\text{Term2} &\geq \exp\left(-\bar{c} \int_{\mathbb{R}^2} \frac{\ell(x)}{\frac{\ell(R)}{\theta} + \ell(x)} \int_{\mathbb{R}^2} \zeta(x+z-y) \zeta(y) dy dx\right) \\
&\geq \exp\left(-\bar{c} \int_{\mathbb{R}^2} \frac{\ell(x)}{\frac{\ell(R)}{\theta} + \ell(x)} (\zeta * \zeta)(x+z) dx\right)
\end{aligned} \tag{4.18}$$

Hence

$$\text{Term2} \geq P_p(\bar{c}\hat{\zeta}^*)$$

□

Since $\zeta \in L_p$, by Young's inequality (30) we have $\hat{\zeta}^* \leq \|\zeta\|_p \|\zeta\|_q$, where $1/p + 1/q = 1$ (conjugate exponents). For $a \geq 1/\sqrt{\pi}$ (Matern) and $\sigma \geq 1/\sqrt{2\pi}$

(Thomas), we get $P_s \geq P_p(\lambda)P_p(\bar{c})$. In general, $\hat{\zeta}^* \leq \|\zeta\|_\infty \|\zeta\|_1$, which is $1/\pi a^2$ for Matern and $1/2\pi\sigma^2$ for Thomas processes. In the latter case, when ζ is Gaussian, $\zeta * \zeta$ is also Gaussian with variance $2\sigma^2$, hence $\hat{\zeta}^* \leq 1/4\pi\sigma^2$. From (12), we get (by change of variables):

$$P_p(\lambda) = \exp\left(-\lambda \int_{\mathbb{R}^2} \beta(R, y) dy\right). \quad (4.19)$$

We have

- for $\ell(x) = \|x\|^{-\alpha}$, $P_p(\lambda) = \exp(-\lambda R^2 \theta^{2/\alpha} C(\alpha))$, where $C(\alpha) = (2\pi\Gamma(2/\alpha)\Gamma(1-2/\alpha))/\alpha = \frac{2\pi^2}{\alpha} \csc(2\pi/\alpha)$. See (12).
- for $\ell(x) = (1 + \|x\|^\alpha)^{-1}$, $P_p(\lambda) = \exp(-\lambda\theta C(\alpha)(\theta + \ell(R))^{2/\alpha-1}\ell(R)^{-2/\alpha})$.

Let $\beta_I = \int_{\mathbb{R}^2} \beta(R, y) dy$, $\hat{\beta} = \sup_{y \in \mathbb{R}^2} \beta(R, y)$ and $\hat{\zeta} = \sup_{y \in \mathbb{R}^2} \zeta(y)$. By Hölders inequality we have $\hat{\beta} \leq \min\{1, \hat{\zeta}\beta_I(R)\}$. Also let $\kappa = \int_{\mathbb{R}^2} \beta(R, y)\zeta(y) dy$.

Lemma 7. [Upper bound]

$$P_s \leq P_p\left(\frac{\lambda}{1 + \bar{c}\hat{\beta}}\right) \quad (4.20)$$

Proof. Neglecting the second term T_2 and using $\exp(-\delta x) \leq 1/(1 + \delta x)$, we have

$$\begin{aligned} P_s &\leq \exp\left(-\lambda_p \int_{\mathbb{R}^2} \left[1 - \frac{1}{1 + \bar{c}\beta(R, y)}\right] dy\right) \\ &= \exp\left(-\lambda_p \int_{\mathbb{R}^2} \frac{\bar{c}\beta(R, y)}{1 + \bar{c}\beta(R, y)} dy\right) \\ &\leq \exp\left(-\frac{\lambda_p \bar{c}}{1 + \bar{c}\hat{\beta}} \int_{\mathbb{R}^2} \beta(R, y) dy\right) \end{aligned}$$

□

From the above two lemmata, we get

$$\mathbb{P}_p(\lambda)\mathbb{P}_p(\bar{c}\hat{\zeta}^*) \leq \mathbb{P}_s \leq \mathbb{P}_p\left(\frac{\lambda}{1+\bar{c}\hat{\beta}}\right) \quad (4.21)$$

from which follows $\mathbb{P}_s \rightarrow \mathbb{P}_p(\lambda)$ as $\frac{\bar{c}}{\sigma}, \frac{\bar{c}}{a} \rightarrow 0$ as expected. In Lemma 7, we have neglected the contribution of the transmitter's cluster. We derive the following upper bound in the proof of Lemma 9,

$$\mathbb{P}_s \leq \mathbb{P}_p(\lambda) \exp\left(\lambda\beta_I\nu(\bar{c}\hat{\beta})\right) \left[1 - \left(1 - \nu(\bar{c}\hat{\beta})\right) \bar{c}\kappa\right] \quad (4.22)$$

where $\nu(x) = (\exp(-x) - 1 + x)/x$. Substituting for $\nu(x)$, we have

$$\mathbb{P}_s \leq \mathbb{P}_p\left(\frac{\lambda(1 - \exp(-\bar{c}\hat{\beta}))}{\bar{c}\hat{\beta}}\right) \left[1 - \left(1 - \exp(-\bar{c}\hat{\beta})\right) \frac{\kappa}{\hat{\beta}}\right] \quad (4.23)$$

(4.23) is a tighter bound than the bound in Lemma 7, but not easily computable due to the presence of κ (for a given R, θ and σ , κ and β^* are constants). In (4.23), the outage due to the interference by the transmitting cluster is also taken into account.

The proof of Lemmata 6 and 7 also indicates that it is only by conditioning on the event that there is a point at the origin that the success probability of Neyman-Scott cluster processes can be lower than in the Poisson process of the same intensity. This implies that the cluster around the transmitter causes the maximum “damage”. So as the receiver moves away from the transmitter, the Neyman-Scott cluster process has a better success probability than the PPP. So, it is not true in general that cluster processes have a lower success probability than PPPs of the same intensity. For example from Figure 4.4, we see that for

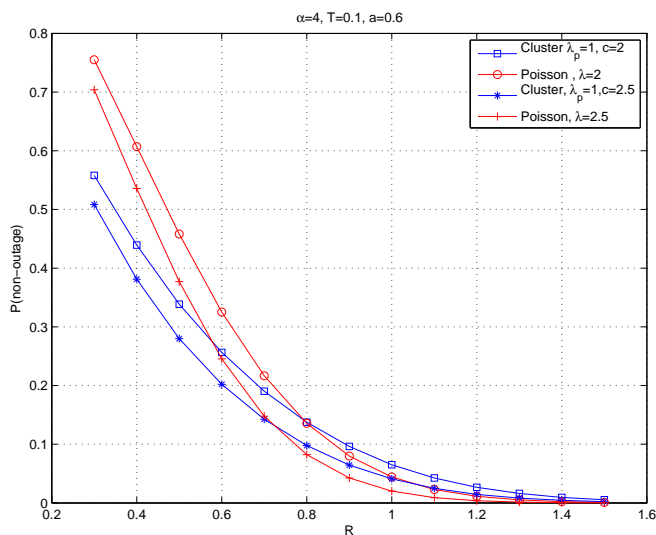


Figure 4.4. Comparison of success probability for cluster and Poisson process of intensity 2 and 2.5.

$R < 0.8$, the PPP has a better success probability than the Matern process. In Subsection 4.3.1 we give a more detailed analysis, which reveals that a PPP with intensity $\lambda_p \bar{c}$ has a lower success probability than a clustered process of the same intensity *for large transmit-receiver distances*. On the other hand, for small R , the success probability of the PPP is higher.

4.3.1 Clustering Gain $G(R)$

In this subsection we compare the performances of a clustered network and a Poisson network of the same intensity with Rayleigh fading. We deduce how the clustering gain depends on the transmitter receiver distance. We use the following

notation,

$$P_1(R, \bar{c}, \lambda_p) \triangleq \exp\left(-\lambda_p \int_{\mathbb{R}^2} 1 - \exp(-\bar{c}\beta(R, y)) dy\right) \quad (4.24)$$

$$P_2(R, \bar{c}) \triangleq \int_{\mathbb{R}^2} \exp(-\bar{c}\beta(R, y)) \zeta(y) dy \quad (4.25)$$

So $P_s = P_1(R, \bar{c}, \lambda_p)P_2(R, \bar{c})$. P_2 is the probability of success due to the presence of the cluster at the origin near the transmitter. P_1 is the probability of success in the presence of other clusters. Interference from these other clusters contributes more to the outage when R is large. This is also intuitive, since as the receiver moves away from the transmitting cluster, the interference from the other clusters starts to dominate. We define the *clustering gain* $G(R)$ as

$$G(R) = \frac{P_1(R, \bar{c}, \lambda_p)P_2(R, \bar{c})}{P_p(\lambda_p \bar{c})}$$

The fluctuation of $G(R)$ around unity indicates the existence of a crossover point R^* below which the PPP performs better than clustered process and vice versa. The values of $G(R)$ at the origin and infinity indicate the gain of scheduling transmitters as clusters instead of being spread uniformly on the plane. So it is beneficial to induce logical clustering of transmitters by MAC if $G(R) > 1$.

We first consider $G(R)$ for large R , *i.e.*, $\lim_{R \rightarrow \infty} G(R)$. By the dominated convergence theorem and (1.1), we have

$$\begin{aligned} \lim_{R \rightarrow \infty} P_2(R, \bar{c}) &= \int_{\mathbb{R}^2} \exp\left(-\bar{c} \int_{\mathbb{R}^2} \lim_{R \rightarrow \infty} \frac{\zeta(x)}{1 + \frac{\ell(R)}{\theta \ell(x-y-R)}} dx\right) \zeta(y) dy \\ &= \exp\left(\frac{-\bar{c}}{1 + \theta^{-1}}\right) \end{aligned} \quad (4.26)$$

Also from the derivation of upper bound we have $P_1(R, \bar{c}, \lambda_p) \leq P_p\left(\frac{\lambda}{1 + \bar{c}\beta}\right)$. Hence

from the definition of $P_p(x)$ we have, $\lim_{R \rightarrow \infty} P_1(R, \bar{c}, \lambda_p) = 0$. Hence for large R ,

$$P_1(R, \bar{c}, \lambda_p) < P_2(R, \bar{c}) \quad (4.27)$$

So for large R , most of the damage is done by transmitting nodes other than the cluster in which the intended transmitter lies.

Lemma 8.

$$\lim_{R \rightarrow \infty} \frac{P_p(\lambda_p \bar{c})}{P_1(R, \bar{c}, \lambda_p)} = 0 \quad (4.28)$$

Proof.

$$\begin{aligned} \frac{P_p(\lambda_p \bar{c})}{P_1(R, \bar{c}, \lambda_p)} &= \exp \left[-\lambda_p \bar{c} \int_{\mathbb{R}^2} \beta(R, y) dy + \lambda_p \int_{\mathbb{R}^2} (1 - \exp[-\bar{c}\beta(R, y)]) dy \right] \\ &= \exp \left[-\lambda_p \int_{\mathbb{R}^2} \underbrace{\{\bar{c}\beta(R, y) - 1 + \exp[-\bar{c}\beta(R, y)]\}}_{\nu(R, y)} dy \right] \end{aligned} \quad (4.29)$$

Since $1 - \exp(-ax) \leq ax$, we have that $\nu(R, y) > 0$. We also have from the dominated convergence theorem and (1.1)

$$\lim_{R \rightarrow \infty} \nu(R, y) = \frac{\bar{c}}{1 + \theta^{-1}} - 1 + \exp\left(-\frac{\bar{c}}{1 + \theta^{-1}}\right) > 0$$

which is a constant. So using Fatou's lemma (30) ($\liminf \int f_n \geq \int(\liminf f_n)$, $f_n > 0$), we have

$$\begin{aligned} \lim_{R \rightarrow \infty} \frac{P_p(\lambda_p \bar{c})}{P_1(R, \bar{c}, \lambda_p)} &= \exp\left[-\lambda_p \lim_{R \rightarrow \infty} \int_{\mathbb{R}^2} \nu(R, y) dy\right] \\ &\leq \exp\left[-\lambda_p \int_{\mathbb{R}^2} \lim_{R \rightarrow \infty} \nu(R, y) dy\right] \\ &= \exp[-\lambda_p \infty] = 0 \end{aligned} \quad (4.30)$$

□

Hence for large R , $\frac{P_p(\lambda_p \bar{c})}{P_1(R, \bar{c}, \lambda_p)} \leq \exp(\frac{-\bar{c}}{1+\theta-1})$. From (4.26) we have $P_p(\lambda_p \bar{c}) \leq P_1(R, \bar{c}, \lambda_p) P_2(R, \bar{c})$, for large R , *i.e.*, $G(R) > 1$ for large transmit-receive distance. We have $\lim_{R \rightarrow \infty} G(R) = \infty$. Hence the *Poisson point process with intensity $\lambda_p \bar{c}$, has a lower success probability than the clustered process of the same intensity for large link distances.*

For small R , $G(R)$ depends on the behavior of the path loss function, $\ell(x)$ at $\|x\| = 0$. We consider the two cases when the channel function is singular at the origin or not.

4.3.1.1 Singular Path-Loss Model: $\lim_{\|x\| \rightarrow 0} \ell(x) = \infty$

In this case we observe that $G(0) = 1$. But at small R , $G(R)$ is less than 1. We have the following lemma.

Lemma 9. *If $(\zeta * \zeta)(x) > \|x\|$ for small $\|x\|$ and $\ell(x) = \|x\|^{-\alpha}$, then for small R ,*

$$P_s \leq P_p(\lambda_p \bar{c}) \tag{4.31}$$

Proof. From (4.37), the probability of success is

$$P_s = P_p(\lambda_p \bar{c}) \underbrace{\exp \left[\lambda_p \bar{c} \int_{\mathbb{R}^2} \beta(R, y) \eta(\bar{c}, R, y) dy \right]}_{T_1} \underbrace{P_2(R, \bar{c})}_{T_2} \tag{4.32}$$

where $\eta(\bar{c}, R, y) = \nu(\bar{c} \beta(R, y))$ and

$$\nu(x) = \frac{\exp(-x) - 1 + x}{x}$$

an increasing function of x . From Young's inequality (30, Sec. 8.7) we have $\beta(R, y) \leq \min\{1, \sup\{\zeta(x)\}R^2\theta^{2/\alpha}C(\alpha)\}$. Hence

$$\eta(\bar{c}, R, y) \leq \nu(\bar{c} \min\{1, \sup\{\zeta(x)\}R^2\theta^{2/\alpha}C(\alpha)\})$$

With a slight abuse of notation, let $\eta(\bar{c}, R) = \nu(\bar{c} \min\{\sup\{\zeta(x)\}R^2\theta^{2/\alpha}C(\alpha), 1\})$.

Hence

$$\begin{aligned} T_1 &\leq \exp[\lambda_p \bar{c} \int_{\mathbb{R}^2} \beta(R, y) \eta(\bar{c}, R) dy] \\ &= \exp[\lambda_p \bar{c} \theta^{2/\alpha} R^2 \eta(\bar{c}, R) C(\alpha)] \end{aligned} \quad (4.33)$$

Also observe that $\eta(\bar{c}, R) \lesssim R^2$. So $T_1 \lesssim \exp(R^4)$.

$$\begin{aligned} T_2 &= \int_{\mathbb{R}^2} 1 - \bar{c}\beta(R, y) + \bar{c}\beta(R, y) \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} (\bar{c}\beta(R, y))^{k-1} \zeta(y) dy \\ &\leq \int_{\mathbb{R}^2} [1 - \bar{c}\beta(R, y) + \bar{c}\beta(R, y) \eta(\bar{c}, R)] \zeta(y) dy \\ &= 1 - [1 - \eta(\bar{c}, R)] \bar{c} \int_{\mathbb{R}^2} \beta(R, y) \zeta(y) dy \end{aligned} \quad (4.34)$$

If one considers x and y as identical and independent random variables with density functions ζ , we then have

$$\int_{\mathbb{R}^2} \beta(R, y) \zeta(y) dy = \mathbb{E} \left[\frac{1}{1 + \frac{\ell(R)}{\theta} \|x - y - R\|^\alpha} \right].$$

Let $0 < \kappa < 1$ be some constant. Using the Chebyshev inequality we get

$$\begin{aligned} \mathbb{E} \left[\frac{1}{1 + \frac{\ell(R)}{\theta} \|x - y - R\|^\alpha} \right] &\geq \kappa \mathbb{P} \left[\frac{1}{1 + \frac{\ell(R)}{\theta} \|x - y - R\|^\alpha} \geq \kappa \right] \\ &= \kappa \mathbb{P} \left[\|x - y - R\| \leq \left(\frac{1}{\kappa} - 1 \right)^{1/\alpha} R \theta^{1/\alpha} \right] \end{aligned}$$

The PDF of $z = x - y$ is given by $(\zeta * \zeta)(z)$, since y is rotation-invariant. Choosing $\kappa = \theta/(1 + \theta)$ we have

$$\begin{aligned} \kappa \mathbb{P} \left[\|x - y - R\| \leq \left(\frac{1}{\kappa} - 1 \right)^{1/\alpha} R \theta^{1/\alpha} \right] &= \frac{\theta}{1 + \theta} \int_{B(R, R)} (\zeta * \zeta)(x) dx \\ &\geq \frac{\theta}{1 + \theta} \int_{B(R, R)} \|x\| dx \\ &= R^3 \underbrace{\frac{\theta}{1 + \theta} \int_{B(1, 1)} \|x\| dx}_{C_2} \quad (4.35) \end{aligned}$$

So we have

$$\begin{aligned} P_2 &\leq 1 - [1 - \eta(\bar{c}, R)] R^3 C_2 \\ &\lesssim 1 - R^3 + R^5 \quad (4.36) \end{aligned}$$

Also we have $T_1 \lesssim \exp(R^4) \lesssim 1 + 1.01R^4$. So we have $P_2 T_1 \lesssim 1 - R^3 + R^5 - 1.01R^7 + 1.01R^9 < 1$ for small $R \neq 0$. Hence for small R we have $P_s \leq P_p(\lambda_p \bar{c})$.

□

Note that $\zeta(x)$ for Matern and Thomas cluster process have the required property. Hence when $\ell(x) = \|x\|^{-\alpha}$, the PPP with intensity $\lambda_p \bar{c}$, has a higher success probability than the clustered process of the same intensity for small transmit re-

ceiver distance. Lemma 9 and the fact that $G(\infty) = \infty$ also indicate the existence of a crossover point R^* between the success curves of the PPP and the cluster process. So it is not true in general that the performance of the clustered process is better or worse than that of the Poisson process. This is because, for the same intensity, a clustered process will have clusters of transmitters (where interference is high) and also vacant areas (where there are no transmitters and interference is low), whereas in a Poisson process, the transmitters are uniformly spread.

4.3.1.2 Bounded Path-Loss Model: $\lim_{\|x\| \rightarrow 0} \ell(x) = \hat{\ell} < \infty$

$P_1(R, \bar{c}, \lambda_p)$ can be written as

$$P_1(R, \bar{c}, \lambda_p) = P_p(\lambda_p \bar{c}) \exp \left(\lambda_p \underbrace{\int_{\mathbb{R}^2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \bar{c}^n \beta(R, y)^n dy}_{>0} \right)$$

Hence $G(R)$ can also be written as follows

$$G(R) = P_2(R, \bar{c}) \exp \left(\lambda_p \bar{c} \int_{\mathbb{R}^2} \beta(R, y) \eta(\bar{c}, R, y) dy \right) \quad (4.37)$$

where $\eta(\bar{c}, R, y) = \nu(\bar{c}\beta(R, y))$, with $\nu(x) = (\exp(-x) - 1 + x)/x$. Observe that $0 \leq \eta(\bar{c}, R, y) \leq 1, \forall x > 0$. If the total density of the transmitters is fixed *i.e.*, $\lambda = \lambda_p \bar{c}$ is constant, how does $G(R)$ behave with respect to \bar{c} ? We have the following lemma which characterizes the monotonicity of $G(R)$ with respect to \bar{c} .

Lemma 10. *Given $\lambda = \lambda_p \bar{c}$ is constant, $G(R)$ is decreasing with \bar{c} , i.e., $\frac{dG(R)}{d\bar{c}} \leq 0$, $\forall \bar{c} > 0$ iff $\lambda \leq \lambda^*(R, T)$, where*

$$\lambda^*(R, T) = \frac{2 \int_{\mathbb{R}^2} \beta(R, y) \zeta(y) dy}{\int_{\mathbb{R}^2} \beta(R, y)^2 dy}$$

Proof. From (4.37),

$$\begin{aligned} G(R) &= P_2(R, \bar{c}) \exp \left[\lambda_p \bar{c} \int_{\mathbb{R}^2} \beta(R, a) \eta(\bar{c}, R, a) da \right] \\ &= \int_{\mathbb{R}^2} \exp \left(-\bar{c} \beta(R, y) + \lambda \int_{\mathbb{R}^2} \beta(R, a) \eta(\bar{c}, R, a) da \right) \zeta(y) dy \quad (4.38) \end{aligned}$$

We have $\frac{d\eta(\bar{c}, R, z)}{d\bar{c}}|_{\bar{c}=0} = \beta(R, z)/2$ and $\frac{d\eta(\bar{c}, R, z)}{d\bar{c}}$ is decreasing in \bar{c} .

$$\begin{aligned} \frac{dG(R)}{d\bar{c}} &= \int_{\mathbb{R}^2} \left[-\beta(R, y) + \lambda \int_{\mathbb{R}^2} \beta(R, a) \frac{d\eta(\bar{c}, R, a)}{d\bar{c}} da \right] \\ &\quad \exp \left(-\bar{c} \beta(R, y) + \lambda \int_{\mathbb{R}^2} \beta(R, a) \eta(\bar{c}, R, a) da \right) \zeta(y) dy \\ &= \exp \left[\lambda \int_{\mathbb{R}^2} \beta(R, a) \eta(\bar{c}, R, a) da \right] \\ &\quad \underbrace{\int_{\mathbb{R}^2} \left[-\beta(R, y) + \lambda \int_{\mathbb{R}^2} \beta(R, z) \frac{d\eta(\bar{c}, R, a)}{d\bar{c}} da \right] \exp \left(-\bar{c} \beta(R, y) \right) \zeta(y) dy}_{T_2(\bar{c})} \end{aligned}$$

Since $\eta'(\bar{c}, R, z)$ is decreasing in \bar{c} , we have $T_2(\bar{c})$ is decreasing in \bar{c} . So a necessary and sufficient condition for $\frac{dG(R)}{d\bar{c}} \leq 0 \forall \bar{c} > 0$ is $T_2(0) \leq 0$. We want

$$\begin{aligned} T_2(0) &= \int_{\mathbb{R}^2} \left[-\beta(R, y) + \frac{\lambda}{2} \int_{\mathbb{R}^2} \beta^2(R, z) dz \right] \zeta(y) dy \leq 0 \\ \Rightarrow \lambda &\leq \frac{2 \int_{\mathbb{R}^2} \beta(R, y) \zeta(y) dy}{\int_{\mathbb{R}^2} \beta^2(R, z) dz} \quad (4.39) \end{aligned}$$

□

Remarks:

1. Since $\beta(0, y) \neq 0$, we have that, $G(0)$ is increasing with λ_p (like $\exp(\lambda_p)$), and hence will be greater than 1 at some λ_p for a fixed \bar{c} .
2. We have $\lim_{\bar{c} \rightarrow 0} G(R) = 1$ and specifically $G(0) = 1$ at $\bar{c} = 0$.

3. From Lemma 10 and Remark 2 we can deduce $G(R) < 1, \forall \bar{c} > 0$ if $\lambda < \lambda^*(R, \theta)$ *i.e.*, the gain $G(R)$ decreases from 1 with increasing \bar{c} if the total intensity of transmitters is less than $\lambda^*(R, \theta)$.
4. Since $G(R)$ is continuous with respect to R , $G(R)$ is close to $G(0)$ for small R .
5. From Figure 4.6, we observe that $G(R)$ increases monotonically with R .

In Figure 4.5, $\lambda^*(0, \theta)$ is plotted against θ . We provide some heuristics as to when logical clustering does not perform better than a uniform distribution of points:

- The exact value of R at which $G(R)$ crosses 1 is difficult to find analytically due to the highly nonlinear nature of $G(R)$. If such a crossover point exists (depends on the path-loss model) we will denote it by R^* .
- If $\ell(x) = \|x\|^{-\alpha}$, it is better to induce logical clustering by the MAC scheme if the link distance is larger than R^* . Otherwise it is better to schedule the transmissions so that they are scattered uniformly on the plane.
- If $\ell(0) < \infty$ and for a constant intensity $\lambda_p \bar{c}$, it is always beneficial to induce clustering for long-hop transmissions. When R is small the answer depends on the total intensity $\lambda_p \bar{c}$. If $\lambda_p \bar{c} < \lambda^*(0, \theta)$ then $G(0) < 1$ by observation 3, and hence $G(R) < 1$ for *small* R by observation 4. Also when $\lambda_p \bar{c} < \lambda^*(0, \theta)$, it is better to reduce logical clustering by decreasing \bar{c} and increasing λ_p , since $G(0)$ is a decreasing function of \bar{c} . From Figure 4.5 we observe that $\lambda^*(0, 0.5) \approx 1.26$ when $\ell(x) = (1 + \|x\|^4)^{-1}$ and $\sigma = 0.25$. In Figure 4.6, $G(R)$ is plotted for $\lambda_p \bar{c} = 0.75, 9$ for the same values of σ, α and the same channel function as of Figure 4.5. When $\lambda_p \bar{c} = 9 > \lambda^*(0, 0.5)$,

we observe that the gain curve $G(R)$ is approximately 10 at the origin and increases. When $\lambda_p \bar{c} = 0.75 < \lambda^*(0, 0.5)$, $G(R)$ starts around 0.25 and crosses 1 at $R \approx 1.2$. We also observe that $G(R)$, for the non-singular $\ell(x)$, seem to increase monotonically. We also observe that the gain function for $\ell(x) = \|x\|^{-\alpha}$ decreases from 1 initially and then increases to infinity.

- For DS-CDMA, the value of θ is smaller by a factor equal to the spreading gain. From Figure 4.5, we observe that the threshold $\lambda^*(0, \theta)$ for clustering to be beneficial at small distances increases with decreasing θ . Hence for a constant intensity of transmissions $\lambda_p \bar{c}$, the benefit of clustering decreases with increasing spreading gain for small link distances. So for DS-CDMA (for a large spreading gain) it is better to make the transmissions uniform on the plane for smaller link distances and cluster the transmitters for long-range communication.
- For FH-CDMA, the total number of transmissions $\lambda_p \bar{c}$ is reduced by the spreading gain while θ remains constant (see Figure 4.5). Hence $\lambda_p \bar{c} < \lambda^*(0, \theta)$ for small distances and one can draw similar conclusions as for DS-CDMA. The relative gain between FH-CDMA and DS-CDMA with clustering is more difficult to characterize analytically.

4.4 Transmission Capacity of Clustered Transmitters

Transmission capacity was introduced in (15; 17; 31) and is defined as the product of the maximum density of successful transmissions and their data rate, given an outage constraint. More formally, if the intensity of the contending transmitters is λ with an outage threshold θ and a bit rate b bits per second per

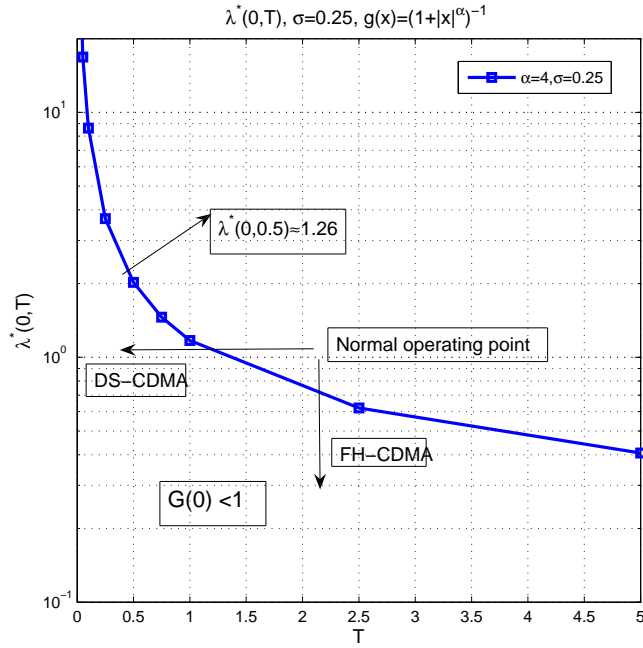


Figure 4.5. $\lambda^*(0, \theta)$ versus θ for $\ell(x) = (1 + \|x\|^4)^{-1}$, $\sigma = 0.25$. The region below the curve consists of all the pairs of $(\theta, \lambda = \lambda_p \bar{c})$ such that $G(0) < 1$. “Normal operating point” denotes a pair (θ, λ) that lies above the curve $(\theta, \lambda^*(0, \theta))$. Suppose we use FH-CDMA, the total intensity decreases by a factor of spreading gain and hence we move vertically downwards into the $G(0) < 1$ region. If DS-CDMA is used, the threshold θ decreases by a factor of spreading gain and hence we move horizontally towards the left into the $G(0) < 1$ region.

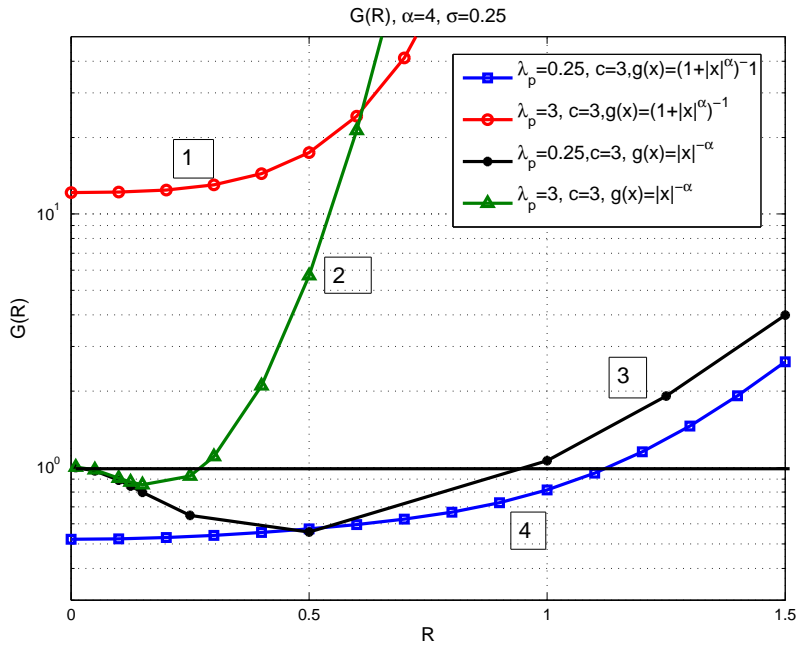


Figure 4.6. $G(R)$ versus R , $\alpha = 4, \sigma = 0.25$. Observe that the gain curves #2 and #3, which correspond to the singular channel, start at 1 decrease and then increase above unity. For the gain curve #4, the total intensity of transmitters is $3 * 0.25 = 0.75$ which is less than the threshold $\lambda^*(0, 0.5) \approx 1.26$. Hence the gain curve for this case starts below unity at $R = 0$ and then increases. For the gain curve #1 the total intensity is $9 > 1.26$. By chance, in the present case the gain curve #1 starts around 10 and increases.

hertz, then the transmission capacity at a fixed distance R is given by

$$C(\epsilon, \theta) = b(1 - \epsilon) \sup_{\lambda} \{ \lambda : \mathbf{P}_s(\lambda, \theta) \geq 1 - \epsilon \} \quad (4.40)$$

where $\mathbf{P}_s(\lambda, \theta)$ denotes the success probability of a given transmitter receiver pair (explicit inclusion of the dependence on θ and λ). More discussion about the transmission capacity and its relation to other metrics like transport capacity is provided in (31). Note that the results proved in (15; 17; 31) are for Poisson arrangement of transmitters.

In this section we evaluate the transmission capacity when the transmitters are arranged as a Poisson cluster process. We prove that for small values of ϵ , the transmission capacity of the clustered process coincides with that of the Poisson arrangement of nodes. We also show that care should be taken in defining transmission capacity for general distribution of nodes. For notational convenience we shall assume $b = 1$. For the clustered process, $\mathbf{P}_s(\lambda, \theta)$ denotes the success probability of the cluster process with intensity $\lambda = \lambda_p \bar{c}$ and threshold θ . Let $\mathbf{P}_l(\lambda, \theta), \mathbf{P}_u(\lambda, \theta)$ denote lower and upper bounds of the success probability $\mathbf{P}_s(\lambda, \theta)$ and the corresponding sets A_l, A_u defined by $A_\chi := \{ \lambda : \mathbf{P}_\chi(\lambda, \theta) \geq 1 - \epsilon \}$ for $\chi \in \{l, u\}$. We then have $A_l \subset A \subset A_u$ which implies

$$\sup A_l \leq \sup A \leq \sup A_u. \quad (4.41)$$

Let $C_l(\epsilon, \theta) = \sup A_l$ and $C_u(\epsilon, \theta) = \sup A_u$ denote lower and upper bounds to the transmission capacity.

For a PPP we have from (4.19), $\mathbf{P}_p(\lambda, \theta) = \exp(-\lambda\beta_I)$ (β_I does not depend on

λ). Hence the transmission capacity of a PPP denoted by $C_p(\epsilon, \theta)$ is given by

$$\begin{aligned} C_p(\epsilon, \theta) &= \frac{1 - \epsilon}{\beta_I} \ln \left(\frac{1}{1 - \epsilon} \right) \\ &\approx \frac{\epsilon(1 - \epsilon)}{\beta_I}, \quad \epsilon \ll 1 \end{aligned} \quad (4.42)$$

For Neyman-Scott cluster processes, the intensity $\lambda = \lambda_p \bar{c}$. We first to try to consider both λ_p and \bar{c} as optimization parameters for the transmission capacity, i.e.

$$C(\epsilon, \theta) := (1 - \epsilon) \sup\{\lambda_p \bar{c} : \lambda_p > 0, \bar{c} > 0, \text{outage-constraint}\} \quad (4.43)$$

without individually constraining the parent node density or the average number of nodes per cluster.

Lemma 11. *The transmission capacity of Poisson clustered processes is lower bounded by the transmission capacity of the PPP,*

$$C(\epsilon, \theta) \geq C_l(\epsilon, \theta) = C_p(\epsilon, \theta) \quad (4.44)$$

Proof. From Lemma 6, we have $P_l(\lambda, \theta) = P_p(\lambda_p \bar{c}) P_p(\bar{c} \hat{\zeta}^*)$. So to get a lower bound, from (4.41) we have to find

$$\sup \left\{ \lambda_p \bar{c} : \lambda_p \bar{c} + \bar{c} \hat{\zeta}^* \leq \frac{1}{\beta_I} \ln \left(\frac{1}{1 - \epsilon} \right) = \frac{C_p(\epsilon, \theta)}{1 - \epsilon} \right\} \quad (4.45)$$

This maximum value of $\lambda_p \bar{c}$ is attained when, $\lambda_p \rightarrow \infty$, while $\bar{c} \rightarrow 0$, such that $\bar{c} \lambda_p = C_p(\epsilon, \theta)(1 - \epsilon)^{-1}$. So we have $C_l(\epsilon, \theta) = C_p(\epsilon, \theta)$. \square

Also observe that $\lambda_p \rightarrow \infty$ and $\bar{c} \rightarrow 0$. This corresponds to the scenario in

which the clustered process degenerated to a PPP. We also have the following upper bound.

Lemma 12. *Let $\rho(\theta) = k/\hat{\beta}$ with $k = \int \beta(R, y)\zeta(y)dy$. For $\epsilon < 1 - e^{-\rho(\theta)}$, we have*

$$C(\epsilon, \theta) \leq C_u(\epsilon, \theta) = C_p(\epsilon, \theta) \quad (4.46)$$

Proof. We find $C_u(\epsilon, \theta)$ and hence upper bound the transmission capacity. We have from the derivation of Lemma 9

$$P_s(\lambda, \theta) \leq P_p(\lambda_p \bar{c}) \exp[\lambda_p \bar{c} \beta_I \eta(\bar{c}, R)] P_2(R, \bar{c}) = P_u(\bar{c} \lambda_p, \theta), \quad (4.47)$$

where $\eta(\bar{c}, R) = (\exp(-\bar{c}\hat{\beta}) - 1 + \bar{c}\hat{\beta})/\bar{c}\hat{\beta}$. With $A_u = \{\lambda_p \bar{c}, P_u(\lambda_p \bar{c}, \theta) \geq 1 - \epsilon\}$, it is sufficient to prove $\sup A_u \leq C_p(\epsilon, \theta)$. Also observe that $P_u(\bar{c} \lambda_p, \theta) \rightarrow 0$ as $\bar{c} \rightarrow \infty$ independent of λ_p . Hence we can assume \bar{c} is finite for the proof. We proceed by contradiction.

Let $\sup A_u > C_p(\epsilon, \theta)$. Hence there exists a $\delta > 0$, $\lambda_p \geq 0$, $\bar{c} \geq 0$ such that $\lambda_p \bar{c} = (C_p(\epsilon, \theta)/(1 - \epsilon)) + \delta \in A_u$. At this value of $\lambda_p \bar{c}$ we have

$$\begin{aligned} 1 - \epsilon \leq P_u(\bar{c} \lambda_p, \theta) &= (1 - \epsilon) P_p(\delta) \exp \left[\eta(\bar{c}, R) \left\{ \ln \left(\frac{1}{1 - \epsilon} \right) + \delta \beta_I \right\} \right] P_2(R, \bar{c}) \\ &= (1 - \epsilon)^{1 - \eta(\bar{c}, R)} \underbrace{P_p(\delta(1 - \eta(\bar{c}, R)))}_{T_1} P_2(R, \bar{c}) \end{aligned} \quad (4.48)$$

From the derivation of Lemma 9, we have $P_2(R, \bar{c}) \leq 1 - [1 - \eta(\bar{c}, R)]\bar{c}k$, with equality only when $\bar{c} = 0$. Hence we have

$$P_u(\bar{c} \lambda_p, T) \leq T_1 (1 - \epsilon)^{1 - \eta(\bar{c}, R)} (1 - [1 - \eta(\bar{c}, R)]\bar{c}k) \quad (4.49)$$

Since $\frac{\exp(-x)-(1-x)}{x} \leq \frac{x}{1+x}$, we have $\eta(\bar{c}, R) \leq \bar{c}\hat{\beta}/(1 + \bar{c}\hat{\beta})$. Using the upper bound for $\eta(\bar{c}, R)$

$$\begin{aligned} \mathbf{P}_u(\bar{c}\lambda_p, \theta) &\leq T_1(1 - \epsilon)(1 - \epsilon)^{-\frac{\bar{c}\hat{\beta}}{1+\bar{c}\hat{\beta}}} \left(1 - \left[1 - \frac{\bar{c}\hat{\beta}}{1 + \bar{c}\hat{\beta}} \right] \bar{c}\hat{\beta}\rho(\theta) \right) \\ &= T_1(1 - \epsilon) \underbrace{(1 - \epsilon)^{-\frac{\bar{c}\hat{\beta}}{1+\bar{c}\hat{\beta}}} \left(1 - \frac{\bar{c}\hat{\beta}\rho(\theta)}{1 + \bar{c}\hat{\beta}} \right)}_{T_2} \end{aligned} \quad (4.50)$$

Using the inequality $1 - ay \leq (1 - b)^y$, $b \leq 1 - e^{-a}$, $y \geq 0$, substituting $y = \frac{\bar{c}\hat{\beta}}{1+\bar{c}\hat{\beta}}$, $b = \epsilon$, $a = \rho(\theta)$, we get $T_2 \leq 1$. Hence we have

$$\mathbf{P}_u(\bar{c}\lambda_p, \theta) \leq (1 - \epsilon)\mathbf{P}_p(\delta(1 - \eta(\bar{c}, R))) \quad (4.51)$$

So if $\delta > 0$, and \bar{c} finite, we also have $\mathbf{P}_p(\delta(1 - \eta(\bar{c}, R))) < 1$. So we have a contradiction from (4.48) and (4.51). Hence there exists no such δ and hence $\sup A_u \leq C_p(\epsilon, \theta)$. We can achieve $C_u(\epsilon, \theta) = C_p(\epsilon, \theta)$, by using $\lambda_p = n \frac{C_p(\epsilon, \theta)}{1 - \epsilon} - 1$, $\bar{c} = 1/n$ for n very large. As $n \rightarrow \infty$, $\mathbf{P}_u(\bar{c}\lambda_p, \theta) \rightarrow \mathbf{P}_p(\bar{c}\lambda_p, \theta)$. \square

Theorem 10. For $\epsilon \leq 1 - e^{-\rho(\theta)}$ we have $C(\epsilon, \theta) = C_p(\epsilon, \theta)$.

Proof. Follows from Lemmata 11 and 12. \square

From the above two proofs, when ϵ is small, the transmission capacity is equal to the Poisson process of same intensity. This capacity is achieved when $\lambda_p \rightarrow \infty$ and $\bar{c} \rightarrow 0$. This is the scenario in which the cluster process becomes a PPP. This is due to the definition of the transmission capacity as $C(\epsilon, \theta) := \sup\{\lambda_p \bar{c} : \lambda_p > 0, \bar{c} > 0, \text{outage-constraint}\}$ where we have two variables to optimize over.

Instead we may fix λ_p as constant and find the transmission capacity with

respect to \bar{c} . So we define constrained transmission capacity as

$$C^*(\epsilon, \theta) := \lambda_p(1 - \epsilon) \sup\{\bar{c} : \bar{c} > 0, \text{outage-constraint}\} \quad (4.52)$$

We have the following bounds for $C^*(\epsilon, \theta)$

Theorem 11.

$$\frac{\lambda_p C_p(\epsilon, \theta)}{\lambda_p + \hat{\zeta}^*} \leq C^*(\epsilon, \theta) \leq \frac{\lambda_p C_p(\epsilon, \theta)}{\max\left\{0, \lambda_p - \frac{\hat{\beta}}{\beta_I} \ln\left(\frac{1}{1-\epsilon}\right)\right\}} \quad (4.53)$$

Proof. From the lower bound on P_s , we have to find

$$\sup\left\{\bar{c} : \lambda_p \bar{c} + \bar{c} \hat{\zeta}^* \leq \frac{1}{\beta_I} \ln\left(\frac{1}{1-\epsilon}\right) = \frac{C_p(\epsilon, \theta)}{1-\epsilon}\right\} \quad (4.54)$$

So we have $C_l^*(\epsilon, \theta) = C_p(\epsilon, \theta)/(\hat{\zeta}^* + \lambda_p)$.

From the upper bound on P_s , we have to find

$$\sup\left\{\bar{c} : \frac{\lambda_p \bar{c}}{1 + \bar{c} \hat{\beta}} \leq \frac{1}{\beta_I} \ln\left(\frac{1}{1-\epsilon}\right) = \frac{C_p(\epsilon, \theta)}{1-\epsilon}\right\} \quad (4.55)$$

□

One can also derive an order approximation to the constrained transmission capacity when ϵ is very small. We have the following order approximation to transmission capacity.

Proposition 2. *When λ_p is fixed, the constrained transmission capacity is given by*

$$C^*(\epsilon, \theta) = (1 - \epsilon) \left(\frac{\epsilon \lambda_p}{\lambda_p \beta_I + k} + o(\epsilon) \right) \quad (4.56)$$

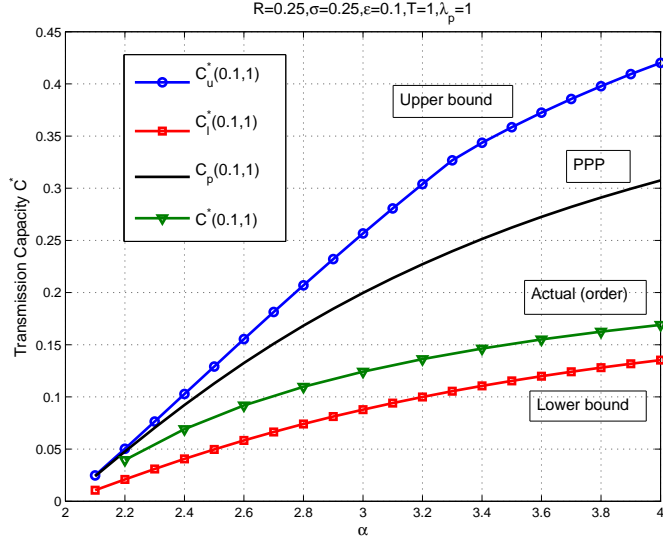


Figure 4.7: Upper and lower bounds of $C^*(\epsilon, \theta)$ versus α , $\ell(x) = \|x\|^{-\alpha}$, $\theta = 1$, $\sigma = 0.25$, $\epsilon = 0.1$, $\lambda_p = 1$

when $\epsilon \rightarrow 0$.

Proof. Let $\gamma(\bar{c})$ denote the outage probability, *i.e.*,

$$\gamma(\bar{c}) = 1 - \exp \left\{ -\lambda_p \int_{\mathbb{R}^2} 1 - \exp[-\bar{c}\beta(R, y)] dy \right\} \int_{\mathbb{R}^2} \exp(-\bar{c}\beta(R, y)) \zeta(y) dy$$

We have $d\gamma(\bar{c})/d\bar{c} > 0$, which implies $\gamma(\bar{c})$ is increasing and invertible and hence $C^*(\epsilon, \theta) = \lambda_p(1 - \epsilon)\gamma^{-1}(\epsilon)$. We approximate $\gamma^{-1}(\epsilon)$ for small ϵ by the Lagrange inversion theorem. Observe that $\gamma(\bar{c})$ is a smooth function of \bar{c} and all derivatives exist. Expanding $\gamma^{-1}(\epsilon)$ around $\epsilon = 0$ by the Lagrange inversion theorem and

using $\gamma(0) = 0$ yields

$$\begin{aligned}
\gamma^{-1}(\epsilon) &= \sum_{n=1}^{\infty} \frac{d^{n-1}}{d\bar{c}^{n-1}} \left(\frac{\bar{c}}{\gamma(\bar{c})} \right)^n \Big|_{\bar{c}=0} \frac{\epsilon^n}{n!} \\
&= \frac{\bar{c}\epsilon}{\gamma(\bar{c})} \Big|_{\bar{c}=0} + o(\epsilon) \\
&\stackrel{(a)}{=} \frac{\epsilon}{\lambda_p \beta_I + k} + o(\epsilon)
\end{aligned} \tag{4.57}$$

where (a) follows by applying de L'Hôpital's rule. \square

We have the following observations

1. The constrained transmission capacity increases (slowly) with λ_p .
2. We also observe that the constrained transmission capacity for the cluster process is always less than that of a Poisson network (see Figure 4.7) and approaches $C_p(\epsilon, \theta)$ as $\lambda_p \rightarrow \infty$.
3. When FH-CDMA with intra-cluster frequency hopping is utilized, we have the cluster intensity \bar{c} reduced by a factor M (spreading gain). One can easily obtain the constrained transmission capacity of this system to be

$$C_{FH}^*(\epsilon, \theta) = (1 - \epsilon) \left(\frac{\epsilon \lambda_p M}{\lambda_p \beta_I + k} + o(\epsilon) \right)$$

When DS-CDMA is used, the constrained transmission capacity is $C_{DS}^*(\epsilon, \theta) = C^*(\epsilon, \theta/M)$. When the transmitters are spread as a Poisson point process, we have from (32; 33)

$$\ln \left(\frac{C_{FH}(\epsilon, \theta)}{C_{DS}(\epsilon, \theta)} \right) = (1 - 2/\alpha) \ln(M).$$

In Figure 4.8, we plot $\ln(C_{FH}^*(\epsilon, \theta)/C_{DS}^*(\epsilon, \theta))/\ln(M)$ with respect to spreading gain M , when the path loss function is $\ell(x) = \|x\|^{-\alpha}$ and $\epsilon = 0.01$. From the figure, we observe a similar $M^{1-2/\alpha}$ gain, even in the case of clustered transmitters.

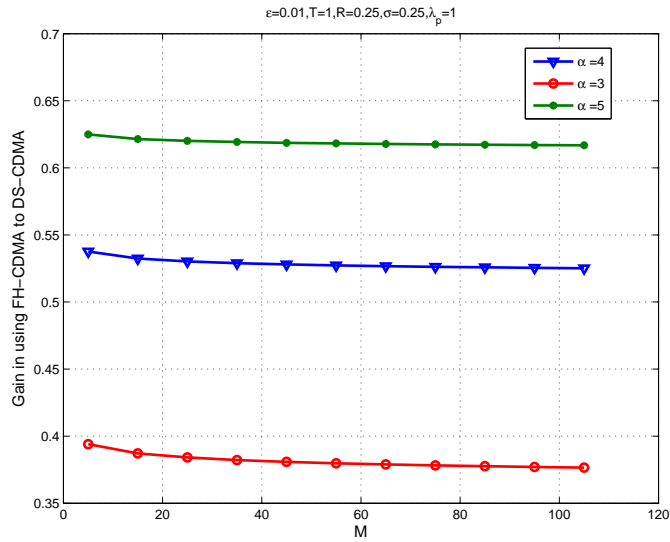


Figure 4.8. $\ln(C_{FH}^*(\epsilon, T)/C_{DS}^*(\epsilon, T))/\ln(M)$ versus M for $\epsilon = 0.01, \lambda_p = 1$

4.5 Chapter Summary

The main focus of this chapter is the probability of success for a typical source-destination pair when the transmitters are spatially distributed as a Poisson cluster process (PCP). A fundamental contribution is the derivation of the conditional

probability generating functional for the Thomas and the Matern PCP's and its use to derive the outage probability. We make the following observations:

1. The performance gain of clustering the transmitters as compared to randomizing the transmitters depends on the source-destination distance R and also critically on the path-loss model. When one considers a singular path-loss model, the gain is always less than one for small R and tends to ∞ for larger R . For a non-singular path-loss model, we have provided criterion for the gain to be larger than unity for small R .
2. The transmission capacity of network when the nodes are distributed as PCP is equal to a network where the nodes are located as a PPP. The equality occurs since PPP can be regarded as a limiting case of a PCP.

CHAPTER 5

TWO-HOP CELLULAR SYSTEMS

5.1 Introduction

Cellular systems are the most widely deployed wireless systems and provide reliable communication services to billions around the world. Cellular systems consist of base stations that serve a geographical area called cell. In most of the present cellular systems, the base station (BS) communicates directly with the mobile users (MS) in its cell. This single-hop architecture makes it difficult for the BSs to communicate with MSs at the cell boundary because of the distance and the inter-cell interference. So a base station will have to increase its power to maintain the rate of transmission. The dead spots problem can be countered by using more base stations, and thereby increasing the spatial reuse. But increasing the number of base stations would increase the interference and degrade the performance of the overall system. This problem can be addressed more effectively by moving away from the paradigm of single-hop communication and permitting the base station to communicate with mobile stations at the boundary by using the other intermediate MSs in its cell in a sequence of hops. Although such multi-hopping requires some significant changes in the present cellular system architecture, it may help to effectively combat the dead spots problem, and hence the cellular multi-hopping problem is worthy to investigate (34; 35). In this

chapter, we analyze the benefits of two-hop cellular communication by comparing its performance with a traditional single-hop cellular system. A two-hop system,

- May provide significant benefits over single-hop communication.
- Does not have the implementation complexity of larger number of hops (in terms of routing and scheduling) and hence is more tractable.

When a BS transmits, multiple MSs will be able to receive the information, and hence these mobile nodes can help the BS to transmit information to the cell edge. Since more than one MS can act as a relay, it is not clear how to choose a subset of these relays in a distributed fashion so as to reduce the interference and increase the probability of packet delivery. In this chapter, we analyze simple relay selection schemes and compare their performance with direct transmission. We account for the inter-cell interference and the spatial structure of the transmitting nodes in the analysis.

We use methods from stochastic geometry and point process theory to model and study the two-hop cellular system. We will provide techniques to use the probability generating functional of a point process to analyze the outage probabilities and also provide asymptotic results for the outage at high SNR and low BS density. The techniques presented in this chapter can be extended to analyze more complicated relay selection schemes, power control mechanisms and other multi-hop techniques. The major emphasis of the chapter is in the methodology and the techniques of the analysis rather than the specifics of the communication system. For example we concentrate only on three specific relay selection methods although many more methods have been proposed in the literature.

5.1.1 Previous Work

The problem of two-hop cellular system has been studied extensively, and a provision for a multi-hop technique has been included in the A-GSM standard (34; 35). In (36), a MS is selected to help the BS depending on the large-scale path-loss on the BS-relay link, and the relay-destination link. (37) considers a similar problem, but the MSs that can act as relays are assumed to be located on a circle around the BS, and the authors provide various power allocation schemes and verify their performance by simulations. The present problem is also very similar to the problem of opportunistic relay selection. In (38; 39) a detailed analysis for obtaining full diversity order using distributed space-time codes has been proposed. But a distributed space-time code requires very tight coordination and precise signaling among the relays, which increases the overhead and complexity in the system. An alternate approach is to choose the *best* relay, and in opportunistic relaying (OR) (40) a relay is chosen so as to maximize the minimum SNR of the source-relay and the relay-destination links. In selection cooperation (SC)(41; 42) the relay with maximum relay-destination SNR is chosen. It has been shown that SC and OR provide a similar diversity order. In (39; 40; 41; 42), distributed relay selection schemes are analyzed and asymptotes of the outage are provided for high SNR. The asymptotes provided are functions of the means of fading coefficients between the source, relays and the destination. Averaging these results with respect to the spatial distribution of the nodes is difficult and hence we use an alternative approach. In our approach we model the node locations in a statistical manner and incorporate this information in the analysis from the start rather than averaging over the spatial locations at the end.

The chapter is organized as follows: In Section 5.2 the system model is intro-

duced, assumptions stated and the metrics used in the chapter defined. In Section 5.3 the outage probability in the direct connection between the BS and its destination is analyzed. In Sections 5.5 and 5.6 the outage probability of the two-hop schemes employing different relay selection schemes are analyzed. The asymptotic gain of using the two-hop schemes over the direct connection is also analyzed in these sections. In Section 5.7 simulation results are provided and compared to the theory.

5.2 System Model

We assume that the BSs (cell towers) are arranged on a lattice. More precisely the base station locations are

$$\Phi_b = \left\{ \frac{x}{\sqrt{\lambda_b}}, \quad x \in \mathbb{Z}^2 \right\},$$

which is a lattice of density λ_b . The analysis in this chapter generalizes in a straightforward manner to any deterministic arrangement of BS. We assume that n_x MSs are available to assist a BS $x \in \Phi_b$. More precisely, the locations of the mobile stations that assist the base station x form a Poisson point process (PPP) Φ_x of density $\lambda_x(y) = \eta(y - x)$. For example choosing $\eta(y) = \mathbf{1}_y([-1/2, 1/2]^2)$ and $\lambda_b = 1$ would lead to a square coverage area for each base station. We use $\mathbf{1}_x(A)$ to denote the indicator function of set A . See Figure 5.1. Observe that it is not necessary for a MS to be associated to its nearest BS, *i.e.*, some MSs may be outside the Voronoi cell of their BS. We also make the following assumptions:

1. The average number of MS that each BS serves is finite, *i.e.*,

$$N = \int_{\mathbb{R}^2} \eta(x) dx < \infty.$$

This assumption implies that Φ_x cannot be homogeneous (2?).

2. The locations of the mobile users associated with different base stations are independent.

Since the number of MSs in each cell is Poisson with mean N , each cell is empty with probability $\exp(-N)$. We shall use μ to denote the probability that a cell is not empty, *i.e.*, $\mu = 1 - \exp(-N)$. Independent Rayleigh fading is assumed between any pair of nodes and also across time, and the power fading coefficient between a node x and node y is denoted by \mathbf{h}_{xy} . Hence \mathbf{h}_{xy} is an exponential random variable with unit mean. The path-loss model is denoted by $\ell(x) : \mathbb{R}^2 \setminus \{o\} \rightarrow \mathbb{R}^+$ and is a continuous, positive, non-increasing function of $\|x\|$ which satisfies

$$\int_{\mathbb{R}^2 \setminus B(o, \epsilon)} \ell(x) dx < \infty, \quad \forall \epsilon > 0,$$

where $B(a, r)$ denotes a disc of radius R centered around a . $\ell(x)$ is usually taken to be a power law in the form

1. Singular path-loss model: $\|x\|^{-\alpha}$.
2. Non-singular path-loss model: $(1 + \|x\|^\alpha)^{-1}$ or $\min\{1, \|x\|^{-\alpha}\}$.

The integrability condition, requires $\alpha > 2$ in all the above models. Assuming simple linear receivers and treating interference as noise, the communication between

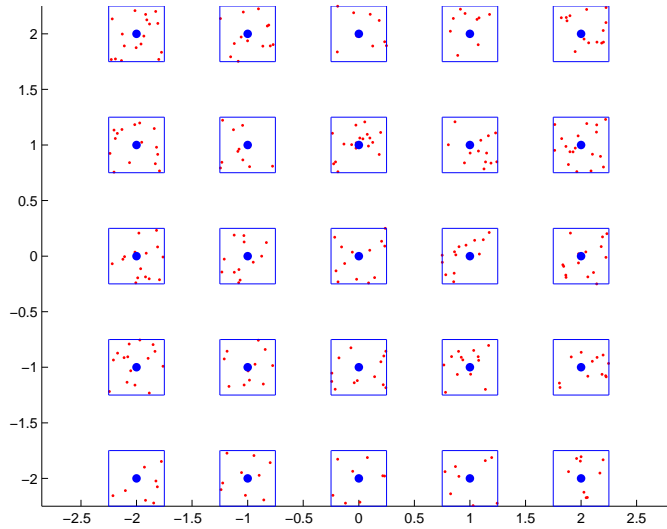


Figure 5.1. Illustration of the cellular system with $\lambda_b = 1$ and $\eta(y) = 50 \cdot \mathbf{1}_y([-0.25, 0.25]^2)$. So on a average there are 12.5 MS per cell. The bold dots represent the BSs and the smaller dots the MSs. In this figure the white spaces between the cells may consist of other cells which transmit at a different frequency. We may model the case where the neighboring cells use the same frequency by choosing $\eta(y) = \mathbf{1}_y([-0.5, 0.5]^2)$.

x and y is successful if

$$\text{SINR}(x, y, \Phi) = \frac{p_x h_{xy} \ell(x - y)}{\sigma^2 + \sum_{z \in \Phi} p_z h_{zy} \ell(z - y)} > \theta. \quad (5.1)$$

We also assume $\theta > 1$ which implies at most one transmitter can connect to a receiver. Here Φ is the set of interfering transmitters and p_z represents the transmission power used by a transmitter located at z . σ^2 represents the additive white Gaussian noise power at the receiver. We make the following assumptions:

1. In the two-hop schemes that will be analyzed, we assume BSs transmit in the even time slots and the MSs transmit in the odd time slots across all

cells.

2. Each base station x has an additional at $r(x)$ with $\|r(x) - x\| = R$, to which the BS wants to transmit information. This additional node just receives and never transmits.
3. All the BSs transmit with equal power P .

Notation:

- Define

$$\mathbf{1}(x \rightarrow y \mid \Phi) = \mathbf{1}(\text{SINR}(x, y, \Phi) > \theta).$$

$\mathbf{1}(x \rightarrow y \mid \Phi)$ is the indicator random variable that is equal to one if a transmitter at x is able to connect to a receiver y when the interfering set is Φ .

- Define

$$\hat{\Phi}(x) = \{y \in \Phi_x : \mathbf{1}(x \rightarrow y \mid \Phi_b \setminus \{x\})\}.$$

$\hat{\Phi}(x)$ is the set of MSs in the cell of BS x to which the BS x is able to connect in the first hop (even time slots).

Metric: Let P_d denote the probability that a BS can connect to its destination directly in the first hop. Since all BSs are identical, we define

$$P_d = \mathbb{E}\mathbf{1}(o \rightarrow r(o) \mid \Phi_b \setminus \{o\}). \quad (5.2)$$

where o denotes the origin $(0, 0)$. A BS can connect to multiple MSs in its cell, and these connected MS are the potential transmitters in the second hop. In the relay selection methods studied in the next section, a subset of these potential

transmitters $R_x \subseteq \hat{\Phi}(x)$ are selected for each $x \in \Phi_b$ to transmit in the next hop. Let the probability that a relay can connect to its intended destination (determined by the source to which it connects in the first hop) in the second hop be P_r , *i.e.*,

$$P_r = 1 - \mathbb{E} \prod_{x \in R_o} 1 - \mathbf{1}(x \rightarrow r(o) \mid \Psi \setminus \{x\}), \quad (5.3)$$

where $\Psi = \bigcup_{x \in \Phi_B} R_x$ is the set of all transmitters in the second hop. Here we are assuming no cooperative communication between nodes which have the same information, and hence relays belonging to the same cluster also interfere with each other in the second hop. Since $\theta > 1$, at most one transmitter can connect to a receiver and thus

$$P_r = \mathbb{E} \sum_{x \in R_o} \mathbf{1}(x \rightarrow r(o) \mid \Psi \setminus \{x\}). \quad (5.4)$$

Hence the probability of success for the two-hop scheme is

$$P_s = 1 - (1 - P_d)(1 - P_r).$$

The BS can potentially transmit in the second hop instead of using the MS as intermediate relays. This retransmission scheme will be used as the base reference, and the performance of the relay selection schemes will be compared with this retransmission scheme. The gain in using the two-hop scheme over the retransmission scheme can be characterized as

$$G(\text{SNR}, \lambda_b) = \frac{(1 - P_d)^2}{(1 - P_d)(1 - P_r)} = \frac{1 - P_d}{1 - P_r}, \quad (5.5)$$

where

$$\text{SNR} = \frac{P\ell(R)}{\sigma^2}$$

is the received SNR for the direct transmission. To compare the direct transmission with the relay selection scheme, power is allocated across the selected relays in the second hop so that the total power is equal to P . Another pertinent metric to capture the performance of the network is the diversity gain, defined as

$$d(\lambda_b) = - \lim_{\text{SNR} \rightarrow \infty} \frac{\log(1 - P_s)}{\log(\text{SNR})}.$$

From the definition of the diversity and the gain, the following relation follows:

$$d_2(\lambda_b) - d_d(\lambda_b) = \lim_{\text{SNR} \rightarrow \infty} \frac{\log(G(\text{SNR}, \lambda_b))}{\log(\text{SNR})},$$

where d_d is the diversity gain for the single-hop retransmission scheme and d_2 is the diversity gain of the two-hop scheme. From the definition of P_s it can be observed that the information received in the two time slots is being decoded independently.

In the next sections, we will analyze the success probability P_r and the diversity order of the relay selection schemes. It is easy to observe that the probability P_r of any relay selection scheme does not tend to one by increasing the SNR because of the interference caused by transmissions in other cells. So as to evaluate the asymptotic performance of the system, we scale the BS density as

$$\lambda_b = \text{SNR}^{-\beta}, \beta \geq 0. \tag{5.6}$$

As will be evident in the next section, if the signal to interference ratio is defined as

$$\text{SIR} = \frac{\ell(R)}{\sum_{x \in \Phi_b \setminus \{o\}} \ell(x - r(o))}, \quad (5.7)$$

the scaling in (5.6) translates to

$$\text{SIR} \sim \text{SNR}^{\frac{\alpha\beta}{2}} \frac{\ell(R)}{C(\alpha)},$$

where $C(\alpha)$ is a constant defined in Lemma 13. So the system is interference-limited when $\beta < 2/\alpha$ and noise-limited otherwise. Hence the scaling in (5.6) helps us in evaluating the performance of the system by varying β . In practice this scaling can be achieved by frequency planning and decreasing the spatial reuse factor. We now begin with the analysis of the direct transmission scheme.

5.3 First Hop: Base Station Transmits

5.3.1 Direct Connection

When the BSs transmit, the inter-cell interference, fading and the noise may cause the transmission to fail. The probability of direct connection is given by

$$P_d = \mathbb{E} \mathbf{1}(o \rightarrow r(o) \mid \Phi_b \setminus \{o\}) \quad (5.8)$$

$$\begin{aligned} &= \mathbb{P} \left(\frac{h_{xy} \ell(R)}{\frac{\sigma^2}{P} + \sum_{y \in \Phi_B \setminus \{o\}} h_{yr(o)} \ell(y - r(o))} > \theta \right) \\ &= \exp \left(-\frac{\theta \sigma^2}{P \ell(R)} \right) \prod_{y \in \Phi_b \setminus \{o\}} \frac{1}{1 + \frac{\theta}{\ell(R)} \ell(y - r(o))} \\ &= \exp \left(-\frac{\theta}{\text{SNR}} \right) \Delta(r(o)) \end{aligned} \quad (5.9)$$

where

$$\Delta(x) = \prod_{y \in \Phi_b \setminus \{o\}} \frac{1}{1 + \frac{\theta}{\ell(x)} \ell(y-x)}.$$

The following lemma is required to analyze the asymptotics of the success probability.

Lemma 13. *When $\ell(x) = \|x\|^{-\alpha}$ or $\ell(x) = 1/(1 + \|x\|^\alpha)$,*

$$\lim_{\lambda_b \rightarrow 0} \frac{1 - \Delta(x)}{\lambda_b^{\alpha/2}} = \frac{\theta C(\alpha)}{\ell(x)},$$

where

$$C(\alpha) = \frac{\xi(\alpha/2, 0) [\xi(\alpha/2, 1/4) - \xi(\alpha/2, 3/4)]}{2^{\alpha-2}}. \quad (5.10)$$

$\xi(s, b) = \sum_{k=0, k \neq -b}^{\infty} (k+b)^{-s}$ is the generalized Riemann zeta function.

Proof. We consider the case of $\ell(x) = \|x\|^{-\alpha}$; the other case follows similarly.

From the definition of $\Delta(x)$ it follows that

$$\exp\left(-\theta \ell(x)^{-1} \sum_{y \in \Phi_b \setminus \{o\}} \ell(y-x)\right) \leq \Delta(x) \leq \left(1 + \theta \ell(x)^{-1} \sum_{y \in \Phi_b \setminus \{o\}} \ell(y-x)\right)^{-1}.$$

We have

$$\begin{aligned} \sum_{y \in \Phi_b \setminus \{o\}} \ell(y-x) &= \sum_{y \in \mathbb{Z}^2 \setminus \{o\}} \ell\left(\frac{y}{\sqrt{\lambda_b}} - x\right) \\ &= \lambda_b^{\alpha/2} \sum_{y \in \mathbb{Z}^2 \setminus \{o\}} \ell(y - x\sqrt{\lambda_b}). \end{aligned}$$

Dividing both sides by $\lambda^{\alpha/2}$ and taking the limit, the result follows from the definition of the Epstein zeta function (43). \square

We have $C(3) \approx 9.03362$ and $C(4) \approx 6.02681$. From the derivation of the

above lemma we observe that $\text{SIR} \sim \text{SNR}^{\alpha\beta/2} \ell(R) C(\alpha)^{-1}$ where the definition of SIR is provided in (5.7). Using the above lemma, the asymptotic expansion of P_d for $\lambda_b = \text{SNR}^{-\beta}, \beta \neq 0$ at high SNR is

$$P_d \sim \begin{cases} 1 - \theta \text{SNR}^{-1} & \alpha\beta > 2 \\ 1 - \theta (1 + C(\alpha) \ell(R)^{-1}) \text{SNR}^{-1} & \alpha\beta = 2 \\ 1 - \theta C(\alpha) \ell(R)^{-1} \text{SNR}^{-\alpha\beta/2} & 0 < \alpha\beta < 2, \end{cases} \quad (5.11)$$

and the diversity gain of the direct transmission is

$$d_d(\text{SNR}^{-\beta}) = \min \left\{ 1, \frac{\beta\alpha}{2} \right\}.$$

So for the direct transmission, $\beta < 2/\alpha$ corresponds to the interference-limited regime and $\beta > 2/\alpha$ corresponds to the noise-limited regime. From Figure 5.2 we observe that the asymptotes in (5.11) are close to the true $1 - P_r$, even at moderate SNR. In the scaling law provided, observe that the distance of the receiver from the BS is fixed.

5.3.2 Properties of the Potential Relay Sets $\hat{\Phi}(x), x \in \Phi_b$.

In this subsection, the properties of the node set that the BS at the origin is able to connect to are analyzed. When the BSs transmit, the interference seen by two MSs is independent. So the set of MSs to which the BS at the origin can connect to is an independent thinning of Φ_o . Hence $\hat{\Phi}(o)$ is also a PPP and since the thinning depends on the position, the resulting process is inhomogeneous. Hence the intensity of $\hat{\Phi}(o)$ is

$$\delta(x) = \eta(x) \mathbb{E} \mathbf{1}(o \rightarrow x \mid \Phi_b \setminus \{o\}).$$

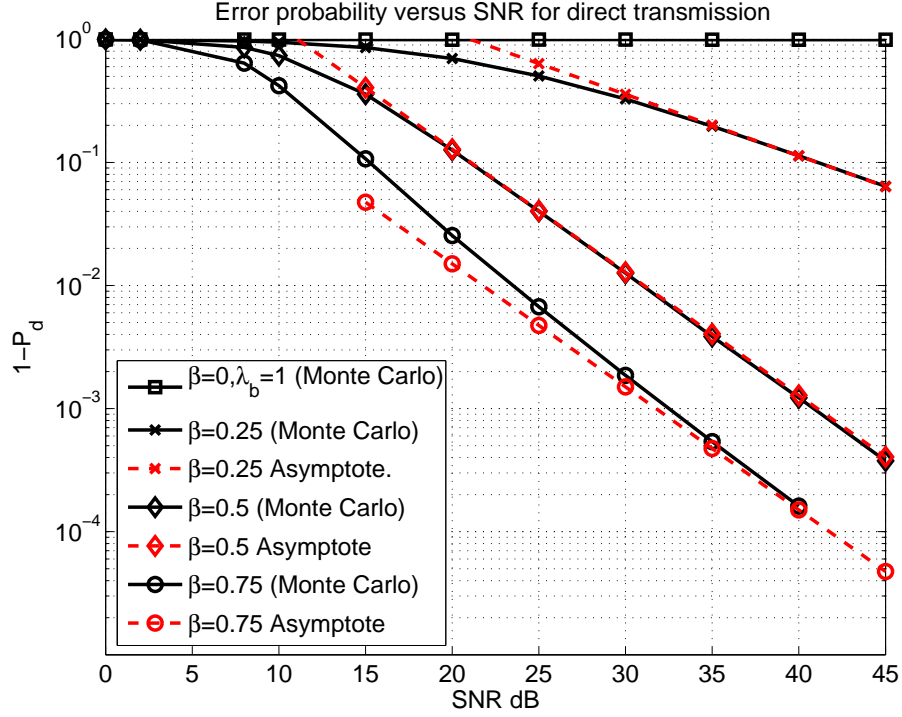


Figure 5.2. Outage probability $1 - P_d$ versus SNR for $\lambda_b = \text{SNR}^{-\beta}$ with different β . The system parameters are $\alpha = 4$, $\theta = 1.5$, $r(o) = (0.5, 0.5)$ and $\ell(x) = (1 + \|x\|^4)^{-1}$. The dashed lines are the asymptotes derived in (5.11). Observe the difference in the slopes of the error curve for $\beta < 0.5$ and $\beta \geq 0.5$.

Following a procedure similar to the derivation of (5.9), the intensity is given by

$$\delta(x) = \eta(x) \exp\left(-\frac{\theta}{\text{SNR}} \frac{\ell(R)}{\ell(x)}\right) \Delta(x). \quad (5.12)$$

The average number of MSs which the BS is able to connect to is

$$\mathbb{E} \sum_{x \in \Phi_o} \mathbf{1}(o \rightarrow x \mid \Phi_b \setminus \{o\}) = \int_{\mathbb{R}^2} \delta(x) dx, \quad (5.13)$$

which follows from the Campbell-Mecke theorem (?). The average distance over which the BS at the origin can connect is

$$L = \frac{\mathbb{E} \sum_{x \in \Phi_o} \|x\| \mathbf{1}(o \rightarrow x \mid \Phi_b)}{\mathbb{E} \sum_{x \in \Phi_o} \mathbf{1}(o \rightarrow x \mid \Phi_b)} \quad (5.14)$$

$$= \frac{\int_{\mathbb{R}^2} \|x\| \delta(x) dx}{\int_{\mathbb{R}^2} \delta(x) dx}. \quad (5.15)$$

In the second hop, a subset of the MSs which were able to receive information in the first-hop transmit. In the next sections we analyze the following three strategies to select a subset $R_x \subset \hat{\Phi}(x)$ to transmit in the second hop (odd time slots):

- All MSs that receive information in the first hop transmit in the second hop. In this relay selection method MSs do not need to have location information or any channel state information.
- The MS closest to the destination and that has received information in the first hop transmits in the second hop. This strategy requires nodes to know their respective locations.
- The MS with the best channel (fading and path-loss) to the destination that has received information in the first hop transmits. This strategy requires the relays to have channel state information.

5.4 Method 1: All Potential Relay Nodes Transmit

We first analyze the scenario in which all the nodes which receive information in the first hop transmit, *i.e.*, $R_x = \hat{\Phi}(x)$. This method also introduces the technique to analyze the success probability.

Equal power transmission: We first analyze the scenario in which all the relays that received information in the first hop transmit with equal power P_1 . From (5.13), the average power used in the second hop is

$$P_1 \int_{\mathbb{R}^2} \delta(x) dx. \quad (5.16)$$

To compare this scheme with the direct transmission we choose P_1 so that the total power to transmit information in the second hop is equal to $P = \sigma^2 \text{SNR} / l(R)$ (*i.e.*, power used by the direct transmission). Hence the power P_1 is set as

$$P_1 = \frac{\text{SNR} \sigma^2}{l(R) \int_{\mathbb{R}^2} \delta(x) dx}. \quad (5.17)$$

From (5.4) we have,

$$P_r = \mathbb{E} \sum_{x \in R_o} \mathbf{1}(x \rightarrow r(o) \mid \Psi),$$

and in this case

$$\Psi = \bigcup_{y \in \Phi_b} R_y = \bigcup_{y \in \Phi_b} \hat{\Phi}_y.$$

where each $\hat{\Phi}_y$ is a PPP with density $\delta(x - y)$. In this case we have

$$\mathbf{1}(x \rightarrow r(o) \mid \Psi) = \mathbf{1} \left(\frac{\mathbf{h}_{xr(o)} \ell(x - r(o))}{\sigma^2 / P_1 + \sum_{y \in \psi \setminus \{x\}} \mathbf{h}_{yr(o)} \ell(y - r(o))} \right).$$

By the Campbell-Mecke theorem it follows that

$$P_r = \int_{\mathbb{R}^2} \delta(x) \mathbb{E}^{!x} \mathbf{1}(x \rightarrow r(o) \mid \psi) dx, \quad (5.18)$$

where $\mathbb{E}^{!x}$ is the expectation with respect to the reduced Palm measure (3). Since the individual clusters R_x are independent, centered around the lattice points,

and are inhomogeneous PPPs, we have $\mathbb{E}^{!x} = \mathbb{E}$. From a similar procedure as in the derivation of P_d we have

$$\mathbb{E}\mathbf{1}(x \rightarrow r(o) \mid \psi) = \exp\left(-\frac{\theta\sigma^2}{P_1\ell(x-r(o))}\right) \underbrace{\mathbb{E} \prod_{y \in \psi} \frac{1}{1 + \frac{\theta}{\ell(x-r(o))}\ell(y-r(o))}}_{p(x)}.$$

where

$$\begin{aligned} p(x) &\stackrel{(a)}{=} \prod_{a \in \Phi_b} \mathbb{E} \prod_{y \in R_a} \frac{1}{1 + \frac{\theta}{\ell(x-r(o))}\ell(y-r(o))} \\ &\stackrel{(b)}{=} \prod_{a \in \Phi_b} \exp\left(-\int_{\mathbb{R}^2} \frac{\delta(y-a)}{1 + \frac{\ell(x-r(o))}{\theta}\ell(y-r(o))^{-1}} dy\right) \\ &= \exp\left(-\int_{\mathbb{R}^2} \frac{\sum_{a \in \Phi_b} \delta(y-a)}{1 + \frac{\ell(x-r(o))}{\theta}\ell(y-r(o))^{-1}} dy\right) \end{aligned}$$

(a) follows from the independence of R_a for $a \in \Phi_b$ and (b) follows from the probability generating functional of the PPP. Hence

$$P_r = \int_{\mathbb{R}^2} \delta(x)p(x) \exp\left(-\frac{\theta\sigma^2}{P_1\ell(x-r(o))}\right) dx.$$

Since there is a finite probability that a cell can be empty,

$$P_r \leq 1 - \exp\left(-\int_{\mathbb{R}^2} \eta(x)dx\right).$$

When the cell is empty, the two-hop technique does not work. The probability of success in two hops conditioned on the event that there the cell is not empty is

obtained as follows:

$$\begin{aligned} \mathbf{P}_r &= (\mathbf{P}_r | (n_o > 0))\mathbb{P}(n_o > 0) + (\mathbf{P}_r | (n_o = 0))\mathbb{P}(n_o = 0) \\ &= (\mathbf{P}_r | (n_o > 0)) \left(1 - \exp \left(- \int_{\mathbb{R}^2} \eta(x) dx \right) \right). \end{aligned}$$

Hence the success probability when the cell is not empty, *i.e.*, $|\Phi_x| > 0$ is equal to

$$\mathbf{P}_r | (n_o > 0) = \frac{\mathbf{P}_r}{\mu}. \quad (5.19)$$

We now analyze the asymptotic gain of this scheme.

Lemma 14. *The gain*

$$G(\text{SNR}, \text{SNR}^{-\beta}) \rightarrow 0$$

as $\text{SNR} \rightarrow \infty$ for all $\beta > 0$.

Proof. The success probability \mathbf{P}_d in the direct connection becomes one when the inter-cell interference becomes zero and SNR becomes infinite. The success probability \mathbf{P}_r does not approach one even when the inter-cell interference becomes zero and SNR becomes infinite. This is because of the intra-cell interference when all the nodes transmit. Indeed when $\text{SNR} \rightarrow \infty$ and $\lambda_b \rightarrow 0$

$$\mathbf{P}_r = \int_{\mathbb{R}^2} \eta(x) \exp \left(- \int_{\mathbb{R}^2} \frac{\eta(y)}{1 + \theta^{-1} \ell(x - r(o)) \ell(y - r(o))^{-1}} dy \right) dx.$$

□

In Figure 5.3 the outage probability of this two-hop scheme is plotted. We observe a non-monotonic behavior with increasing SNR: when the SNR is low only a few relays close to the BS can decode and hence in the second hop there

might be no potential relays. On the other hand, at high SNR many relays will be able to decode the information but would interfere with each other in the second hop. Hence we can observe that there is a sweet spot at some intermediate SNR. From the above lemma we observe that there is no gain when the inter-

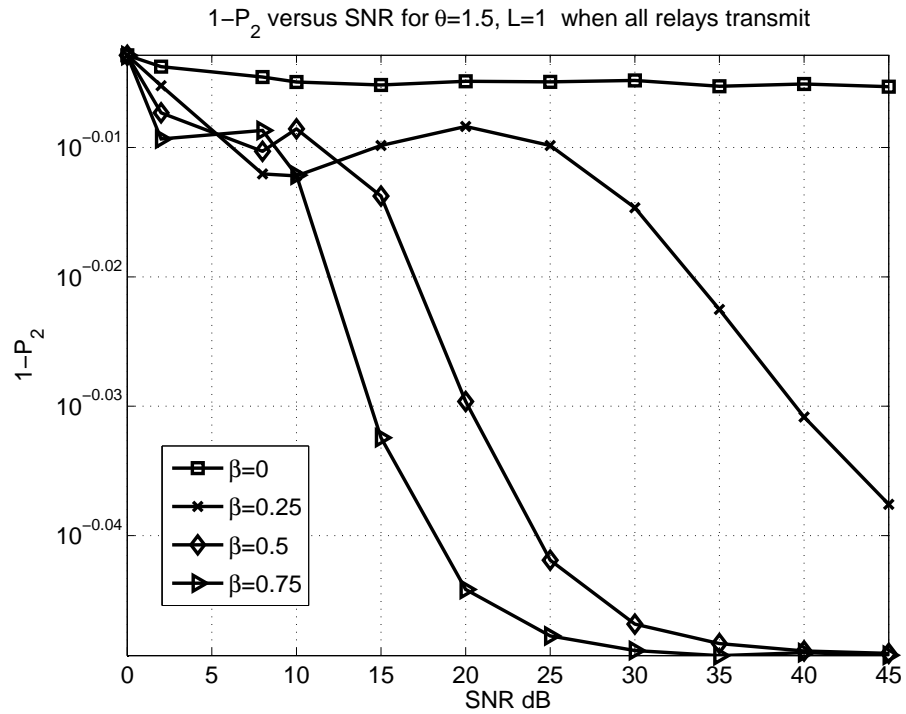


Figure 5.3. Outage probability $1 - P_r | (n_o > 0)$ versus SNR for $\lambda_b = \text{SNR}^{-\beta}$ and different β . We have $\alpha = 4$, $\theta = 1.5$, $z = (0.5, 0.5)$, $\ell(x) = (1 + \|x\|^4)^{-1}$ and $\eta(y) = 5\mathbf{1}_y([-0.5, 0.5]^2)$.

cell interference becomes zero. So the diversity achieved when all the relays that received information transmit in the second hop is equal to zero. Observe that

in these methods the information from the direct path is not combined with the information received from the relay. The diversity is 0 on the relay-to-destination connection because of the presence of interference in the second hop, and increasing the SNR will not change the interference. From the above derivation, it can be observed that thinning the MS that transmit in the second hop would not improve the diversity. In essence the second hop is interference-limited, *i.e.*, the probability of error never approaches zero even for a large SNR. In contrast, if the BS at o directly transmits the same information to the MS at z for two time slots it would achieve a first order diversity with majority decoding and second order diversity when MRC is used. When the destination does not receive a packet from the direct transmission, it would in general send a NACK (assuming that it knows it should receive something). This NACK can be used by the MS to estimate the large-scale path-loss from the destination to themselves. If channel reciprocity is assumed the MS can estimate the path-loss gain to the destination and each MS can compensate for the large scale path-loss to the destination, *i.e.*, each $x \in R_o$ chooses a power equal to $\sigma^2 \ell(x - r(o))^{-1}$.

Theorem 12 (Power compensation). *When each MS that successfully received information from some BS does a power compensation to their respective destination, the probability of success is*

$$P_r = \varrho \exp \left(-\theta - \frac{\theta}{1 + \theta} \int_{\mathbb{R}^2} \delta(x) dx \right) \int_{\mathbb{R}^2} \delta(x) dx,$$

where

$$\varrho = \exp \left(- \sum_{a \in \Phi_b \setminus \{o\}} \int_{\mathbb{R}^2} \frac{\delta(y - a)}{1 + \frac{\ell(y - r(a))^{-1}}{\theta} \ell(y - r(o))^{-1}} dy \right).$$

Proof. Let the number of points in R_o be equal to n . For notational convenience let h_i represent the fading coefficient $h_{x_i r(o)}$. Conditioned on the number of points in R_o ,

$$\begin{aligned} P_r &= \mathbb{E} \sum_{i=1}^n \mathbf{1} \left(\frac{h_i}{1 + \sum_{k \neq i} h_k + I} > \theta \right) \\ &= n \exp(-\theta) \left(\frac{1}{1 + \theta} \right)^{n-1} \mathbb{E} \exp(-\theta I). \end{aligned}$$

I is the interference caused by the MS in the cells other than the one at the origin. $\mathbb{E} \exp(-\theta I)$ can be calculated by a method similar to the derivation of the success probability without power compensation. Since n is a Poisson random variable with mean $\int \delta(x) dx$, taking the average with respect to n , the result follows. \square

The total average power used in the second hop is

$$\sigma^2 \text{SNR} \int_{\mathbb{R}^2} \frac{\delta(x)}{\ell(x - r(o))} dx.$$

5.5 Method 2: Nearest Relay to the Destination Transmits

In this relay selection method, the node $x \in \hat{\Phi}(a)$, $a \in \Phi_b$ closest to $r(a)$ is selected to transmit in the second hop. To do this each node should contain its own location and each packet should have location information about its destination. For fair comparison with the direct transmission we assume that the selected relay transmits with power $\text{SNR}\sigma^2/\ell(R)$. The probability of success in this relay selection method is equal to

$$P_r = \mathbb{P} (h_{x,r(o)}\ell(r) > \theta(\sigma^2 + I)),$$

where I is the inter cell interference at $r(o)$. r is the distance from the relay in the set $\hat{\Phi}(o)$ that is nearest to $r(o)$. More precisely

$$r = \begin{cases} \inf_{x \in \hat{\Phi}(o)} \|x - r(o)\|, & |\hat{\Phi}(o)| > 0 \\ \infty, & |\hat{\Phi}(o)| = 0. \end{cases}$$

$\hat{\Phi}(o)$ can be empty because of the following two reasons:

1. The cell has no MS to begin with. The probability of this happening is equal to $1 - \mu$.
2. The BS was not able to connect to any MS in the first time slot.

For a fair comparison with direct transmission, we condition on the cell at the origin having at least one MS to begin with, *i.e.*, $n_o > 0$. So

$$\mathbb{P}_r \mid (n_o > 0) = \mathbb{P}_r \mu^{-1}.$$

Let $F_o(r, \text{SNR}, \lambda_b)$ denote the CDF of the first contact distribution of $\hat{\Phi}(o)$ from $r(o)$. It is given by

$$F_o(r, \text{SNR}, \lambda_b) = 1 - \exp\left(-\int_{B(r(o), r)} \delta(x) dx\right). \quad (5.20)$$

Observe that $F_o(r, \text{SNR}, \lambda_b)$ is a defective distribution, *i.e.*, $F_o(\infty, \text{SNR}, \lambda_b) < 1$.

Let

$$f_o(r, \text{SNR}, \lambda_b) = -\frac{\partial}{\partial r} F_o(r, \text{SNR}, \lambda_b)$$

denote the PDF of the first contact distribution. Hence

$$P_r = \int_0^\infty \exp\left(-\frac{\theta\ell(R)}{\text{SNR}\ell(r)}\right) \underbrace{\mathbb{E}\left(\exp\left(-\frac{\theta\ell(R)}{\text{SNR}\sigma^2\ell(r)}I\right)\right)}_{T_1(\lambda_b, r)} f_o(r, \text{SNR}, \lambda_b) dr.$$

where I is the interference at $r(o)$ by transmitters in other cells. Even though $\int f_o(r, \text{SNR}, \lambda_b) dr < 1$ the above average is correct since the integrand is zero at $r = \infty$ where the remaining mass of the first contact distribution lies. We

now evaluate $T_1(\lambda_b, r)$. Let $f_a(x), a \neq 0, x \in \mathbb{R}^2$, denote the PDF of the nearest neighbor of $r(a)$ in the set $\hat{\Phi}(a)$ relative to a conditioned on the event $|\hat{\Phi}(a)| > 0$.

We then have

$$T_1(\lambda_b, r) = \prod_{a \in \mathbb{Z}^2 \setminus \{o\}} \int_{\mathbb{R}^2} \mathbb{E} \left[\frac{f_a(x)}{1 + \frac{\theta}{\ell(r)} \ell\left(\frac{a}{\sqrt{\lambda_b}} + x - r(o)\right) \mathbf{1}(|\hat{\Phi}(a)| > 0)} \right] dx$$

Taking the average with respect to $|\hat{\Phi}(a)|$ we have

$$\prod_{a \in \mathbb{Z}^2 \setminus \{o\}} \int_{\mathbb{R}^2} \mathbb{E} \left[1 - \mathbf{1}(|\hat{\Phi}(a)| > 0) + \frac{f_a(x) \mathbf{1}(|\hat{\Phi}(a)| > 0)}{1 + \frac{\theta}{\ell(r)} \ell\left(\frac{a}{\sqrt{\lambda_b}} + x - r(o)\right)} \right] dx$$

$$\prod_{a \in \mathbb{Z}^2 \setminus \{o\}} 1 - \int_{\mathbb{R}^2} \frac{f_a(x) (1 - \exp(-\int \delta(y) dy))}{1 + \frac{\ell(r)}{\theta} \ell\left(\frac{a}{\sqrt{\lambda_b}} + x - r(o)\right)^{-1}} dx.$$

$f_a(x)$ depends on the geometry of each cell, $\delta(x)$, $r(a)$ and is easy to calculate once these quantities are known. We now calculate the asymptotic of P_r and the asymptotic gain.

Asymptotic gain: In this part we scale the BS density as $\lambda_b = \text{SNR}^{-\beta}$. It is easy to observe that the average number of MS in each cell that are potential relays,

i.e., $\int \delta(x)dx$ scales as

$$\int \delta(x)dx \sim \int \eta(x)dx - \frac{\theta\ell(R)}{\text{SNR}} \int \frac{\eta(x)}{\ell(x)}dx - \frac{\theta C(\alpha)}{\text{SNR}^{\alpha\beta/2}} \int \frac{\eta(x)}{\ell(x)}dx. \quad (5.21)$$

It can also be verified that

$$\sup_r |F_o(r, \text{SNR}, \text{SNR}^{-\beta}) - F_o(r, \infty, 0)| \rightarrow 0$$

as $\text{SNR} \rightarrow \infty$, which implies $F_o(r, \text{SNR}, \text{SNR}^{-\beta})$ converges uniformly to $F_o(r, \infty, 0)$.

Hence we can interchange the derivative and the limit in the asymptotic analysis.

We have

$$f_o(r, \text{SNR}, \text{SNR}^{-\beta}) = \exp\left(-\int_{B(r(o),r)} \delta(x)dx\right) \frac{\partial}{\partial r} \int_{B(r(o),r)} \delta(x)dx.$$

From (5.21) and the fact that $\exp(-x) \sim 1 - x$ for small x it follows that,

$$f_o(r, \text{SNR}, \text{SNR}^{-\beta}) \sim \begin{cases} \exp(-f(r)) \left(\frac{\partial}{\partial r} f(r) - \frac{\theta\ell(R)}{\text{SNR}} g(r) \right) & \alpha\beta > 2 \\ \exp(-f(r)) \left(\frac{\partial}{\partial r} f(r) - \frac{\theta C(\alpha)}{\text{SNR}^{\alpha\beta/2}} g(r) \right) & \alpha\beta < 2 \end{cases}$$

where

$$g(r) = \frac{\partial}{\partial r} \int_{B(r(o),r)} \frac{\eta(x)}{\ell(x)}dx - \int_{B(r(o),r)} \frac{\eta(x)}{\ell(x)}dx \frac{\partial}{\partial r} f(r),$$

and

$$f(r) = \int_{B(r(o),r)} \eta(x)dx.$$

The following limit follows similar to the asymptotic analysis of \mathbf{P}_d

$$\lim_{\text{SNR} \rightarrow \infty} \frac{1 - T_1(\text{SNR}^{-\beta}, r)}{\text{SNR}^{-\beta\alpha/2}} = \frac{\theta C(\alpha)\mu}{\ell(r)},$$

where $C(\alpha)$ is given by (5.10). By some basic algebraic manipulations the asymptotic expansion of the error probability with respect to SNR with $\lambda_b = \text{SNR}^{-\beta}, \beta > 0$ is equal to

$$1 - P_2 | (n_o > 0) \sim \begin{cases} \frac{\theta \ell(R)}{\text{SNR}^\mu} \int_0^\infty \exp(-f(r)) \{g(r) + \ell(r)^{-1} \frac{\partial}{\partial r} f(r)\} dr & \alpha\beta > 2 \\ \frac{\theta C(\alpha)}{\text{SNR}^{\alpha\beta/2} \mu} \int_0^\infty \exp(-f(r)) \{g(r) + \mu \ell(r)^{-1} \frac{\partial}{\partial r} f(r)\} dr & \alpha\beta < 2. \end{cases} \quad (5.22)$$

These asymptotes are plotted in Figure 5.4. From (5.22) the asymptotic gain is asymptotically equal to

$$\begin{cases} \mu \ell(R)^{-1} \left(\int_0^\infty \exp(-f(r)) \{g(r) + \ell(r)^{-1} \frac{\partial}{\partial r} f(r)\} dr \right)^{-1} & \alpha\beta > 2 \\ \mu \ell(R)^{-1} \left(\int_0^\infty \exp(-f(r)) \{g(r) + \mu \ell(r)^{-1} \frac{\partial}{\partial r} f(r)\} dr \right)^{-1} & \alpha\beta < 2. \end{cases}$$

Remarks:

- We observe that the gain is higher in the interference-limited regime than the noise-limited regime. This is because in this relay selection method, some of the cells may not be able to transmit because they do not contain any MS and this happens with probability $1 - \mu$.
- Since the gain does not scale with SNR, the diversity of this scheme is also equal to $\min\{1, \beta\alpha/2\}$. See Figure 5.4 for the error plot obtained by Monte Carlo simulations and the above asymptotes obtained theoretically.

5.6 Method 3: Relay With Best Channel to the Destination (Selection Cooperation) Transmits.

In this selection procedure, the fading between a potential relay and the destination is also included in the criterion for the relay selection. The relay with

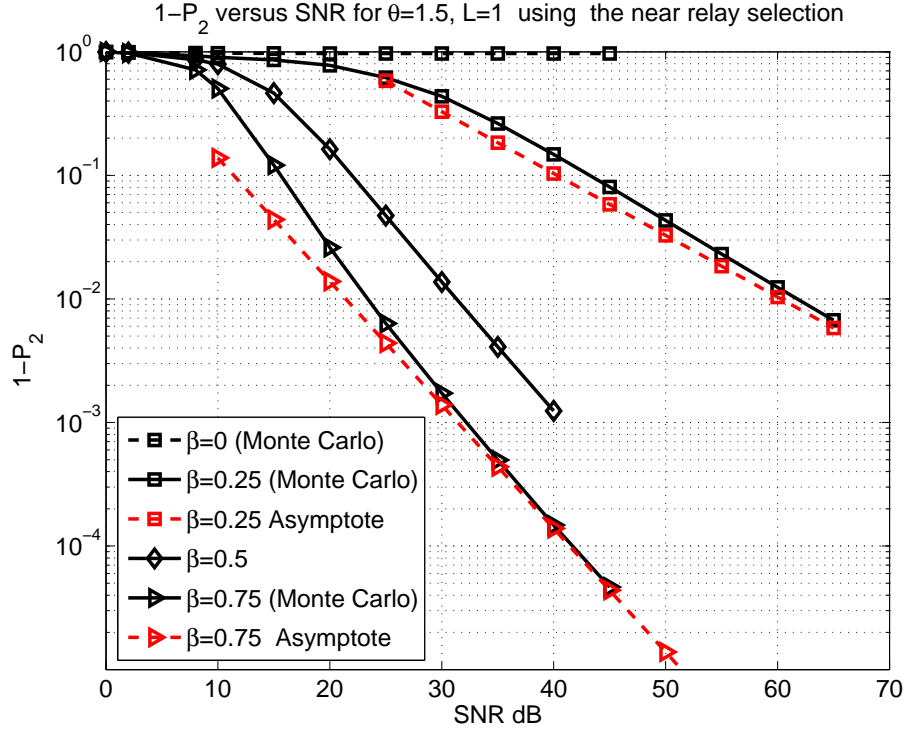


Figure 5.4. Outage probability $1 - P_r | (n_o > 0)$ versus SNR for $\lambda_b = \text{SNR}^{-\beta}$ for different β . The system parameters are $\alpha = 4$, $\theta = 1.5$, $r(o) = (0.5, 0.5)$, $\ell(x) = (1 + \|x\|^4)^{-1}$ and $\eta(y) = 51_y([-0.5, 0.5]^2)$. The dashed lines are the asymptotes derived in (5.22). The dashed lines are the asymptotes derived in (5.22) and are approximately equal to $10.351\text{SNR}^{-0.5}$ (interference-limited) and 1.387SNR^{-1} (noise-limited).

the best channel to the destination is selected. This method of relay selection is called selection cooperation. In the second hop, each relay of the set $\hat{\Phi}(o)$ can send a channel estimation packet to the destination in an orthogonal fashion and the destination can choose the relay with the best channel. Otherwise if channel reciprocity is assumed, the relays can estimate the channel between themselves and the destination using a NACK and use this information to elect the best relay in a distributed fashion.

As in the previous section we shall find the success probability conditioned on the cell at the origin being non-empty, *i.e.*, $\mathbf{P}_r \mid n_o > 0$. As indicated earlier

$$\mathbf{P}_r \mid n_o > 0 = \mu^{-1} \mathbf{P}_r.$$

Hence we shall first calculate the unconditional probability \mathbf{P}_r and then multiply it with μ^{-1} . The relay that is selected is mathematically described by

$$\arg \max_{x \in \hat{\Phi}(o)} \{ \mathbf{h}_{xr(o)} \ell(x - r(o)) \}.$$

The exact analysis of this relay selection in the presence of interference is difficult and hence our aim in this section is to obtain the scaling of $G(\text{SNR}, \text{SNR}^{-\beta/2})$. Let k denote the cardinality of the set $\hat{\Phi}(o)$. Since the connectivity in the first hop is independent across relays, k is a binomial random variable with mean

$$\mathbb{E}[k] = \int_{\mathbb{R}^2} \delta(x) dx.$$

To make the comparison with the direct transmission easier, we assume that each node transmits with power $P = \text{SNR} \sigma^2 / \ell(R)$. The probability of error is

$$1 - \mathbf{P}_r = \mathbb{P} \left(P \max_{x \in \hat{\Phi}(o)} \{ \mathbf{h}_{xr(o)} \ell(x - r(o)) \} < \theta(I + \sigma^2) \right),$$

where I is the interference at $r(o)$ caused by concurrent transmissions in other

cells. Conditioning on the point set $\hat{\Phi}(o)$ we have

$$\begin{aligned} 1 - \mathbb{P}_r \mid \hat{\Phi}(o) &= \mathbb{P} \left(P \max_{x \in \hat{\Phi}(o)} \{h_{xr(o)} \ell(x - r(o))\} < \theta(I + \sigma^2) \mid \hat{\Phi}(o) \right) \\ &= \mathbb{E} \left[\prod_{x \in \hat{\Phi}(o)} 1 - \exp \left(-\frac{\theta(\sigma^2 + I)}{P \ell(x - r(o))} \right) \mid \hat{\Phi}(o) \right]. \end{aligned}$$

Since $\hat{\Phi}(o)$ is a PPP with intensity function $\delta(x)$, conditioning on there being k points in the set, each node in the set is independently distributed with density $\kappa(x) = \frac{\delta(x)}{\int_{\mathbb{R}^2} \delta(y) dy}$. Removing the conditioning on the locations of $\hat{\Phi}(o)$, we obtain

$$1 - \mathbb{P}_r \mid (|\hat{\Phi}(o)| = k) = \mathbb{E} \left[1 - \int_{\mathbb{R}^2} \exp \left(-\frac{\theta(\sigma^2 + I)}{P \ell(x - r(o))} \right) \kappa(x) dx \right]^k \quad (5.23)$$

Using binomial expansion,

$$1 - \mathbb{P}_r \mid (|\hat{\Phi}(o)| = k) = \underbrace{\mathbb{E} \left[\sum_{m=0}^k (-1)^m \binom{k}{m} \mathbb{E} \left[\int_{\mathbb{R}^2} \exp \left(-\frac{\theta(\sigma^2 + I)}{P \ell(x - r(o))} \right) \kappa(x) dx \right]^m \right]}_{a_k}.$$

Hence $1 - \mathbb{P}_r \mid (|\hat{\Phi}(o)| = k)$ is equal to

$$\begin{aligned} &\sum_{m=0}^k (-1)^m \binom{k}{m} \int_{\mathbb{R}^{2m}} \nu(x_1, \dots, x_m) \exp \left(-\frac{\theta \sigma^2 \sum_{b=1}^m \ell(x_b - r(o))^{-1}}{P} \right) \\ &\cdot \prod_{b=1}^m \kappa(x_b) dx_1 \dots dx_m, \end{aligned}$$

where

$$\begin{aligned}
\nu(x_1, \dots, x_m) &= \mathbb{E} \left[\prod_{b=1}^m \exp \left(-\frac{I\theta}{\ell(x_b - r(o))} \right) \right] \\
&= \mathbb{E} [\exp(-I\theta \varrho(x_1^m))] \\
&= \mathbb{E} \left[\exp \left(-\theta \varrho(x_1^m) \sum_{a \in \mathbb{Z}^2} \mathbf{h}_{y(a)r(o)} \ell(y(a) - r(o)) \mathbf{1}(|\hat{\Phi}(a)| > 0) \right) \right]
\end{aligned}$$

where $y(a)$ denotes the location of the selected relay in the cell at a and $\varrho(x_1^m) = \sum_{b=1}^m \ell(x_b - r(o))^{-1}$. Let $g_a(x)$ denote the PDF of $a - x$ where

$$x = \arg \max_{x \in \hat{\Phi}(a)} \{ \mathbf{h}_{xr(a)} \ell(x - r(a)) \}.$$

$g_a(x)$ is difficult to calculate and is the reason of resorting to asymptotics. Since $\mathbf{h}_{y(a)r(o)}$ is exponential it follows that

$$\nu(x_1, \dots, x_m) = \prod_{a \in \mathbb{Z}^2} 1 - \int_{\mathbb{R}^2} \frac{g_a(y) (1 - \exp(-\int \delta(x) dx))}{1 + \theta^{-1} \varrho(x_1^m)^{-1} \ell(y + \frac{a}{\sqrt{\lambda_b}} - r(o))^{-1}} dy.$$

Hence the unconditional probability of error is

$$\mathbf{P}_r = \mu^{-1} \left[1 - \exp \left(- \int_{\mathbb{R}^2} \delta(x) dx \right) \sum_{k=0}^{\infty} a_k \frac{(\int \delta(x) dx)^k}{k!} \right].$$

Asymptotic gain: The above expansion is too unwieldy to yield any asymptotics.

We shall use (5.23) to obtain the gain in the high SNR and low interference regime.

Removing the conditioning in (5.23) we have

$$1 - \mathbf{P}_r = \mathbb{E} \exp \left[- \int_{\mathbb{R}^2} \exp \left(-\frac{\theta(\sigma^2 + I)}{P\ell(x - r(o))} \right) \delta(x) dx \right].$$

The above result follows from the generating function of a Poisson random

variable. Hence the required conditional probability is

$$\mathbb{P}_r | n_o > 0 = \mu^{-1} \left(1 - \mathbb{E} \exp \left[- \int_{\mathbb{R}^2} \exp \left(- \frac{\theta(\sigma^2 + I)}{P\ell(x - r(o))} \right) \delta(x) dx \right] \right).$$

An upper bound follows from Jensen's inequality:

$$\mathbb{P}_r | (n_o > 0) \leq \mu^{-1} \left(1 - \exp \left[- \int_{\mathbb{R}^2} \mathbb{E} \exp \left(- \frac{\theta(\sigma^2 + I)}{P\ell(x - r(o))} \right) \delta(x) dx \right] \right).$$

Similarly a lower bound can be obtained by using the inequality $\exp(-x) \geq 1 - x$:

$$\mathbb{P}_r | (n_o > 0) \geq \mu^{-1} \left(1 - \exp \left(- \int_{\mathbb{R}^2} \delta(x) dx \right) \mathbb{E} \exp \left(\int_{\mathbb{R}^2} \frac{\theta(\sigma^2 + I)}{P\ell(x - r(o))} \delta(x) dx \right) \right).$$

To evaluate the upper and lower bounds we observe that we will have to find $\mathbb{E}[\exp(-sI)]$, which follows from the derivation similar to $\nu(x_1, \dots, x_m)$:

$$\mathbb{E}[\exp(-sI)] = \prod_{a \in \mathbb{Z}^2} 1 - \int_{\mathbb{R}^2} \frac{g_a(y)(1 - \exp(-\int \delta(x) dx))}{1 + s^{-1}\ell(y + \frac{a}{\sqrt{\lambda_b}} - r(o))^{-1}} dy.$$

Recall that $\delta(x)$ is equal to

$$\eta(x) \exp \left(- \frac{\theta}{\text{SNR}} \frac{\ell(R)}{\ell(x)} \right) \prod_{y \in \mathbb{Z}^2 \setminus \{o\}} \frac{1}{1 + \frac{\theta}{\ell(x)} \ell(y/\sqrt{\lambda_b} - x)}.$$

We now find the asymptotic lower and upper bound when $\lambda_b = \text{SNR}^{-\beta}$ for large SNR. We first observe that

$$\delta(x) \sim \eta(x) \left(1 - \frac{\theta\ell(R)}{\ell(x)} \text{SNR}^{-1} - \frac{\theta}{\ell(x)} C(\alpha) \text{SNR}^{-\alpha\beta/2} \right).$$

It is also easy to obtain that

$$\mathbb{E}[\exp(-sI)] \sim 1 - \mu s C(\alpha).$$

After basic algebraic manipulation, it is established that both the upper and the lower bounds exhibit the same scaling which is

$$P_r | (n_o > 0) \sim \begin{cases} 1 - \text{SNR}^{-1} \left(\frac{1-\mu}{\mu} \right) \theta \ell(R) \int_{\mathbb{R}^2} \left[\frac{1}{\ell(x-r(o))} + \frac{1}{\ell(x)} \right] \eta(x) dx & \alpha\beta > 2 \\ 1 - \text{SNR}^{-\alpha\beta/2} \left(\frac{1-\mu}{\mu} \right) \theta C(\alpha) \int_{\mathbb{R}^2} \left[\frac{\mu}{\ell(x-r(o))} + \frac{1}{\ell(x)} \right] \eta(x) dx & \alpha\beta < 2. \end{cases} \quad (5.24)$$

Hence the gain is

$$\lim_{\text{SNR} \rightarrow \infty} G(\text{SNR}, \text{SNR}^{-\beta}) = \begin{cases} \frac{\mu}{1-\mu} \ell(R)^{-1} \left[\int_{\mathbb{R}^2} \left[\frac{1}{\ell(x-r(o))} + \frac{1}{\ell(x)} \right] \eta(x) dx \right]^{-1} & \alpha\beta > 2 \\ \frac{\mu}{1-\mu} \ell(R)^{-1} \left[\int_{\mathbb{R}^2} \left[\frac{\mu}{\ell(x-r(o))} + \frac{1}{\ell(x)} \right] \eta(x) dx \right]^{-1} & \alpha\beta < 2 \end{cases} \quad (5.25)$$

Hence the diversity of this scheme is

$$d_2(\text{SNR}^{-\beta}) = \min \left\{ 1, \frac{\alpha\beta}{2} \right\}.$$

In the above analysis we assumed that the cell is non-empty and hence obtained a maximum diversity of 1.

5.7 Simulation Results and Observations

In this section the gain of the proposed methods over direct transmission is obtained by Monte-Carlo simulations. For the purpose of simulation we truncate the BS lattice to $\lambda_b^{-1/2} \{-2, -1, 0, 1, 2\}^2$ and $\theta = 1.5$ is used as the decoding threshold. The cells are modeled as squares and the destination of each BS is

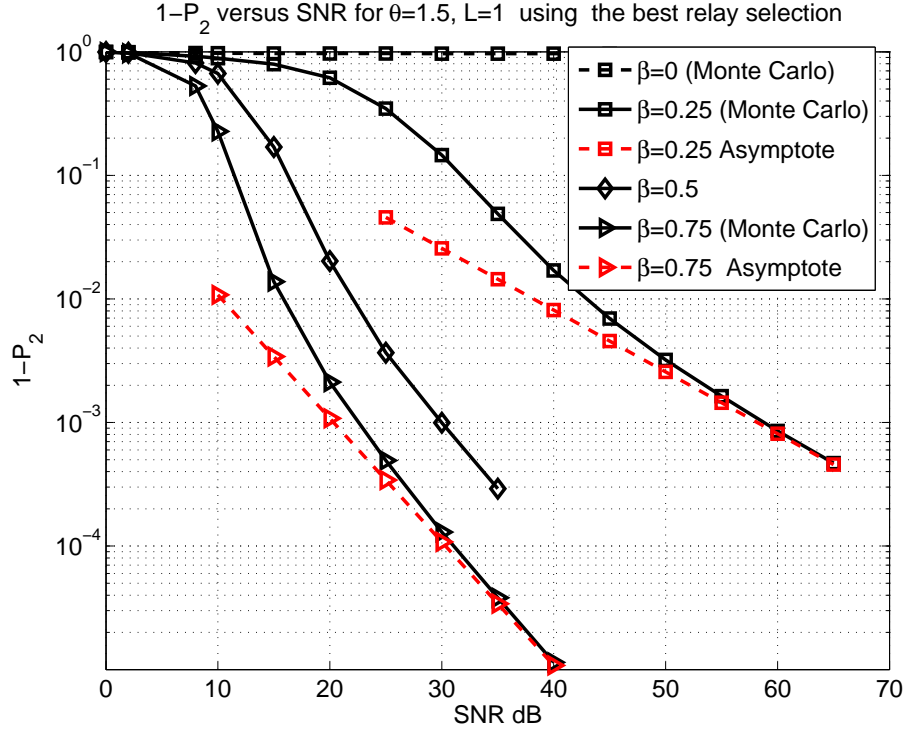


Figure 5.5. Outage probability $1 - P_r | (n_o > 0)$ versus SNR for $\lambda_b = \text{SNR}^{-\beta}$ and various β . The system parameters are $\alpha = 4$, $\theta = 1.5$, $z = (0.5, 0.5)$, $\ell(x) = (1 + \|x\|^4)^{-1}$ and $\eta(y) = 5\mathbf{1}_y([-0.5, 0.5]^2)$. The dashed lines are the asymptotes derived in (5.24) and are approximately equal to $0.812\text{SNR}^{-0.5}$ (interference limited) and 0.108SNR^{-1} (noise limited).

located at a random vertex of the square. The spatial density used is equal to

$$\eta(y) = \lambda_m \mathbf{1}_y([-L/2, L/2]^2).$$

If not specified we use $\lambda_m = 5$ and $L = 1$.

In Figures 5.4 and 5.5 the error probability of the schemes employing nearest relay to the destination and the best relay are plotted. We observe that the

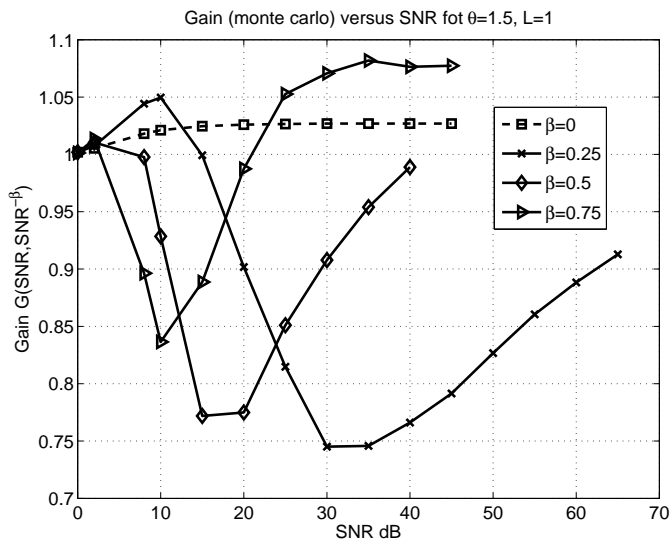


Figure 5.6. $G(\text{SNR}, \text{SNR}^{-\beta})$ versus SNR for various β . Relay closest to the destination is selected.

asymptotes obtained from theory match perfectly with the simulation results. As predicted by theory, the diversity obtained is 1 when $\alpha\beta > 2$ and is equal to $\alpha\beta/2$ otherwise. From Figure 5.6 and 5.7, it can be seen that the gain reaches a constant when $\text{SNR} \rightarrow \infty$. We observe that the best-relay selection scheme performs the best.

In Figure 5.9, we observe that the asymptotic gain increases exponentially with λ_m because of the $(1 - \mu)/\mu$ factor in the expression for the asymptotic gain. Setting $\lambda_b = \text{SNR}^{-\beta}$ reduces the spatial reuse factor as the SNR increases. The effective throughput density of the network is equal to $P_2 \log(1 + \theta) \text{SNR}^{-\beta}$ and the maximum of this throughput density is the transmission capacity. In Figure 5.10, we plot $(P_2 | n_o > 0) \log(1 + \theta) \text{SNR}^{-\beta}$ versus SNR for various β . We observe that for each SNR there is an operating β which maximizes the throughput density, and that as $\text{SNR} \rightarrow \infty$, the maximizing β tends to 0,

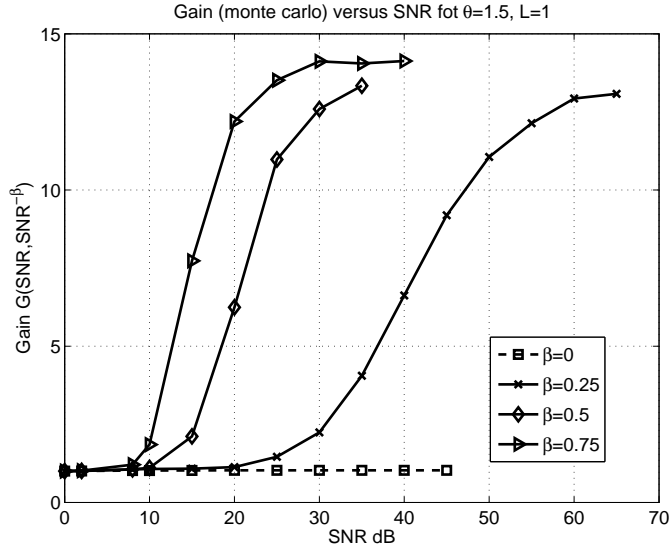


Figure 5.7. $G(\text{SNR}, \text{SNR}^{-\beta})$ versus SNR for various β . Relay with the best channel to the destination is selected.

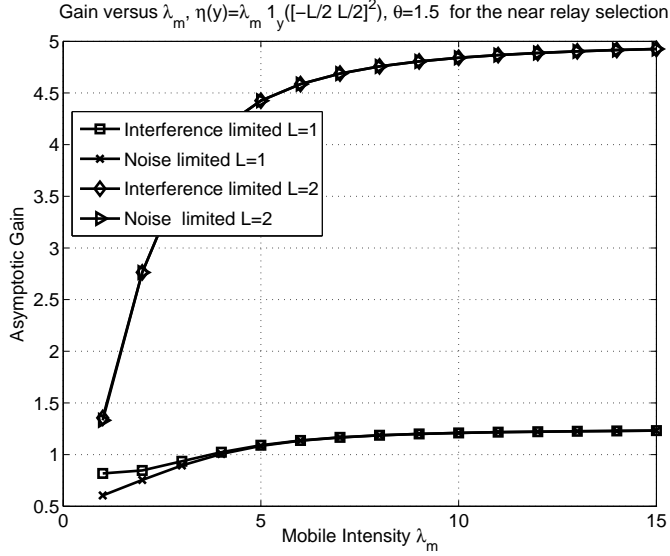


Figure 5.8. Asymptotic gain versus λ_m where λ_m is the intensity in $\eta(y) = \lambda_m \mathbf{1}_y([-0.5, 0.5]^2)$

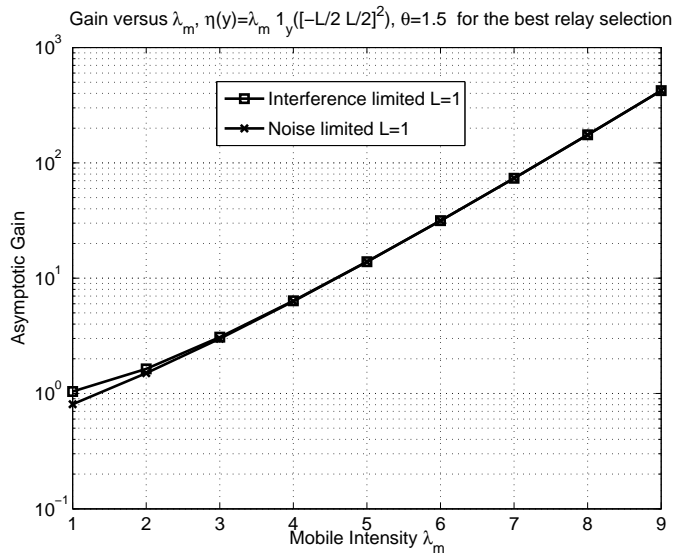


Figure 5.9. Asymptotic gain versus λ_m

which is intuitive. The figure indicates that a throughput density of $\approx 0.1\text{bps}/\text{m}^2$ is achieved at low SNR and it increases with SNR.

5.8 Chapter Summary

In this chapter we have analyzed the outage in a two-hop cellular systems inclusive of all the node location statistics. In particular we have analyzed two relay selection schemes in the context of cellular architecture. We have provide the success probability using tools from stochastic geometry and point process theory. We also have provided asymptotics of the outage with respect to SNR when the density of BSs scale like $\lambda_b = \text{SNR}^{-\beta}$ and proved that the diversity in the two relay selection schemes is $\min\{1, \alpha\beta/2\}$ where α is the path-loss exponent. We provided computable asymptotic results for the error probability and verified them with simulations. From simulations we observe that there is substantial

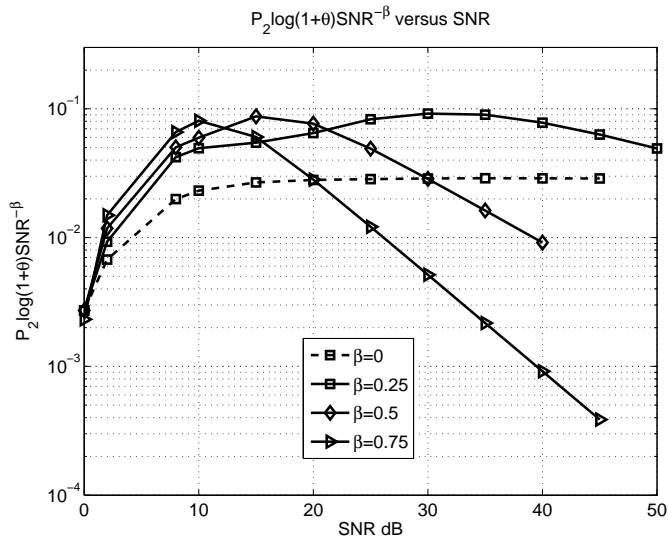


Figure 5.10. $(P_2 | n_o > 0) \log_2(1 + \theta) \text{SNR}^{-\beta}$ versus SNR for various β . The best relay selection scheme is used.

gain in using two-hop architecture compared to direct retransmissions and the gain increases with the BS-destination distance. The techniques introduced in this chapter can be easily extended for the spatial analysis of other relay selection schemes.

CHAPTER 6

MATHEMATICAL PRELIMINARIES

6.1 Point Processes

Let \mathbf{N} be the set of all sequences of points in \mathbb{R}^d , such that any sequence $\phi \in \mathbf{N}$:

- is finite, *i.e.*, has only a finite number of points in any bounded subset of \mathbb{R}^d
- is simple, *i.e.*, $x \neq y$ for any $x, y \in \phi$.

Point process. We shall use the notation $\phi(B)$, $\phi \in \mathbf{N}$ and $B \subset \mathbb{R}^d$, to denote the number of points of ϕ in B . Let \mathcal{N} denote the smallest sigma algebra so that the maps $\phi \rightarrow \phi(B)$ are measurable for all Borel subsets B of \mathbb{R}^d . \mathcal{N} is the equivalent of the Borel sigma algebra on the real line for the set of point sequences \mathbf{N} .

Definition 12. A point process Φ on \mathbb{R}^d is a measurable mapping from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to $(\mathbf{N}, \mathcal{N})$, *i.e.*,

$$\Phi : \Omega \rightarrow \mathbf{N}$$

So a point process is a random variable that takes values in the set of sequences \mathbf{N} . Each elementary outcome $\omega \in \Omega$ determines an entire point sequence $\Phi(\omega)$. The *distribution* of Φ is

$$\mathbf{P}(E) = \mathbb{P} \circ \Phi^{-1}(E) = \mathbb{P}(\Phi \in E) \quad \forall E \in \mathcal{N}.$$

Measurability requires that $N^{-1}(E) \in \mathcal{A}$. An element of \mathcal{N} is an event and can be viewed as a property of the point sequence. For example in a wireless network, $Y \in \mathcal{N}$ may represent the event that there are 10 wireless nodes in a unit ball around the origin, or it can represent the event that the minimum distance between any pair of nodes is greater than unity.

6.1.1 Poisson Point Process (PPP)

A stationary PPP of density (or intensity) λ is characterized by the following two properties:

- The number of points in any set $B \subset \mathbb{R}^d$ is a Poisson random variable with mean $\lambda|B|$.
- The number of points in disjoint sets are independent random variables.

From the definition we observe that

$$\mathbb{P}(\Phi(B) = k) = \exp(-\lambda|B|) \frac{(\lambda|B|)^k}{k!},$$

and in particular the void probability is given by $\exp(-\lambda|B|)$. The *thinning* of a PPP (*i.e.*, selecting a point of the process with probability p independently of the other points and discard it with probability $1 - p$) results in two *independent* PPP's of intensity measures $p\lambda$ and $(1 - p)\lambda$. An inhomogeneous PPP of intensity measure Λ is defined in a similar manner as the stationary PPP, except that the number of points in a set B is a Poisson random variable with mean $\Lambda(B)$. For example using $\Lambda(B) = (2\pi)^{-1} \int_B \exp(-\|x\|^2/2) dx$ results in a PPP with Gaussian density. A PPP with density λ , restricted to bounded domain A results in an

inhomogeneous PPP with $\Lambda(B) = \lambda|A \cap B|$. Observe that a stationary PPP is a special case of the inhomogeneous PPP with $\Lambda(B) = \lambda|B|$.

6.1.2 Poisson Cluster Process (PCP)

A PCP consists of a parent PPP $\Phi_p = \{\mathbf{x}_1, \mathbf{x}_2, \dots\}$ of density λ_p . The clusters are of the form $N^{\mathbf{x}_i} = N_i + \mathbf{x}_i$ for each $\mathbf{x}_i \in \Phi_p$. The N_i are a family of identical, independently distributed point sets and also independent of the parent process. The complete process Φ is given by

$$\Phi = \bigcup_{\mathbf{x} \in \Phi_p} N^{\mathbf{x}}.$$

The daughter points of the representative cluster N_o are scattered independently and with a identical spatial distribution

$$\Xi(A) = \int_A \zeta(x) dx, A \subset \mathbb{R}^2$$

around the origin. The number of points in a cluster may be random and we denote its mean by \bar{c} . We now provide some basic definitions and intensity measures associated with point processes.

Definition 13. *A point process $\Phi = \{x_n\}$ is said to be stationary if*

$$\mathbb{P}(\Phi \in Y) = \mathbb{P}(\Phi_x \in Y)$$

for all $Y \in \mathcal{N}$, where $\Phi_x = \{x_n + x\}$. A point process Φ is said to be isotropic if

$$\mathbb{P}(\Phi \in Y) = \mathbb{P}(\Phi_x \in rY),$$

where r is a rotation in \mathbb{R}^d .

A point process which is both stationary and isotropic is said to be motion-invariant. From the above definition we observe that a BPP is not a stationary point process, and that it is rotationally invariant when the domain is rotationally invariant. A PPP with constant intensity measure, *i.e.*, $\Lambda = \lambda$ is stationary and also rotationally invariant. We now define an equivalent of the mean of random variables for the point processes.

Definition 14. *The intensity measure of a point process Φ is equal to the average number of points in a set $B \subset \mathbb{R}^d$.*

$$\Lambda(B) = \mathbb{E}(\Phi(B))$$

If Φ is stationary, then $\Lambda(B) = \lambda|B|$ where λ is called the intensity (density) of the stationary point process Φ . We have

- For a stationary PPP, the intensity measure is equal to $\lambda|B|$.
- For a PCP, the intensity measure is equal to $\lambda_p \bar{c}|B|$.

The next theorem helps in evaluating sums of function over the point process.

Theorem 13. *Campbell's Theorem: Let $f(x) : \mathbb{R}^d \rightarrow [0, \infty]$ be a measurable function. Then*

$$\mathbb{E}\left(\sum_{x \in \Phi} f(x)\right) = \int_{\mathbb{R}^d} f(x) \Lambda(dx).$$

When Φ is stationary with intensity λ the right side is equal to $\lambda \int_{\mathbb{R}^d} f(x) dx$.

Definition 15. *The second order product density $\varrho^{(2)}$ of a point process is defined*

by the following relation:

$$\mathbb{E} \left(\sum_{\substack{\neq \\ \mathbf{x}_1, \mathbf{x}_2 \in \Phi}} f(\mathbf{x}_1, \mathbf{x}_2) \right) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x_1, x_2) \varrho^{(2)}(x_1, x_2) dx_1 dx_2$$

for any non-negative and measurable function f .

$\sum_{x,y}^{\neq}$ represents summation over distinct x and y . When Φ is stationary, $\varrho^{(2)}(x_1, x_2)$ depends only on $x_1 - x_2$. $\varrho^{(2)}(x_1, x_2) dx_1 dx_2$ can be interpreted as the probability that there exists two points of Φ in the infinitesimal regions dx_1 and dx_2 . We now provide an equivalent of the moment generating functional for the point process.

Definition 16 (Probability generating functional (PGFL)). *Let $\nu(x) : \mathbb{R}^d \rightarrow [0, \infty)$ be measurable. The PGFL of the point process Φ is defined as*

$$\mathcal{G}[\nu] = \mathbb{E} \prod_{\mathbf{x} \in \Phi} \nu(\mathbf{x}).$$

Observe that the PGFL is a functional, *i.e.*, acts on a function and when the function is a multivariate, a dot '·' is used to represent the variable that the PGFL acts on. For example $\mathcal{G}[\nu(\cdot + y)] = \mathbb{E} \prod_{\mathbf{x} \in \Phi} \nu(\mathbf{x} + y)$. The probability generating functional of a PPP it is equal to

$$\mathcal{G}[\nu] = \exp \left(- \int_{\mathbb{R}^d} (1 - \nu(x)) \Lambda(dx) \right). \quad (6.1)$$

The probability generating functional of the PCP is given by

$$\mathcal{G}[\nu] = \exp \left(- \lambda_p \int_{\mathbb{R}^d} 1 - M \left(\int_{\mathbb{R}^d} \nu(x + y) \zeta(x) dx \right) \right) \quad (6.2)$$

where $M(z)$ is the moment generating function of the number of points in the representative cluster. In general, Campbell's theorem is used to evaluate the average of a sum and the PGFL for the average of a product of a function over the point process. In the next section, we provide the equivalent of conditional probability for the point process.

6.2 Palm Distributions

Palm distributions are the counterparts to the conditional distributions for the point processes, and they arise when the point process is conditioned to have a point at $x \in \mathbb{R}^d$. The use of Palm measures arises in a wireless network when we calculate outage probabilities which requires conditioning on either the receiver or the transmitter location. We provide the definition of Palm distribution in terms of the Campbell measure, which is a measure on $\mathbb{R}^d \times \mathbf{N}$.

Definition 17. *The reduced Campbell measure of a point process is defined as*

$$C^!(A \times Y) = \mathbb{E} \left[\sum_{x \in \Phi \cap A} \mathbf{1}(\Phi \setminus \{x\} \in Y) \right]$$

for any Borel set $A \subset \mathbb{R}^d$ and $Y \in \mathcal{N}$.

An immediate consequence of this definition is the following theorem:

Theorem 14 (Mecke). *Let $f(x, \phi)$ be a measurable function on $\mathbb{R}^d \times \mathbf{N}$. Then*

$$\mathbb{E} \sum_{x \in \Phi} f(x, \Phi \setminus \{x\}) = \int_{\mathbb{R}^d} \int_{\mathbf{N}} f(x, \phi) dC^!(x, \phi)$$

When $C(\cdot \times Y)$ is absolutely continuous with respect to the intensity measure Λ , we have by the Radon-Nikodym theorem

$$C^!(A \times Y) = \int_{\mathbb{R}^d} \mathbf{P}^{!x}(Y) d\Lambda(x) \quad (6.3)$$

$\mathbf{P}^{!x}$ is called the reduced Palm measure of the process Φ . Intuitively this is equal to conditioning on the point process having a point at the origin but not counting it. From (6.3), Mecke's theorem and the definition of the Campbell measure, we have

$$\mathbb{E} \sum_{x \in \Phi} f(x, \Phi \setminus \{x\}) = \int_{\mathbb{R}^d} \mathbb{E}^{!x}(f(x, \Phi)) \Lambda(dx). \quad (6.4)$$

We now provide a brief description of the reduced Palm probability measure for PPP and PCP conditioned on a point being at the origin.

Theorem 15 (Slivnyak). *The Palm measure of a PPP is given by*

$$\mathbf{P}^{!o} = \mathbf{P}.$$

This is also a complete characterization of the PPP. It says that an additional point at o does not change the distribution of the other points of the PPP. Hence for a stationary PPP, the Mecke theorem reads as:

$$\mathbb{E} \sum_{x \in \Phi} f(x, \Phi \setminus \{x\}) = \lambda \int_{\mathbb{R}^d} \mathbb{E} f(x, \Phi) dx.$$

Theorem 16. *The Palm measure of a PCP is given by*

$$\mathbf{P}^{!o} = \mathbf{P} * \Omega^{!o},$$

where Ω^{lo} is the reduced Palm measure of the representative cluster N_o , and is given by

$$\Omega^{lo}(Y) = \frac{1}{\bar{c}} \mathbb{E} \sum_{\mathbf{x} \in N_o} \mathbf{1}_Y((N_o - \mathbf{x}) \setminus \{o\}).$$

* denotes the convolution of the distributions, which corresponds to the superposition of the two point measures.

Definition 18. For a point process ϕ , the second order moment measure is defined as

$$\mathcal{K}_2(B) = \mathbb{E}^{lo} \sum_{\mathbf{x} \in \Phi} \mathbf{1}_B(\mathbf{x}) \quad (6.5)$$

for any Borel set $B \subset \mathbb{R}^d$.

It is equal to the average number of points in the set B given that there is a point at the origin but without counting the point. We also have the following relation between the second order moment measure and the second order product measure

$$\lambda^2 \mathcal{K}_2(B) = \int_B \rho^{(2)}(x) dx. \quad (6.6)$$

An important characteristic of a stationary point process is Ripley's K-function defined as $K(r) = \mathcal{K}(b(o, r))$. For a PPP is equal to $c_d r^d$, where c_d is equal to the volume of the unit ball in d -dimensions. For any stationary point process $K(r) \sim c_d r^d$, $r \rightarrow \infty$. Similar to the reduced second moment measure, the reduced n -th factorial moment measure (2; 3) of a point process Φ , is defined as

$$\lambda^{n-1} \mathcal{K}_n(B) = \mathbb{E}^{lo} \left[\sum_{\substack{x_i \neq x_j \\ x_1, \dots, x_{n-1} \in \Phi}} \mathbf{1}_B(x_1, \dots, x_{n-1}) \right], \quad (6.7)$$

where $B = B_1 \times \dots \times B_{n-1}$, $B_i \subset \mathbb{R}^2$. When $\mathcal{K}_n(B)$ are absolutely continuous with respect to the Lebesgue measure, we denote the densities(44?) as $\rho^{(n)}$, the

exact relation being (in the stationary case)

$$\mathcal{K}_n(B) = \frac{1}{\lambda^n} \int_B \rho^{(n)}(x_1, x_2, \dots, x_{n-1}) dx. \quad (6.8)$$

Similar to the definition of the PGFL, the conditional PGFL is defined as :

Definition 19 (Conditional PGFL). *Let $v(x) : \mathbb{R}^d \rightarrow (0, \infty)$. The conditional PGFL is*

$$\tilde{\mathcal{G}}[v] = \mathbb{E}^{l_0} \left[\prod_{x \in \Phi} v(x) \right]. \quad (6.9)$$

For a PPP, by the Slivnyak theorem the conditional PGFL is equal to the PGFL, *i.e.*,

$$\tilde{\mathcal{G}}[v] = \exp \left(- \int_{\mathbb{R}^d} (1 - v(x)) \Lambda(dx) \right).$$

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