Linear and Rotational Kinematics

**Linear** \((a = \text{const})\)

\[
\begin{align*}
V &= V_0 + at \\
x &= x_0 + V_0 t + \frac{1}{2} at^2 \\
v^2 &= v_0^2 + 2a(\Delta x)
\end{align*}
\]

**Rotational** \((\alpha = \text{const})\)

\[
\begin{align*}
\omega &= \omega_0 + \alpha t \\
\theta &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \\
\omega^2 &= \omega_0^2 + 2\alpha(\Delta \theta)
\end{align*}
\]
Starting from rest, a disk takes 10 revolutions to reach an angular velocity \( \omega \). If the angular acceleration is constant throughout, how many additional revolutions are required to reach an angular velocity of \( 2\omega \)? (\( 2\pi \) (rev))

a) 10 revolutions

\[ \omega^2 = \omega_0^2 + 2\alpha (\Delta \theta_1) = 2\alpha (2\pi) \]

b) 20 revolutions

\[ (2\omega)^2 = \omega_0^2 + 2\alpha (\Delta \theta_1), 4\omega^2 = 2\alpha (\Delta \theta_2) \]

\[ \Delta \theta_2 = 4\Delta \theta_1 \rightarrow 40 \text{ rev.} \]

\[ \Delta \theta_2 - \Delta \theta_1 = 30 \text{ rev.} \]

c) 30 revolutions

d) 40 revolutions

e) 50 revolutions

\[ \therefore \text{solve for } t_1, \rightarrow \alpha \]

\[ \omega_f = \omega_0 + \alpha t, \quad \omega = \alpha t, \quad t = \frac{\omega}{\alpha}, \]

\[ \Delta \theta = \omega_0 t + \frac{1}{2} \alpha t^2, \quad 20\pi = \frac{1}{2} \alpha \left( \frac{\omega}{\alpha} \right)^2 = \frac{1}{2} \frac{\omega^2}{\alpha}, \quad \alpha = \frac{\omega^2}{40\pi} \]

\[ \omega_f^2 = \omega_0^2 + 2\alpha (\Delta \theta_2), \quad (2\omega)^2 = \omega^2 + 2 \left( \frac{\omega^2}{40\pi} \right) \Delta \theta_2, \quad 4\omega^2 = \omega^2 + \frac{\omega^2}{20\pi} \Delta \theta_2 \]

\[ 3\omega^2 = \frac{\omega^2}{20\pi} \Delta \theta_2, \quad \Delta \theta_2 = 60\pi = 30(2\pi) = 30 \text{ rev.} \]
A wheel has eight spokes and a radius of 30 cm. It is mounted on a fixed axle and is spinning at 2.5 rev/s. You want to shoot a 24-cm arrow parallel to this axle and through the wheel without hitting any of the spokes. Assume that the arrow and the spokes are very thin.

(a) What minimum speed must the arrow have to pass through without contact?

(b) Does it matter where you aim the arrow between the axle and the rim? If so, where is the best location? Why?

\[ \text{rev} = \frac{1}{2.5 \text{ rev/s}} = \frac{1}{2.5} \text{ sec}, \ 8 \text{ wedges} \]
\[ \text{r} \text{wedge} = \frac{1}{8} \cdot \frac{1}{2.5} = \frac{1}{20} \text{ sec} \]
\[ V_{\text{arrow}} T_{\text{wedge}} = L = 0.24 \text{ m}, \]
\[ V_{\text{arrow}} = \frac{0.24}{\frac{1}{20}} = \frac{0.24}{0.05} = 4.8 \text{ m/s} \]
At some point during its motion, a certain wheel turns through 90 rev in 15 s; its angular speed is 10 rev/s at the end of this period.

(a) What was the angular speed of the wheel at the beginning of the 15 s time interval, assuming constant angular acceleration?

(b) How much time had elapsed between the time the wheel was at rest and the beginning of the 15 s interval?

\[ \Delta \theta = \omega_0 t + \frac{(\omega_f - \omega_0)}{2} t, \quad 2\Delta \theta = (\omega_f + \omega_0) t, \quad \omega_0 t = \frac{2\Delta \theta - \omega_f t}{t} \rightarrow \omega_0 = \frac{\omega_f - \omega_0}{t} = \frac{4\pi}{15} \text{ rad/s} \]

\[ \alpha = \frac{\omega_f - \omega_0}{t} \]

\[ \Delta \theta = \omega_0 t + \frac{1}{2} \alpha t^2 \]

\[ d\theta = \omega_0 t + \frac{1}{2} (\omega_f - \omega_0) t^2 \]

\[ \frac{\Delta \theta}{t} = \frac{\omega_0}{2} + \frac{\omega_f}{2} \]

\[ \frac{2\Delta \theta}{t} - \omega_f = \omega_0 \text{ in } 4\pi/\text{rad} \]

\[ \omega_f = \omega_0 + \alpha t \]

\[ \alpha = \frac{2\pi r - 4\pi}{15} = \frac{15}{15} \pi \text{ rad/s}^2 \]

\[ \frac{4\pi}{15} = \theta + \frac{16}{15} \pi t \]
A gyroscope flywheel of radius 2.83 cm is accelerated from rest at 14.2 rad/s² until its angular speed is 2760 rev/min. → 289 rad/s

(a) What is the tangential acceleration of a point on the rim of the flywheel?

\[ a_t = r \alpha \] , \[ a_t = (0.0283)(14.2) = 0.4 \text{ m/s}^2 \]

(b) What is the radial acceleration of this point when the wheel is spinning at full speed?

\[ a_r = r \omega^2 = \frac{v^2}{r} \] (\( v = r \omega \)) \[ a_r = (0.0283)(289)^2 = 2364 \text{ m/s}^2 \]

(c) Through what distance does a point on the rim move during the acceleration?

\[ \omega_f^2 = \omega_0^2 + 2 \alpha (\Delta \theta) \] , \[ 2941 \text{ rad}, \sim 470 \text{ rev.} \]

\[ \Delta s = r \Delta \theta \] , \[ \Delta s = (0.0283)(2941) = 83.2 \text{ m} \]
Rotational Dynamics

Rotational analogue of force = "Torque"

\[ \tau = F \cdot l, \]

\( l \) is perp. distance to pivot.

F produces a torque \( \tau \).
You are using a wrench and trying to loosen a rusty nut. Which of the arrangements shown is most effective?

\[ b < a = d < c \]

\[
\text{most effective}
\]
Torque and Newton’s 2nd Law for Rotation

\[ \tau = F_T \cdot r = ma \cdot r = mr^2 \alpha \]

\( \tau = \frac{\dot{\alpha}}{I} \) \quad \text{"Moment of Inertia"}

\( \tau = \frac{\dot{\alpha}}{I} \) \quad \text{Newton’s 2nd Law for Rotation}

All quantities calculated w.r.t. one axis of rotation.

\[ F_T = 1 \tilde{F} \cdot \sin \phi \quad , \quad \tau_{\perp} = 1 \tilde{\tau} \cdot \sin \phi \]

\[ \tau = F_T |\tilde{r}| = 1 \tilde{F} |\tilde{r}| \sin \phi \]

\[ \tau = |\tilde{F}| |\tilde{r}| \sin \phi \]

\( (\vec{\tau} = \vec{r} \times \vec{F}) \quad \left[ |A \times B| = |A||B| \sin \phi \right] \)
Moment of Inertia and Newton’s 2\textsuperscript{nd} Law

Two objects, stuck together

\[ I = m r^2 \]

\[ \sum \tau = F_{1T} \cdot r_1 + F_{2T} \cdot r_2 = m_1 a_{T_1} \cdot r_1 + m_2 a_{T_2} \cdot r_2 \]

\[ = m_1 r_1^2 \alpha_1 + m_2 r_2^2 \alpha_2 = (m_1 r_1^2 + m_2 r_2^2) \alpha \]

\[ \alpha_1 = \alpha_2 = \alpha \text{ const across the object} \]

\[ = (I_1 + I_2) \alpha = I_{\text{tot}} \alpha \]

\[ I_{\text{tot}} = \sum_i m_i r_i^2 \]

\[ \sum \mathcal{L}_{\text{ext}} = I_{\text{tot}} \alpha \]

\[ N \text{ 2\textsuperscript{nd} Rot.} \]
A force $F$ is applied to a dumbbell for a time interval $\Delta t$, first as in (a) and then as in (b). In which case does the dumbbell acquire the greater center-of-mass speed?

1. (a)
2. (b)
3. no difference
4. The answer depends on the rotational inertia of the dumbbell.

$$\vec{F}_{\text{net}} = M\vec{a}_{\text{CM}}$$
Calculating Moments of Inertia:

A set of four atoms are bound into a square with sides of length $a$. If a constant torque is applied to one of the masses so that the system will rotate about the axes shown, which of the following configurations will have the largest angular acceleration?

(a) $I_a = 2m_i r_i^2 = 4m \left(\frac{a}{2}\right)^2 = ma^2$

(b) $I_b = 2ma^2 = 2ma^2$

(c) $I_c = 4m \left(\frac{a}{\sqrt{2}}\right)^2 = 4m \frac{a^2}{2} = 2ma^2$

$I_a$ is smallest.
A constant *tangential* force is applied to one of atoms in configurations a) and b), below, forcing the square to spin about the axes shown.

The angular acceleration of square b) is

1.) half  
2.) the same as  
3.) twice  
4.) four times

that of square a).

\[
\tau_a = F \left( \frac{a}{2} \right) = I_a \alpha_a = \frac{Fa}{2} = ma^2 \alpha_a
\]

\[
\alpha_a = \frac{F}{2ma} \quad \leftarrow \quad \alpha_b = \frac{F}{2ma}
\]

\[
\tau_b = Fa = I_b \alpha_b
\]

\[
Fa = 2ma^2 \alpha_b
\]
Moments of Inertia for Macroscopic Objects

\[ I = \sum_i m_i \cdot r_i^2 = \int r^2 \, dm(r) \quad \text{2nd moment} \]

\[ \text{CM} = \int r \, dm \quad \text{1st moment} \]

\[ M \times _{cm} = \]

\[ M = \int dm \quad \text{0th moment} \]
Calculate the moment of inertia of a hoop of radius $R$ and mass $M$ about an axis passing through the center of the hoop and perpendicular to the plane of the hoop.

\[ I = \int r^2 \, dm \]
\[ = \int R^2 \, dm = R^2 \int \text{object} \, dm \]
\[ = MR^2 \]

\( \text{equivalent to a point of mass } M \, @ \, R \).
Calculate the moment of inertia of a disk of radius $R$ and mass $M$ about an axis passing through the center of the disk and perpendicular to the plane of the disk.

\[ I = \int r^2 dm \]

\[ = \int_0^R r^2 dm = \int_0^R \frac{2M}{R^2} (r^2) r \, dr \]

\[ = \frac{2M}{R^2} \int_0^R r^3 \, dr = \frac{2M}{R^2} \left( \frac{1}{4} R^4 \right) \]

\[ I = \frac{1}{2} MR^2 \]

\[ dA = 2\pi r \, dr \]

\[ \Rightarrow dm = \frac{M}{\text{Area}} \, dA = \frac{M}{\pi R^2} (2\pi r \, dr) = \frac{2M}{R^2} r \, dr \]

\[ A = \pi R^2, \quad \rho = \frac{M}{A} = \frac{M}{\pi R^2} \]
The Parallel Axis Theorem

\[ I_{\text{new}} = I_{cm} + M L^2 \]
Two identical solid spheres of mass $M$ and radius $R$ are joined together, and the combination is rotated about an axis tangent to one sphere and perpendicular to the line connecting them. What is the moment of inertia of the combination? (Inertia Table says $I_{\text{sphere}} = \frac{2}{5}MR^2$ about any diameter.)

\[ I_{\text{total}} = I_1 + I_2 \]

\[ I_1 = I_{\text{cm}} + MR^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2 \]

\[ I_2 = \frac{2}{5}MR^2 + M (3R)^2 = \frac{2}{5}MR^2 + 9MR^2 = \frac{47}{5}MR^2 \]

\[ I_1 + I_2 = \frac{54}{5}MR^2 \]

\[ \left( "M" (2R)^2 \right. \quad \left. \frac{54}{5}MR^2 \right) \]

\[ \left. \quad "M" \neq 2M \right) \]
Now, onto Dynamics:

A constant force $F$ pulls a string attached to the rim of a disk of radius $R$ and mass $M$ which is free to spin on an axis through its center. Find the angular acceleration, velocity, and position as a function of time, assuming the disk starts from rest.

\[ \tau = I \alpha = \frac{1}{2} MR^2 \alpha \]
\[ FR = \frac{1}{2} MR^2 \alpha \quad \Rightarrow \quad \alpha = \frac{2F}{MR} \]

\[ \omega = \omega_0 + \alpha t = \frac{2F}{MR} t \]

\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 = \frac{F}{MR} t^2 \]
A mass \( M \) hangs on a massless rope attached to the rim of a vertical wheel of radius \( R \). The wheel is fixed but is free to rotate about its center, and it has a moment of inertia \( I \). a) Draw free-body diagrams for the wheel and the block, and write Newton’s second law appropriate to each object. b) When the mass \( m \) is released from rest, it falls a distance \( D \) in time \( t \). Find the acceleration of the block and the angular acceleration of the wheel in terms of \( D \), and \( R \). c) Find the tension \( T \) in the rope as the mass is falling.

\[ \begin{align*}
\text{a. Disk: } & \quad I \alpha = T R = \frac{1}{2} M R^2 \alpha \\
\text{block: } & \quad F = m g = m g - T \\
\text{b. } & \quad y = \frac{1}{2} a_y t^2, \quad D = \frac{1}{2} a_y t^2, \quad a_y = \frac{2D}{t^2}, \quad a_x = a_T = R \alpha \\
\alpha & = \frac{a_x}{R} = \frac{2D}{R t^2} \\
\text{c. } & \quad T \cdot R = \frac{1}{2} M R^2 \cdot \frac{2D}{R t^2} = \frac{M D}{t^2}, \quad T = \frac{I}{R} \frac{2D}{R t^2} = \frac{I}{R^2 t^2} \\
m \frac{2D}{t^2} = m g - T, \quad T = m g - m \left( \frac{2D}{t^2} \right)
\end{align*} \]
A cylinder of radius $R$ and mass $M$ has a string wrapped around it. It is released from rest, and the string is pulled upward so that the center of mass of the cylinder does not move. Find a) the tension in the string and b) the angular acceleration of the cylinder.

\[
\begin{align*}
\text{a)} \quad & \tau_F \mathbf{y} = Ma_{\text{om}} = 0 = T - Mg \\
& T = Mg \\
\text{b)} \quad & \tau \mathbf{c}_m = I \alpha = T \cdot R = MgR \\
& (mg \text{ has } r=0, \text{ passes through } C_{\text{om}}) \\
& I_{\text{om}} = \frac{1}{2}MR^2 \\
& (\tfrac{1}{2}MR^2) \alpha = MgR \\
& \alpha = \frac{2g}{R}
\end{align*}
\]
Same problem, except that now the string is held fixed and the cylinder falls. Find the tension in the string and the linear acceleration of the center of mass.

\[ \sum F_y = Ma_y = Mg - T \]

\[ \sum T = I_{cm} \alpha = \frac{1}{2} MR^2 \alpha = T \cdot R \]

\[ R \alpha_{cm} = a_{cm}, \quad \alpha_{cm} = \frac{a_{cm}}{R} \]

\[ \frac{1}{2} MR^2 \frac{a}{R} = T \cdot R, \quad T = \frac{Ma}{2} \]

\[ \sum F \Rightarrow Ma = Mg - \frac{Ma}{2}, \quad \frac{3}{2} Ma = Mg \]

\[ a = \frac{2}{3} g \]

\[ \frac{2}{3} Mg = Mg - T, \quad T = \frac{1}{3} mg \]
Falling Bar: at what angle does the vertical acceleration of the tip of the bar exceed g?

\[ \Sigma F = Mg \cos \theta \frac{L}{2} = I \alpha = \frac{1}{3} ML^2 \alpha \]

\[ \alpha = \frac{3}{2} \frac{g}{L} \cos \theta \]

Linear acceleration of end:

\[ a_T = L \alpha = \frac{3}{2} g \cos \theta \]

\[ a_y = \text{vert. comp.} \quad a_T \cos \theta = \frac{3}{2} g \cos^2 \theta \]

\[ \frac{3}{2} g \cos^2 \theta > g, \quad \cos^2 \theta > \frac{2}{3}, \quad \theta \approx 35^\circ \]
Energy of Rotation

\[ KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \ldots \]

for pure rotation, \( w \) is uniform, \( v_i = r_i \cdot w \)
velocities are tangential.

\[ KE = \frac{1}{2} m_1 r_1^2 w^2 + \frac{1}{2} m_2 r_2^2 w^2 + \frac{1}{2} m_3 r_3^2 w^2 \]

\[ = \frac{1}{2} \left( m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 \right) w^2 \]

\[ = \frac{1}{2} I w^2 \]
Energy of Rotation, cont.

\[
KE = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I_{cm} \omega_{cm}^2
\]

\[ \vec{r}_n = \vec{r}_{cm} + \vec{r}_n \]

\[ \vec{v}_n = \vec{v}_{cm} + \vec{v}_n \]

\[ |\vec{r}_n| = \text{const for rigid object} \]

\[ |\vec{v}_n|^2 = \vec{v}_{ni} \cdot \vec{w}_{cm} \to \text{tangential} \]

So,

\[
KE_n = \frac{1}{2} m_n \vec{v}_n \cdot \vec{v}_n = \frac{1}{2} m_n \left( v_{cm}^2 + 2 \vec{v}_{cm} \cdot \vec{v}_n + v_{n i}^2 \right)
\]

\[
KE_{total} = \sum_n \frac{1}{2} m_n \left( v_{cm}^2 + 2 \vec{v}_{cm} \cdot \vec{v}_n + v_{n i}^2 \right)
\]

\[
= \frac{1}{2} \left( \sum_n m_n \right) v_{cm}^2 + \frac{1}{2} \left( \sum m_n \vec{r}_n^2 \right) \omega_{cm}^2 + \sum \frac{1}{2} m_n \left( 2 \vec{v}_{cm} \cdot \vec{v}_n \right)
\]

\[
= \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I_{cm} \omega_{cm}^2 + \vec{v}_{cm} \cdot \left( \sum m_n \vec{v}_n \right)
\]