Objectives
- To understand oscillation in relation to equilibrium of conservative forces
- To manipulate the independent variables of oscillation: the amplitude, the frequency, and the phase
- To predict the equation of motion of an oscillating system
- To understand conservation of energy in an oscillating system and calculate the total energy of such a system.

Introduction
In any system in which a conservative force acts on an object, we can define a potential energy function as part of describing how that force will affect the object's motion. As you have learned, the potential energy $U$ in a one-dimensional system is related to the force by

$$\frac{dU(x)}{dx} = -F(x) \quad (1)$$

This tells us that wherever there is a local minimum in a potential energy vs. displacement graph, the force at that point is zero. (Technically, this is also true of a local maximum; maxima of gravitational potential energy are common, but those of other force types are less likely to be encountered. You will only be dealing with minima in today's lab.) In an experiment where all forces are conservative, then, the point of maximum kinetic energy should equal the point of minimum potential, and thus the point of force equilibrium.

In Fig 6-1 at right, all forces are balanced when the object is at point $x_0$. When it moves away from $x_0$, forces will decelerate it until it stops, returns, and passes $x_0$. In the absence of friction, this process would keep going forever, as the body will ride up the potential curve until all the kinetic energy it had at point $x_0$ has been converted to potential energy.

Whenever a potential energy vs. displacement graph displays a "potential well" like this, we can use a Taylor approximation to estimate the potential energy very close to the point of equilibrium:

$$U(x - x_0) = U(x_0) + U'(x_0)(x - x_0) + \frac{1}{2} U''(x_0)(x - x_0)^2 + \cdots \approx U(x_0) + \frac{1}{2} k(x - x_0)^2 \quad (2)$$

This is a good approximation because $U'(x_0) = 0$, and sufficiently close to $x_0$ the second derivative $U''$ is approximately constant. We'll call that constant $k$.

Within the domain in which this holds (which must be determined for any particular problem or
experiment) the force as the body moves away from \( x_0 \) is therefore

\[
F(x) = -\frac{dU(x)}{dx} = -k(x - x_0) \tag{3}
\]

As you know, when a force varies directly with the displacement from stable equilibrium, we call the resulting oscillation “simple harmonic motion”. As this derivation shows, any time there is a local minimum in potential energy, sufficiently small oscillations will be simple harmonic motion.

**Oscillation on a spring**

The simplest setup to use for observing simple harmonic motion is a spring with a mass suspended from one end. By Hooke’s Law, the force exerted by the spring varies directly with distance from equilibrium – which is precisely the definition of simple harmonic motion! In other words, equation 3 above holds for an object on a spring at all points, not merely within a small neighborhood of \( x_0 \).

\[
F_{spring} = k(x - x_0) \tag{4}
\]

As we have done before when we do spring experiments, we set the equilibrium point \( x_0 \) as the origin \( (x = 0) \) for simplicity.

To find the displacement-time equation for this situation, we first find the acceleration at any point:

\[
F = ma = m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{5}
\]

To solve the equation on the right requires knowledge of advanced differential equations, but the solutions take the following general form:

\[
x(t) = A\sin(\omega t - \delta) \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}
\tag{6}
\]

\( A \) and \( \delta \) are arbitrary constants introduced when the equation is integrated. You can verify yourself that this is a solution to equation 5.

The position of the body, when plotted against time, will look something like this:

![Fig 6-2](image)

Each of the three variables \( A, \omega, \) and \( \delta \) modifies the function in an important way. As you can see from the algebra, multiplying \( \sin \omega t \) by a constant \( A \) will stretch the function vertically.

![Fig 6-3](image)
\( \omega \) determines the period of the oscillation:

\[ x(t) = A \sin(\omega t) \]

\( \delta \) indicates the initial phase of the oscillation. A positive value for \( \delta \) shifts the graph to the right:

\[ v(t) = \omega A \cos(\omega t) \]

As a final note, observe the similarity between the graphs of position and velocity in an oscillating system:

The velocity graph is the first derivative of the position graph, as you would expect. Note that at \( t=0 \), when position is also 0, the velocity is at its maximum.

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Note: the time \( t_0 \) can be obtained by noting that \( \sin(\omega t_0 - \delta) = 0 \), so \( (\omega t_0 - \delta) = 0, \pm \pi, \text{ etc.} \). This allows you to solve for \( t_0 \):

\[ t_0 = \frac{\delta}{\omega}, \text{ or} \]

\[ t_0 = \left( \frac{\delta \pm \pi}{\omega} \right), \text{ depending on the initial conditions.} \]

What is the correct choice here?
INVESTIGATION 1: BASIC HARMONIC MOTION

You will need the following materials for this investigation:

- Harmonic motion assembly with a vertically hanging spring
- Two hanging masses
- Motion detector
- Meterstick

Activity 1-1: The Equation of Harmonic Motion

1. Find the mass of each of your mass bobs.
   \[ m_1 = \]
   \[ m_2 = \]

2. Measure how far each mass stretches the spring. Use this displacement to calculate k, the spring constant. Remember to use correct units (N/m).
   \[ \Delta y_1 = \]
   \[ k = \]
   \[ \Delta y_2 = \]
   \[ k = \]

Question 1-1: How well do these measurements of k agree? You should redo the measurements if they differ by more than 10%.

3. Open a new activity in Data Studio. You should only have a motion detector hooked up to the computer. In Setup, set the detector to read 20 data points per second and check “position”, “velocity”, and “acceleration”. Open a graph of position vs. time.

4. Position the motion detector carefully under the hanging spring. Make sure that the detector is as close to vertical as possible.

5. Use the lighter mass bob for this experiment.

6. Do a practice run on the computer: pull the mass bob up until the spring is fully compressed and let it fall. Make sure that it does not hit the motion detector. With the bob now oscillating up and down, start the Data Studio run and make sure that the detector can see the mass bob along its whole path. Make sure the bob is oscillating vertically and has no swinging motion.

7. When you get a clean sinusoid on your position graph, add plots of velocity and acceleration on the same graph window. Print out this graph. Once you have the sheet of paper:
   a. Draw vertical lines connecting the maxima of the position graph to the velocity and acceleration. When the position is at its extrema, what is the velocity and acceleration?
   b. On the acceleration graph, draw the direction of the force where the acceleration is at its largest positive or negative values. Is it along the direction of motion (velocity), or against it? Does it have the same sign as the displacement, or opposite?
8. Set the meterstick up next to the hanging spring. Note the height of the mass bob at equilibrium as measured by the meter stick, and read off the graph the equilibrium height as measured by the motion sensor. These will be different; the motion sensor value is what will be reported:

\[
\text{equilibrium height (meter stick)} = \]

\[
\text{equilibrium height (motion sensor)} = \]

**Prediction 1-1:** Give the equation of the motion of this system if you set it to oscillating by pulling the bob down 20 cm and releasing. Use the value of \( k \) that you derived on the previous page and your measured \( m \) to calculate the frequency. Getting the phase right is the hard part. Sketch what the graph will look like on the axes below. Refer to (eq.6) in the introduction to this lab if you are confused. Label your graph with values for the amplitude, period, and phase offset. Use positive values for positions above equilibrium and negative values below.

\[
y(t) = \]

9. To un-clutter you window, delete the velocity and acceleration graphs. Now, pull the mass bob down 20 cm from equilibrium. Start recording data, then release the mass bob. Record data for at least 10 seconds.

Click and drag to select your data beginning when you released the mass bob.

10. Under Fit, select “Sine”.

**Question 1-2:** The sine-fitting procedure in Data Studio will return four constants: A, B, C, and D. What do these constants mean? You must find out by double-clicking the box containing the fit parameters. Write the equation below, using A, B, C, and D in their proper place, and give the physical meaning (e.g., amplitude, phase, etc.) of each. You may want to express these in terms of \( \omega \) and \( \delta \) as we have defined them:

\[
y(t) = \]

\[
A = \]

\[
B = \]

\[
C = \]

\[
D = \]

Also, in terms of A, B, C, and/or D, what are \( \omega \) and \( \delta \)?

\[
\omega = \]

\[
\delta = \]

**Question 1-3:** The phase-offset parameter of the sine function (\( \delta \) in (eq.6)) which your graph yields will probably not be equal to the one you predicted in Prediction 1-1. This is because you did not start
recording data at time \( t = 0 \), but held the mass bob until you knew the recording had already begun. The DataStudio sine fit has an option that will shift the fit over and take the offset out for you. Double click on the fit parameter box again, and select “Fit with first selected data point at X=0.” You have to do this EVERY TIME YOU FIT. (It’s simpler that way...) Calculate values of Amplitude, \( \omega \) and \( \delta \) from the fit parameters, and give them here:

Question 1-4: How well did you predictions match your experimental results?

11. Print out a copy of your graph, including your fit curve and data, to turn in at the end of lab.

INVESTIGATION 2: ADJUSTING DEPENDENT AND INDEPENDENT VARIABLES

There are only three variables in the classical harmonic-motion equation (6), and four in the sine-fit routine which Data Studio applies to an experiment. The fourth is merely a vertical translator: in (6), it is assumed that the system is at equilibrium at position = 0, while in the experiments you are running today, the zero position is occupied by the motion detector.

As you saw in the previous investigation, two of these variables are completely controlled by the experimenter: amplitude and phase. The third, period, a function of \( \omega \), is completely dependent on the experimental setup.

This means that you should be able to adjust each of these variables and not affect any of the others. You will test this hypothesis in this investigation. You will use your data from the previous investigation, and compare with the data which results when you change one of these variables at a time.

Activity 2-1: Varying Amplitude

1. In this experiment you will record the motion of the mass bob with different amplitudes. You will once again use the lighter of the two masses.

Prediction 2-1a: Predict the equation for the position-time graph which would result if you pulled the mass bob down to only 10 cm. Draw it on the following axes, with your predictions for the graph’s amplitude, period, and phase offset.

\[
y(t) =
\]

\[
y
\]

\[
t
\]
2. Perform the experiment using the exact same procedure you did in the previous investigation, except this time, you will pull it down only 10 cm.

3. Using the sine fit, find the equation for the bob’s motion. Make sure that you have selected data with the mouse, and that you tell the fit to start with the first point at x=0.

   \[ y(t) = \]

**Question 2-1:** Compare this equation with the one you determined in Investigation 1. Are your values for \( \omega \) the same? For \( \delta \)? Why or why not?

**Question 2-2:** Compare your values of \( \omega \) and \( \delta \) in Investigation 1 and this activity: how did changing the function’s amplitude affect the period and phase?

**Activity 2-2: Varying \( \omega \)**

In the last activity, we played with the amplitude of our motion and asked if changing it affected the rest of the experiment. In this activity, we will play with the quantity \( \omega \), which the theory says should equal \( \sqrt{\frac{k}{m}} \). As we are using only one spring, we will vary \( \omega \) by using different masses.

1. Take the heavier mass and hang it from the spring.

**Prediction 2-2:** Predict the equation of the position-time graph which will result from releasing the heavier mass bob from a spring extension of 20 cm. Draw it on the following axes, with your predictions for the graph’s amplitude, period, and phase-offset.

   \[ y(t) = \]

2. Hold the mass bob 15-20 cm below equilibrium, start recording data, and release the bob.

3. Fit your sine curve to the data. Record it below. Make sure you shifted the x=0 of the fit.

   \[ y(t) = \]

4. Print out a copy of your graph to turn in at the end of lab.
Question 2-4: How well did the data match your predictions? Explain any discrepancies.

Question 2-5: Compare the bob’s motion in this experiment to its motion in Investigation 1: was it affected in any way? Was the amplitude or the phase affected by the change in mass?

Activity 2-3: Varying Phase

In this activity we will compare what happens when you start the bob oscillating from different points in its trajectory.

1. Use the lighter mass bob for this activity.

Prediction 2-3: Predict the equation of the position-time graph which will result from releasing the mass bob from 20 cm above equilibrium. Draw it on the following axes, with your predictions for the graph’s amplitude, period, and phase-offset.

\[ y(t) = \]

2. Hold the mass bob 20 cm above equilibrium. Start recording data, then release the bob. Record data for at least 10 seconds.

3. Fit a sine curve to your data. Make sure you shift the first point to \( x=0 \). Record its equation:

\[ y(t) = \]

4. Print out a copy of your graph to turn in at the end of lab.

Question 2-6: Was your prediction correct? Explain any discrepancies.

Question 2-7: Compare this equation to the equation from Investigation 1. What is different?

5. For the last experiment in this investigation, you will start the mass bob oscillating from the equilibrium point by giving it a push up or down. The amplitude of the oscillation should be close...
to 20 cm, though it need not be exact. You may need to try a few times before you get a good oscillation.

**Prediction 2-4**: Predict the equation of the position-time graph which will result from starting the mass bob’s oscillation at the equilibrium point. Draw it on the following axes, with your predictions for the graph’s amplitude, period, and phase-offset.

\[
y(t) = \]

6. Start recording data, then set the system oscillating as described above. Try to set the bob oscillating as sharply as possible (i.e. take as little time as possible in accelerating the bob), because it is only when you release the bob that it will exhibit true sinusoidal motion.

7. When you get an amplitude close to 20 cm, click and drag to select data from when you set the bob in motion. Record the equation of the curve (make sure you make the first point \(x=0\)):

\[
y(t) = \]

8. Print out a copy of your graph to turn in at the end of lab.

**Question 2-8**: Was your prediction correct? Explain any discrepancies.

**Question 2-9**: Compare your two graphs from this Activity and the graph from Investigation 1. As you varied the initial phase of the oscillation, how did the equation change?
INVESTIGATION 3: ENERGY IN AN OSCILLATING SYSTEM

Let’s go back to the idea of the “potential well” discussed in the introduction. The shape of such a well indicates what kind of force we are dealing with: gravity, which is (almost exactly) constant over short distances, yields a straight-line potential graph with slope g. A spring force, which varies linearly with distance from equilibrium, yields a parabolic potential graph with the vertex at the equilibrium point.

This is important, because only a parabolic potential function will yield true sinusoidal harmonic motion at all amplitudes. Other potentials will approximate sinusoidal motion at small amplitudes, but as the body swings further from equilibrium, its motion will begin to diverge from a true sine curve more and more.

The experimental setup you are using today gives results which map to sine functions with very little deviation. This indicates that the potential energy under which the bob is acting is quadratic with respect to distance from equilibrium.

As it turns out, we can disregard gravity in calculating the potential energy of a spring supporting a hanging mass because of the linear nature of a spring force. Whether your spring is sitting at its “true equilibrium” or supports a mass bob which stretches it a distance of $x_0$ to a new equilibrium, stretching or compressing that spring an additional distance $y$ will result in the same force $ky$ being exerted on the mass.

The effective potential energy at equilibrium is 0 (this is precisely what equilibrium means); the gravitational potential downward is balanced by the spring’s potential energy, essentially the chemical potential energy holding the metal together and pulling the end of the spring upward when it is stretched. As the string stretches farther, $E_{GP}$ does not increase, but $U_{spring}$ does, meaning the net tendency is back toward equilibrium; if the spring is compressed past equilibrium, the spring’s potential decreases (relative to its “true” equilibrium point), and the gravitational potential pulls the mass bob back down.

Activity 3-1: Kinetic Energy from Spring Potential

1. Under Setup, select the checkboxes next to “position” and “velocity” under the motion sensor.
2. Using the calculator, program two equation measurements into the experiment: kinetic energy and spring potential energy. Remember that, in giving the computer the equation for potential energy, you must find the equilibrium point and define an equation to find the bob’s distance from that point.
3. Define a third function composed simply of the sum of the other two functions: this is the total mechanical energy of the system.

Enter your equations here:

$KE =$
U = 
ME = 

4. You may use whichever mass bob you like for this Investigation. Make sure you enter the correct mass in your equations.

**Prediction 3-1:** Predict the maximum and minimum kinetic energies of the system if it is set into oscillation with an amplitude of 20 cm.

**Prediction 3-2:** Predict the behavior of the system's mechanical energy as you have defined it in your calculated function. Will it be constant, or will it vary? If it varies, predict what pattern it will follow.

5. Test your predictions. Set the system in motion by holding the mass bob 20 cm below equilibrium, starting the data recorder, and releasing.

6. Print out graphs of position and velocity on one sheet and the three calculated energy functions on another sheet. “Scroll” the axes so you can clearly see the potential and kinetic energies oscillating back and forth.

**Question 3-1:** Were your predictions correct? Did your potential and kinetic energy calculations display conservation of energy? Why or why not? Explain any discrepancies with your predictions.

**End-of-lab Checklist:**

Make sure you turn in:

- Your lab manual sheets with all predictions and questions answered and data filled in.
- Graphs for the following activities: 1-1, 2-2, 2-3 (2 graphs), 3-1 (2 sheets)