behavior of non-rotating fluid

$\nabla \times u = \omega$ vorticity

- if $\omega = 0 \longrightarrow$ irrotational

1 axis $\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}$ 2 axis $\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$ 3 axis $\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$

def) Potential function - (scalar) $\Phi(x, y, z, t)$

$u = \nabla \Phi$ true for inviscid fluid

$u_1 = \frac{\partial \phi}{\partial x_1}, \; u_2 = \frac{\partial \phi}{\partial x_2}, \; u_3 = \frac{\partial \phi}{\partial x_3}$ $\qquad \frac{\partial^2 \phi}{\partial x_2 \partial x_3} = 0$

u = gradient of scalar $= \nabla \Phi$

* $\nabla u = 0$ (continuity eqn. for incompressible fluid)

$$\frac{\partial}{\partial x_1}\frac{\partial \phi}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial \phi}{\partial x_3} = 0 \rightarrow \boxed{\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0}$$

$\nabla^2 \phi = 0$ Laplace eqn. - flow field

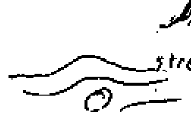
(i) - solve that instead of other one $\left(\frac{\partial u}{\partial t} + u \cdot \nabla u = \frac{-\nabla P}{\rho} + g\right)$

Lecture starts here

last time $\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \bar{\nabla} \bar{u} = -\frac{1}{\rho} \bar{\nabla} P$ 11/9/00

when $\omega = 0$ $\quad u = \nabla \phi \quad \nabla^2 \phi = 0$ solve Laplace to get potential

Stream function - everywhere tangent to velocity



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ChE: ③ ②

must have 2D planar flow or 2D axisymmetric flow

$\nabla \cdot u = 0 \qquad u_x = +\frac{\partial \psi}{\partial y}, \quad u_y = \frac{-\partial \psi}{\partial x}$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial \psi}{\partial x}\right) = 0 \quad \text{or} \quad \frac{\partial^2 \psi}{\partial x \partial y} + \frac{-\partial^2 \psi}{\partial x \partial y} = 0$$

want eqn for ψ — use the fact that flow is irrotational

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$$

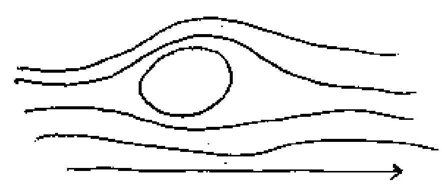
cheg forever

$$\frac{-\partial}{\partial x}\frac{\partial \psi}{\partial x} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial y} = 0 \longrightarrow -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \boxed{\nabla^2 \psi = 0}$$

- Solving this eqn. allows us to skip solving the nonlinear PDE

We have the Laplace equation which is what we wanted

2D flow past a circle



- as fluid flows past sphere speed ↑, pressure ↓
- calculating flow variables (flow field) allows us to obtain the pressure distribution

use cylindrical coords.

$\nabla^2 \psi = 0$ define stream functions (in cyl. coords)

$$V_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad V_\theta = \frac{\partial \psi}{\partial r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

How many boundary conds do we need? (dead silence)

[redacted]

(Q) (3)

there's no "no slip" condition (neglecting viscous terms).

- only need 2 BC's since problem is 2nd order

- No flow across streamlines

BC's | 1) no flow thru surface : at $r = R$, $V_r = 0$

2) as $r \to \infty$, $u = u_\infty$

get limits to get u_r, u_θ
$u_r = \pm u_\infty$, $\theta = 0, \pi$ (use cos)
$u_r = 0$, $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$

$u_r = u_\infty \cos\theta$, $u_\theta = -u_\infty \sin\theta$

Now solve PDE by changing it to an ODE

$\psi = F(r)\, G(\theta) = F(r) \sin\theta$

$\frac{1}{r} \sin\theta \left[F'(r) + r F''(r) \right] - \frac{F(r)}{r^2} \sin\theta = 0$

$$\boxed{F''(r) + \frac{F'(r)}{r} + \frac{F(r)}{r^2} = 0}$$ Euler Diff EQ

or

$r^2 F''(r) + r F'(r) + F(r) = 0$

looking at it we see that the derivatives of polynomial functions have similar form

$F(r) = r^\alpha$

$r^\alpha \alpha(\alpha - 1) + r^\alpha \alpha - r^\alpha(1) = 0$

$\alpha^2 - \alpha + \alpha - 1 = 0$

$\alpha^2 - 1 = 0$

$F(r) = ar + \frac{b}{r}$

$\alpha = \pm 1$

using BC's

$$u_r = u_\infty \left(1 - \frac{R^2}{r^2}\right) \cos\theta$$

$$u_\theta = -u_\infty \left(1 + \frac{R^2}{r^2}\right) \sin\theta$$

$$\boxed{\Psi = u_\infty \left(r - \frac{R^2}{r}\right) \sin\theta}$$

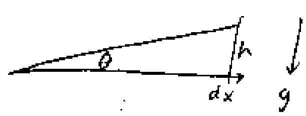
Chegs rule

- So what's the pressure ???

- integrate Euler's eqn " to get Bernoulli's eqn. " $\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_x$

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_x$$

get gravity g_x → slope of down angle in plane that gravity acts.



$\frac{\partial h}{\partial x}$ = slope (mult by gravity)

$$g_x = g \frac{\partial h}{\partial x}$$

use $u = \nabla\phi$. $\quad \frac{1}{2}\frac{\partial}{\partial x} u^2 = \cancel{2} u \frac{\partial u}{\partial x}$

$$\frac{\partial}{\partial t}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial z}\frac{\partial \phi}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g\frac{\partial h}{\partial x}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 + \frac{P}{\rho} + -gh\right] = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{u^2 + v^2 + w^2}{2} + \frac{P}{\rho} + -gh = F(t)$$

— Bernoulli's Eqn. —

use for real life situations eg. pipe flow rate
Pipe flow + fudge factor ⟶ friction factor

References for this lecture and some of the material from Tuesday and Thursday morning.

These books are on reserve in the EG library

Denn, Process Fluid mechanics, pp233-237, 239, 277-281, 282-284.

Panton, Incompressible flow: pp315-316, 544-546.