Introduction to the Mechanics of Waves

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Contents

Preface

This is a set of notes written for seniors or beginning graduate students in engineering, especially for those in the mechanical sciences like mechanical, chemical, civil, or aerospace engineering. The material can be covered as an introduction at the beginning of a course on any one of the specialized applications such as elasticity, acoustics, or water-surface waves. It is also suitable for self-study by working engineers or for those for whom a classroom course is not readily available. It is assumed that the reader has a basic background in undergraduate mathematics including multi-variable and vector calculus, linear algebra, and ordinary differential equations. Knowledge of partial differential differential equations would be a plus, but is not essential since most of the details are included here. Some useful background topics are included in the Appendix.

The notes emphasize the generic nature of wave theory that is common to most applications. Most of this relates to the kinematics of waves and how they travel. For the incorporation of material parameters and the generation and damping of waves, however, one has to recur to the dynamics of the specific physical processes that enables them. In this context physical applications are also introduced, more for the purpose of pointing out common features than to enter in depth in each one, and the reader interested in an application should continue to one of the many specialized books on each. For this part of the notes, it is assumed that the reader is familiar with undergraduate statics, dynamics, solid mechanics, fluid mechanics, and heat transfer.

Chapter 1

Introduction

Humans have interacted with some form of waves for a very long time, though of course they were not known as such. Speech and sound, for example, which enabled communication among ourselves and the creation of music, date from tens of thousands of years ago. Examples are the acoustics of amphitheaters[1] and musical instruments[2]. However, the science of the subject was really studied relatively recently, culminating in classic books like Rayleigh (1877), Lamb (1910) and Jeans (1937). Interaction between temperature differences and sound was discovered by Rijke in 1859, when he found a way of using heat to sustain a sound in a cylindrical tube open at both ends.[3] Water waves is another area that was familiar to the ancients, and its recent history is well documented[4]. Significant contributions were made by Airy, Stokes, and Scott[5], among others. Electromagnetism is another field that has made significant contributions to the general theory of waves. Huygens in 1690 explained the behavior of light in terms of waves, and in 1865 Maxwell was able to calculate its speed from known material properties. This topic, not being a mechanical wave, will only have a passing reference in the Appendix, though its contribution to the fundamental ideas of waves cannot be overestimated.

There are many kinds of waves that mechanical engineers deal with. Some of them are in solids and others in liquids and gases; some are on the surface of the material and some in its interior. The basic quantity to be studied in a wave, that we will call u, will depend on the physical applications. For example, for a water wave it is the instantaneous local height of the water surface above a mean level, and for acoustics it is the instantaneous local pressure. In either case u is a function of space x and time t.

[1] A.F Bilsen, Repetition pitch glide from the step pyramid at Chichen Itza, *J. Acoustical Society of America,* Vol. 120, pp. 594-596, 2006; N.F. Declercq and C.S. Dekeyser, Acoustic diffraction effects at the Hellenistic amphitheater of Epidaurus: Seat rows responsible for the marvelous acoustics, *J. Acoustical Society of America,* Vol. 121, pp. 2011-2022, 2007.

[2] T. Higham, L. Basellb, R. Jacobi, R. Wood, C.B. Ramsey, N.J.Conard, Testing models for the beginnings of the Aurignacian and the advent of figurative art and music: The radiocarbon chronology of Geißenklösterle, *J. Human Evolution,* Vol. 62, pp. 664-676, 2012

[3] P.L. Rijke, On the vibration of the air in a tube open at both ends, *Philosophical Magazine,* vol. 17, pp. 419-422, 1859

[4] A.D.D. Craik, The Origins of Water Wave Theory, *Annual Review of Fluid Mechanics*, Vol. 36: 1-28, 2004.

[5] J.S. Russell, 1845, Report on Waves, *Report of the fourteenth meeting of the British Association for the Advancement of Science,* York, pp. 311-390, September 1844, John Murray, London.

Chapter 2

Wave kinematics

2.1 What is a wave?

A wave is a spatial form that translates in space while maintaining its shape. In general, a wave traveling in the x-direction can be represented by the function of the form $f(\xi)$, where $\xi = x - ct - x_0$, so that

$$u(x,t) = f(x - ct - x_0), \tag{2.1}$$

where c and x_0 are constants, and u is whatever scalar physical quantity that constitutes the wave. For the moment c has no physical meaning but has units of velocity, and x_0 of length. x_0 can arbitrarily be absorbed in the independent variables x or t, i.e. by defining $x' = x - x_0$ or $t' = t + x_0/c$. What Eq. (2.1) signifies is that at different instants of time, t_1 and $t_1 + \Delta t$ say, the two functions $u(x, t_1)$ and $u(x, t_1 + \Delta t)$ are identical in shape, but are displaced in the x direction by a distance Δx, where $\Delta x = c\Delta t$, as shown in Fig. 2.1. Depending on the sign of c, the function will be displaced in the positive or negative x-direction.

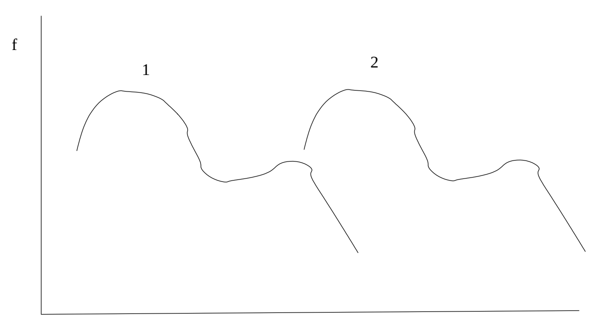


Figure 2.1: Functions $f(x, t_1)$ and $f(x, t_1 + \Delta t)$.

If we have a moving sensor with position $x_s(t)$, then it will see[1]

$$u_s(t) = f(x_s(t) - ct - x_0).$$

[1]Note that x_s is a dependent variable, as opposed to x which is an independent variable.

If now the sensor moves such that the argument $\xi = x_s(t) - ct - x_0$ is constant, it will have to move at velocity c. For a coordinate system moving with this velocity, the function in Eq. (2.1) always has the same form. This is called the phase velocity.

The distance shift between two waves

$$u_1 = f(x - ct - x_{0,1}), \tag{2.2a}$$
$$u_2 = f(x - ct - x_{0,2}), \tag{2.2b}$$

is $\Delta x = x_{0,2} - x_{0,1}$. This quantity is useful only if there is more than one wave, and if the shapes of the two waves are the same, i.e. if f and c are the same. Furthermore, one of the x_0s can be absorbed in x or t, so that the other will represent the distance shift. When dealing with many waves, we can use one wave as a reference and measure the distance shifts for all the others from this. For a single wave, we can omit x_0 without loss of generality and write

$$u(x,t) = f(x - ct). \tag{2.3}$$

The quantity traveling as a wave could be a vector $\mathbf{u}$. For this the wave is

$$\mathbf{u}(x,t) = \mathbf{f}(x - ct).$$

If the wave motion $\mathbf{u}$ is normal to or along the direction of propagation of the wave, it is called a transverse or longitudinal wave, respectively.

■ **Example**

Q: Show that $u(x,t) = A\left(\sin kx \cos \omega t - \cos kx \sin \omega t\right)$, where k and ω are constants, is a wave.
A: Using the trigonometric relation

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

we can write

$$\begin{aligned} u(x,t) &= A \sin\left(kx - \omega t\right), \\ &= A \sin k \left(x - \frac{\omega}{k} t\right) \end{aligned}$$

which is in the form of Eq. (2.3), and is hence a wave.

2.2 Equations for waves

To find the differential equation for which Eq. (2.3) is a solution, we can differentiate it partially once w.r.t. x and t independently to get

$$\begin{aligned} \frac{\partial u}{\partial t} &= -cf', \\ \frac{\partial u}{\partial x} &= f'. \end{aligned}$$

where the primes indicate derivatives w.r.t. ξ, and the chain rule of differentiation is used. From these, we get

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{2.4}$$

We can stop here, but let us see if there are other equations with the same solution. In fact further differentiations will lead us to

$$\frac{\partial^2 u}{\partial t^2} = c^2 f'',$$
$$\frac{\partial^2 u}{\partial x^2} = f'',$$

from which

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{2.5}$$

Though one solution of Eq. (2.5) is indeed Eq. (2.3), it has another solution also, as we will see later. Furthermore, we can easily show that Eq. (2.3) is also a solution to higher-order wave equations.

■ **Example**

Q: Find a lowest-order differential equation with mixed derivatives that satisfies Eq. (2.3).
A: The lowest-order mixed derivative is

$$\frac{\partial^2 u}{\partial x \partial t} = -c f'',$$

so that

$$\frac{\partial^2 u}{\partial t^2} + c \frac{\partial^2 u}{\partial x \partial t} = 0.$$

In fact this can be written as

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0.$$

This is the time derivative of Eq. (2.4). The spatial derivative of the same equation will also satisfy Eq. (2.3).

2.3 Harmonic waves

A harmonic version of Eq. (2.3) is

$$u(x,t) = A\cos\left(kx - \omega t\right), \tag{2.6}$$

for a scalar, where A is the scalar amplitude. This is an infinite wave train. A sine function could have been chosen instead of a cosine, i.e.

$$u(x,t) = A\sin\left(kx - \omega t\right). \tag{2.7}$$

The two are not identical since, there being a $-90°$ phase difference between them, as will be shown below.

For a vector

$$\mathbf{u} = \mathbf{A}\cos(kx - \omega t).$$

The amplitude $\mathbf{A}$ is now a vector.

We can give physical meaning to the quantities in the harmonic wave.

- At a certain instant in time, i.e. if we take a snapshot at $t = t_0$, the harmonic wave $u(x, t_0)$ is a sinusoidal function in x. There are k peaks in 2π units of length; k is thus called a wave-number.
- Similarly, when we consider a given point in space, i.e. we watch the harmonic wave at a location $x = x_0$, then $u(x_0, t)$ will be seen as oscillating sinusoidally in time. There will be ω peaks in a 2π period of time; ω is thus called the radian frequency of the wave.
- The non-dimensional quantity $\phi = kx - \omega t$ is called the phase.

Other commonly-used quantities that can be defined from these are

- The wavelength is $\lambda = 2\pi/k$. This is the distance between adjacent peaks of a wave at any instant of time.
- The cyclic frequency is $n = \omega/2\pi$, and is usually measured in cycles per second, or Hertz (Hz). The cyclic frequency ω and the radian frequency n, though proportional to each other, should not confused.
- The period is $1/n$. This is the time interval between peaks at any given position in space.
- The phase difference between the two waves

$$u_1(x,t) = A\cos(kx - \omega t),$$
$$u_2(x,t) = A\cos(kx - \omega t + \Delta\theta).$$

is the angle $\Delta\theta$. For example, since $\cos(kx - \omega t - 90^\circ) = \sin(kx - \omega t)$, the phase difference between the waveforms in Eqs. (2.6) and (2.7) is -90°. The distance shift in Eq. (2.2) is related to the phase difference by $\Delta x = \Delta\theta/k$.

Writing the wave function in the form

$$u(x,t) = A\cos\left\{k\left(x - \frac{\omega}{k}t\right)\right\},$$

we see that it is in the same form as Eq. (2.3), with

$$c = \frac{\omega}{k}. \tag{2.8}$$

Thus the phase velocity can also be written as

$$c = n\lambda.$$

In terms of complex numbers, Eq. (2.6) can be written as

$$f(x,t) = Ae^{i(kx - \omega t)}.$$

■ **Example**

Q: If the height is a water wave is given by $u(x,t) = 15\sin(1.5x - 7.5t)$ m, where x is in cm and t is s, find the wavelength, cyclic frequency and period of the wave.
A: Given $k = 1.5$ cm^{-1}, $\omega = 7.5$ rad/s, so that the wavelength $\lambda = 2\pi/k = 4.19$ cm, cyclic frequency $n = \omega/2\pi = 1.194$ Hz, period $1/n = 0.838$.

2.4 Solitary waves

These are waves that have a single peak and decay on either side of that. An example is

$$u(x,t) = A \exp\left\{-\frac{1}{2}\left(\frac{x-ct}{\sigma}\right)^2\right\}.$$

This is a Gaussian with amplitude A and standard deviation σ that is traveling to the right with velocity c.

2.5 Shock wave

This is a wave that is defined by

$$u(x,t) = \begin{cases} A & \text{if } x < ct \\ B & \text{if } x \geq ct \end{cases}$$

A and B are different constants, so that the wave is a constant on either side of the point that is traveling to the right with a velocity c. Another way to write this is

$$u(x,t) = (A-B)\{1 - H(x-ct)\} + B, \tag{2.9}$$

where $H(x)$ is the Heaviside step function[2]. It is now in the form of Eq. (2.3). It is a single wave as opposed to being a wave train.

■ **Example**

Q: Find the equation for a single symmetrical triangular wave of width δ that is traveling at velocity c.
A: A triangle centered on the origin $\xi = 0$ is

$$u(\xi) = \begin{cases} 0 & \text{if } \xi < -\delta/2 \\ 1 + 2\xi/\delta & \text{if } -\delta/2 < \xi < 0 \\ 1 - 2\xi/\delta & \text{if } 0 < \xi < \delta/2 \\ 0 & \text{if } \xi > \delta/2 \end{cases}$$

We can make the origin move by taking $\xi = x - ct$. Thus

$$u(x,t) = \begin{cases} 0 & \text{if } x - ct < -\delta/2 \\ 1 + 2(x-ct)/\delta & \text{if } -\delta/2 < x - ct < 0 \\ 1 - 2(x-ct)/\delta & \text{if } 0 < x - ct < \delta/2 \\ 0 & \text{if } x - ct > \delta/2 \end{cases}$$

is the triangular wave with phase velocity c.

2.6 Sigmoid wave

This wave is of the form

$$u(x,t) = \frac{B-A}{1 + \exp\{-(x-ct)/x_0\}} + A.$$

[2] Defined to be zero if its argument is negative and unity if it is non-negative.

which goes smoothly from a value of A on the left to B on the right of a moving point $x = ct$. It becomes steeper and approaches the shock as $x_0 \to 0$.

■ Example

Q: Interpret x_0 physically.
A: Taking the derivative of the sigmoid

$$u(\xi) = \frac{B - A}{1 + \exp\{-\xi/x_0\}} + A,$$

we get

$$u'(\xi) = \frac{(B - A)e^{-\xi/x_0}}{x_0(1 + \exp\{-(\xi/x_0)^2\}} + A.$$

As $x_0 \to 0$, this becomes Wrong!

$$u'(0) = \frac{B - A}{x_0}.$$

For large x_0, $1/x_0$ is proportional to the slope of the sigmoid at $\xi = 0$. As $x_0 \to 0$, the sigmoid approaches a Heaviside step function.

Exercises

1. For a wave of the form
$$u(x, t) = A\cos(kx - \omega t), \tag{2.10}$$
what will a moving sensor with position $x_s(t) = a \sin t$ read?
2. Show by substitution[3] that the shock wave Eq. (2.9) is a solution of Eq. (2.4).

[3]The derivative of $H(x - x_0)$ is the delta function $\delta(x - x_0)$. Both H and δ are actually not functions but distributions or generalized functions.

Chapter 3

Properties of waves

3.1 Ray tracing

http://en.wikipedia.org/wiki/Ray_tracing_%28physics%29

$$\frac{d\mathbf{s}}{dt} = \mathbf{c}$$

3.2 Dispersion relation

We can relax the independent k and ω assumptions in Eq. (2.6) and take

$$\omega = \omega(k).$$

This is called the *dispersion relation*, and the waves are *dispersive*. Though k and ω are still constants, they are now dependent on each other. The phase velocity $c(k) = \omega(k)/k$ is then a function of k; or, if one wants to think of it this way, it is a function of ω. A special case is when c is independent of k; such waves are then *non-dispersive*.

The phase velocity c depends on the physical properties of the material through which the wave is propagating and the kind of wave. This will be taken up when we discuss specific applications.

http://en.wikipedia.org/wiki/Group_velocity

■ **Example**

Q: Show that the phase and group velocities are the same for a non-dispersive wave.
A: From Eq. (2.8) we have that $\omega = ck$. Since c is independent of k for a non-dispersive wave, the group velocity is $c_g = d\omega/dk = c$.

3.3 Energy flow

The instantaneous energy flow is

$$E(t) = Au + B\frac{\partial u}{\partial t}.$$

For a complex waveform

$$u(x,t) = \int_{-\infty}^{\infty} U(k) e^{i(kx-\omega t)}\ dk \tag{3.1}$$

where $U(k)$ is the temporal Fourier transform of $u(x,t)$. In the neighborhood of $k = k_0$, a Taylor series expansion of the dispersion relation gives

$$\omega = \omega_0 + \omega_{k,0}\,(k - k_0) + \dots$$

where $\omega_0 = \omega(k_0)$, and $\omega_{k,0} = (d\omega/dk)_{k=k_0}$. Substituting in Eq. (3.1), we get

$$\begin{aligned} u(x,t) &= e^{i(k_0\omega_{k,0}-\omega_0)t} \int_{-\infty}^{\infty} U(k) e^{i(k_0\omega_{k,0}-\omega_0)t}\ dk, \\ &= e^{i(k_0\omega_{k,0}-\omega_0)t} u\left(x - \omega_{k,0} t, t\right)\ dk. \end{aligned}$$

The argument of u on the right shows that it moves with a velocity

$$c_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}. \tag{3.2}$$

This is called the group velocity, and is the velocity at which the energy of the wave moves.

■ **Example**

Q: Show that the phase and group velocities are the same for a non-dispersive wave.
A: For a non-dispersive wave c does not depend on k. Thus

$$\begin{aligned} \omega &= ck, \\ c_g &= \frac{d\omega}{dk}, \\ &= c. \end{aligned}$$

As a simple example of a wave group, take two slightly different waves

$$\begin{aligned} u_1(x,t) &= A\cos\left\{(k + dk/2)x - (\omega + d\omega/2)t\right\}, \\ u_2(x,t) &= A\cos\left\{(k - dk/2)x - (\omega - d\omega/2)t\right\}. \end{aligned}$$

Superposition of these gives

$$\begin{aligned} u &= u_1 + u_2, \\ &= A\cos\left\{(k + dk/2)x - (\omega + d\omega/2)t\right\} + A\cos\left\{(k - dk/2)x - (\omega - d\omega/2)t\right\}, \\ &= 2A\cos(kx - \omega t)\cos(dk\ x - d\omega\ t), \end{aligned}$$

where we have used the identity

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}.$$

This is a harmonic wave group with an envelope that travels at speed $c_g = d\omega/dk$.

Exercises

1. Show that $c_g = c + k\ dc/dk$ for a dispersive wave.
2. Show by substitution that $u(x,t) = A\cos(kx - \omega t)$, $A\sin(kx - \omega t)$, and $Ae^{i(kx-\omega t)}$ are all separately solutions of the first-order one-dimensional wave equation.
3. Write the equation of a single unit pulse of width δ traveling at velocity c.

Chapter 4

Wave-like forms

It is not very common to see a harmonic wave train like Eq. (2.6) since it is over an infinite domain of time and space, and does not grow or decay. Even the general wave represented by Eq. (2.3) goes on for ever in time and space. More commonly we deal with situations in which $u(x,t)$ is not exactly of either form, but close to it to be recognizable as a wave.

4.1 Complex wave forms

If two individual waves

$$u_1(x,t) = f_1(x - c_1 t),$$
$$u_2(x,t) = f_2(x - c_2 t),$$

where the waves are different, are added, then $u = u_1 + u_2$ is not of the form of Eq. (2.3), and is hence strictly not a wave. However, it may be possible to visually identify the two waves from the signal $u(x,t)$. It is much more difficult if many waves are added, as for instance if

$$u(x,t) = \sum_{i=1}^{N} f_i(x - c_i t).$$

The total sum, in this case, is not a true wave while each one of the N different components is.

The continuum version of the sum of discrete waves is

$$u(x,t) = \int_{-\infty}^{\infty} f_i(x - ct)\ dc.$$

4.2 Standing waves

Take two equal harmonic waves traveling in opposite directions are

$$u_1(x,t) = A\cos(kx - \omega t),$$
$$u_2(x,t) = A\cos(kx + \omega t).$$

The sum is

$$\begin{aligned} u &= u_1 + u_2, \\ &= A\cos(kx - \omega t) + A\cos(kx + \omega t), \\ &= 2\cos kx \cos \omega t. \end{aligned}$$

4.3 Damped and growing waves

Often a function of the form

$$u(x,t) = e^{-\alpha x} f(x - ct) \tag{4.1}$$

or

$$u(x,t) = e^{-\alpha t} f(x - ct)$$

is known as a damped or growing wave if $\alpha > 0$ or < 0, respectively . This is not a true wave in the sense of a function translating itself in space and maintaining its shape intact, since it changes in magnitude. Of course the special case of $\alpha = 0$ gives a true wave as defined by Eq. (2.3).

Differentiating Eq. (4.1), we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= -ce^{-\alpha x} f', \\ \frac{\partial u}{\partial x} &= e^{-\alpha x} f' - \alpha e^{-\alpha x} f, \end{aligned}$$

from which

$$\begin{aligned} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial t} &= -ce^{-\alpha x} f' + c\left(e^{-\alpha x} f' - \alpha e^{-\alpha x} f\right), \\ &= -c\alpha e^{-\alpha x} f, \\ &= -c\alpha u. \end{aligned}$$

Thus a damped or growing wave is a solution of

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial t} + c\alpha u = 0.$$

Of course there are also higher-order equations of which Eq. (4.1) is a solution.

4.4 Modulated waves

There are a number of ways in which the wave

$$u(x,t) = A\cos\left(kx - \omega t - \theta\right)$$

may be modulated.

Amplitude modulation

The expression

$$u(x,t) = f_m(x,t)\ \cos\left(kx - \omega t\right)$$

may be thought of as harmonic wave with wavenumber and frequency k and ω, but with its amplitude modulated by $f_m(x,t)$. The modulation may itself be a harmonic wave if, for example,

$$f_m(x,t) = A\cos(k'x - \omega' t),$$

where the envelope travels at velocity $c' = \omega'/k'$. Other modulation functions can be used, such as

$$f_m(x,t) = A\exp\left\{-(x - c't)^2\right\}.$$

This function defines an envelope enclosing a wave packet. Notice that the speed of the envelope c' and that of the individual waves $c = \omega/k$ may be different.

Frequency modulation

In this case

$$u(x,t) = A\cos\left(kx - \omega(t)t\right).$$

Phase modulation

For this

$$u(x,t) = A\cos\left(k_x - \omega t - \theta(t)\right).$$

4.5 Nonlinear waves

Consider a variant of Eq. (2.3)

$$u(x,t) = f(x + c(u)t),$$

in which the phase velocity depends on u. We can show that

$$\begin{aligned}\frac{\partial u}{\partial t} &= (1 - c't\frac{\partial u}{\partial x})f',\\ \frac{\partial u}{\partial x} &= (-c't\frac{\partial u}{\partial t} - c)f',\end{aligned}$$

from which

$$\begin{aligned}\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= (1 - c't\frac{\partial u}{\partial x})f' - c\left\{(-c't\frac{\partial u}{\partial t} - c)\right\}f',\\ &= \left\{-c't\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)\right\}f'.\end{aligned}$$

Thus

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

Exercises

1.

Chapter 5

Waves in inhomogeneous media

5.1 Slowly-varying media

If we assume that $c(x)$ is slowly varying, i.e. if

$$\frac{\lambda}{c}\,\frac{dc}{dx} \ll 1$$

then the solution of

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0$$

and one solution of

$$\frac{\partial^2 u}{\partial t^2} - c(x)\frac{\partial^2 u}{\partial x^2} = 0.$$

is still Eq. (2.3).

If the slowly varying condition is not satisfied, then I don't know what happens.

5.2 Random media

Exercises

1.

Chapter 6

Wave equations

6.1 First-order equations

Characteristics

If c is a constant, the solution of the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

is

$$dt = \frac{dx}{c} = \frac{du}{0},$$

from which

$$\begin{aligned} du &= 0, \\ u &= C_1, \end{aligned}$$

and

$$\begin{aligned} dx - c\ dt &= 0, \\ x - ct &= C_2. \end{aligned}$$

Putting $C_1 = f(C_2)$, we get the general solution

$$u = f(x - ct).$$

■ Example

Q: Solve

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

where $c = c(x)$.

A: The characteristic equation is

$$dt = \frac{dx}{c(x)} = \frac{du}{0}.$$

We have $u = C_1$, and

$$t - \int \frac{dx}{c(x)} = C_2,$$

so that the solution is

$$u = f\left(t - \int \frac{dx}{c(x)}\right).$$

6.2 Second-order equations

The general solution of

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = 0. \tag{6.1}$$

is

$$u = f(x + ct) + g(x - ct), \tag{6.2}$$

where $f(x + ct)$ is a wave running to the left, and $g(x - ct)$ is to the right.

In fact, Eq. (6.1) can be written as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0,$$

which is the same as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = v, \tag{6.3a}$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0. \tag{6.3b}$$

■ Example

Q: Solve Eqs. (6.3) in sequence as first-order equations. Check!
A: The solution to the second equation is

$$dt = -\frac{dx}{c} = \frac{dv}{0},$$

from which

$$C_1 = ct + x,$$
$$C_2 = v.$$

Thus

$$v = f(ct + x),$$

The solution to the first is

$$dt = \frac{dx}{c} = \frac{du}{f(ct+x)},$$

from which

$$\begin{aligned} C_1 &= ct - x, \\ C_2 &= u - \int f(2ct - C_1)\ dt. \end{aligned}$$

Thus

$$\begin{aligned} u &= \int f(2ct - C_1)\ dt + g(ct - x), \\ &= \frac{1}{2c}\int f(\zeta)\ d\zeta + g(ct - x),\ \text{where } \zeta = 2ct - C_1 \\ &= h(\zeta) + g(ct - x), \\ &= h(2ct - C_1) + g(ct - x), \\ &= h(ct + x) + g(ct - x). \end{aligned}$$

This can be transformed to Eq. (6.2) with suitable manipulation.

6.2.1 D'Alembert's solution

With the initial conditions

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned}$$

the solution of Eq. (6.1) is

$$u(x,t) = \frac{1}{2}\left\{f(x-ct) + f(x+ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

which is known as D'Alembert's solution.

6.2.2 Riemann-Volterra solution (Sneddon)

6.2.3 Telegraph equation

This is

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu.$$

6.3 Korteweg-de Vries equation

This is the non-linear equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0.$$

A solitary wave solution is

$$u(x,t) = \frac{1}{2} c \operatorname{sech}^2 \left[\frac{\sqrt{c}}{2}(x - ct - a) \right]$$

where c is the phase velocity and a is a constant.

A cnoidal wave solution is

$$\eta(x,t) = \eta_2 + H \operatorname{cn}^2 \left(2\, K(m) \frac{x - c\, t}{\lambda} \;\middle|\; m \right),$$

where $\operatorname{cn}(\cdot)$ is one of the Jacobi elliptic functions defined by

$$u = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - m \sin^2 \theta}},$$
$$\operatorname{cn} u = \cos \phi.$$

6.4 Fourth-order equations

Biharmonic equation

$$\nabla^4 u = -\frac{1}{p^2} \frac{\partial^2 y}{\partial t^2}$$

Exercises

1.

Chapter 7

Wave generation

Waves can be generated in two different ways through a time-dependent external forcing $F(t)$. The first is by injecting energy in the interior, and the other is to introduce it from the boundary.

7.1 Interior sources

For example, we can have

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = g(x),$$

with the solution

$$u(x,t) = f(x-ct) + \frac{1}{c}\int g(x)\ dx. \tag{7.1}$$

The solution of

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = g(t),$$

is

$$u(x,t) = f(x-ct) + \int g(t)\ dt. \tag{7.2}$$

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = F(t).$$

The Green's function of

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u = \delta(\mathbf{x}, t)$$

is

?

7.2 Boundary sources

In this case we may have

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = 0,$$

with $u(0,t) = F(t)$. To generate harmonic waves we can take $F = A\cos\Omega t$.

Exercises

1. Show Eqs. (7.1) and (7.2).

Chapter 8

Harmonic waves

Harmonic waves are the most common, so several of their properties will be mentioned here.

8.1 Traveling and standing waves

8.2 Beating

http://en.wikipedia.org/wiki/Beat_%28acoustics%29

8.3 Refraction

8.4 Reflection

8.5 Interference

8.6 Diffraction

8.7 Doppler effect

8.8 Scattering

Exercises

1.

Chapter 9

Multi-dimensional waves

A scaler quantity traveling as a wave in three-dimensional physical space is

$$u(\mathbf{x}, t) = f(\mathbf{x} - \mathbf{c}t). \tag{9.1}$$

Note that the phase velocity $\mathbf{c}$ is a vector. From time instant $t = t_1$ to $t = t_1 + \Delta t$, this wave travels a distance $\Delta\mathbf{x} = \mathbf{c}\ \Delta t$. Similarly, a traveling vector quantity has the representation

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x} - \mathbf{c}t).$$

The directions of $\mathbf{u}$ and $\mathbf{c}$ are, in general, unrelated and may be different.

The equation that Eq. (9.1) satisfies can be easily found. We write

$$u(x, y, z, t) = f(\xi_x, \xi_y, \xi_z)$$

where $\xi_1 = x - c_x t$, etc. The derivatives are

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -c_x \frac{\partial f}{\partial \xi_x} - c_y \frac{\partial f}{\partial \xi_y} - c_z \frac{\partial f}{\partial \xi_z}, \\
\frac{\partial u}{\partial x} &= \frac{\partial f}{\partial \xi_x}, \\
\frac{\partial u}{\partial y} &= \frac{\partial f}{\partial \xi_y}, \\
\frac{\partial u}{\partial z} &= \frac{\partial f}{\partial \xi_z},
\end{aligned}$$

so that

$$\frac{\partial u}{\partial t} + c_x \frac{\partial f}{\partial \xi_x} + c_y \frac{\partial f}{\partial \xi_y} + c_z \frac{\partial f}{\partial \xi_z} = 0.$$

This can also be compactly written as

$$\frac{\partial u}{\partial t} + \mathbf{c} \cdot \nabla_\xi f = 0,$$

where

$$\nabla_\xi = \mathbf{i}\frac{\partial}{\partial \xi_x} + \mathbf{j}\frac{\partial}{\partial \xi_y} + \mathbf{k}\frac{\partial}{\partial \xi_z}.$$

The second-order multi-dimensional wave equation for a scalar u is[1]

$$\frac{\partial^2 u}{\partial t^2} - c\nabla^2 u = 0.$$

and for a vector $\mathbf{u}$ is[2]

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - c\nabla^2 \mathbf{u} = 0.$$

The harmonic version of the scaler wave is

$$u(\mathbf{x}, t) = A\cos(\mathbf{k}\cdot\mathbf{x} - \omega t), \tag{9.2}$$

where $\mathbf{k}$ is a vector wavenumber. From time instant $t = t_1$ to $t = t_1 + \Delta t$, this wave travels a distance $\Delta\mathbf{x}$, where $\mathbf{k}\cdot\Delta\mathbf{x} = \omega\ \Delta t$. Another way of writing Eq. (9.2) is

$$u(\mathbf{x}, t) = A\cos(k_x x + k_y y + k_z z - \omega t).$$

We can also write

$$u(\mathbf{x}, t) = A\cos\left\{\mathbf{k}\cdot(\mathbf{x} - \frac{\mathbf{k}}{k^2}\omega t)\right\}.$$

where $k = |\mathbf{k}|$, so that

$$\mathbf{c} = \frac{\mathbf{k}}{k^2}\omega.$$

The velocity vector $\mathbf{c}$ and the vector wavenumber $\mathbf{k}$ are both in the direction of travel of the wave.

The generalization of the group velocity in Eq. (3.2) to multiple dimensions is

$$\mathbf{c}_g = \nabla_{\mathbf{k}}\,\omega \tag{9.3}$$

where $\nabla_{\mathbf{k}} = \mathbf{i}\partial/\partial k_x + \mathbf{j}\partial/\partial k_y + \mathbf{k}\partial/\partial k_z$.

Exercises

1. Prove Eq. (9.3).

[1] The operators $\nabla u = \mathbf{e}_x\partial u/\partial x + \mathbf{e}_y\partial u/\partial y + \mathbf{e}_z\partial u/\partial z$ and $\nabla^2 u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2$ in Cartesian coordinates.

[2] Here, $\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u})$.

Chapter 10

Transverse waves in a string and membrane

10.1 String

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

10.2 Membrane

$$\nabla_H^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Exercises

1.

Chapter 11

Elastic waves in solids

http://en.wikipedia.org/wiki/Seismic_wave

http://en.wikiversity.org/wiki/Waves_in_composites_and_metamaterials/Waves_in_layered_media_and_point_sources

11.1 Longitudinal waves

11.2 Shear waves

11.3 Thermoelastic waves

Exercises

1.

Chapter 12

Surface waves in solids

http://en.wikipedia.org/wiki/Surface_acoustic_wave
http://en.wikipedia.org/wiki/Rayleigh_wave

Exercises

1.

Chapter 13

Phonons in solids

http://en.wikipedia.org/wiki/Phonon

13.1 Single atom type

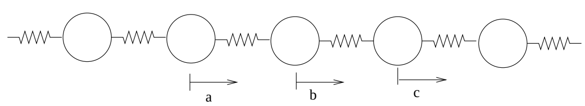


Figure 13.1: Lattice of atoms of a single type

A lattice of atoms of a single type is shown in Fig. 13.1. The mass of each atom is m, the spring constants are r, and a is the mean distance between the atoms. For a typical atom n, Newton's second law gives

$$\begin{aligned} m\frac{d^2x_n}{dt^2} &= r(x_{n+1} - x_n) - r(x_n - x_{n-1}) \\ &= r(x_{n+1} - 2x_n + x_{n-1}). \end{aligned}$$

http://en.wikipedia.org/wiki/Lennard-Jones_potential

Let

$$x_i = \widehat{x}e^{i(nka-\omega t)},$$

then the dispersion relation is

$$\omega = \left(\frac{2r}{m}\right)^{1/2} (1 - \cos ka)^{1/2}.$$

The phase velocity is

$$c = \left(\frac{2r}{mk^2}\right)^{1/2} (1 - \cos ka)^{1/2},$$

and the group velocity is

$$c_g = \left(\frac{r}{2m}\right)^{1/2} \frac{a \sin ka}{(1-\cos ka)^{1/2}}.$$

For $ka \to 0$, we have

$$v_g = a\left(\frac{r}{m}\right)^{1/2}.$$

13.2 Two atom types

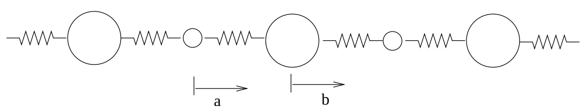


Figure 13.2: Lattice of atoms of two different type

Newton's second law gives

$$\begin{aligned} m_1 \frac{d^2 x_i}{dt^2} &= r(y_i - x_i) - r(x_i - y_{i-1}) \\ &= r(y_i - 2x_i + y_{i-1}), \\ m_2 \frac{d^2 y_i}{dt^2} &= r(x_{i+1} - y_i) - r(y_i - x_i) \\ &= r(x_{i+1} - 2y_i + x_i). \end{aligned}$$

Let

$$\begin{aligned} x_i &= \widehat{x} e^{i(nka-\omega t)}, \\ y_i &= \widehat{y} e^{i(nka-\omega t)}, \end{aligned}$$

so that

$$\begin{aligned} -m_1 \widehat{x}\omega^2 &= r\left(\widehat{y} - 2\widehat{x} + \widehat{y}e^{-ika}\right), \\ -m_2 \widehat{y}\omega^2 &= r\left(\widehat{x}e^{ika} - 2\widehat{y} + \widehat{x}\right), \end{aligned}$$

which can also be written as

$$\left[\begin{array}{cc} 2r - m_1\omega^2 & -r(1+e^{-ika}) \\ -r(1+e^{ika}) & 2r - m_2\omega^2 \end{array}\right] \left[\begin{array}{c} \widehat{x} \\ \widehat{y} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right].$$

This means that

$$(2r - m_1\omega^2)(2r - m_2\omega^2) - r^2(1+e^{-ika})(1+e^{ika}) = 0,$$

which simplifies to

$$m_1 m_2 \omega^4 - 2r(m_1 + m_2)\omega^2 + 2r^2(1 - \cos ka) = 0.$$

The solution is

$$\omega^2 = \frac{1}{2m_1 m_2}\left[2r(m_1 + m_2) \pm 2r\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos ka}\right].$$

The positive sign corresponds to the optical and the negative to the acoustic mode.
`http://demonstrations.wolfram.com/PhononDispersionRelationInBrillouinZone/`

Exercises

1.

Chapter 14

Thermal waves

Exercises

1.

Chapter 15

Water surface waves

http://en.wikipedia.org/wiki/Airy_wave_theory
http://en.wikipedia.org/wiki/Stokes_wave
http://en.wikipedia.org/wiki/Clapotis

15.1 Governing equations

For potential flow

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{15.1}$$

where $\phi(x, z, t)$ is the velocity potential. The velocity components are

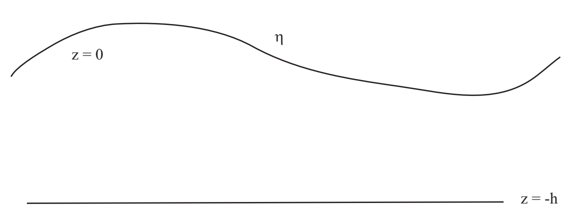
$$u = \frac{\partial \phi}{\partial x}, \tag{15.2a}$$

$$w = \frac{\partial \phi}{\partial z}. \tag{15.2b}$$

15.1.1 Boundary condition at lower surface

The lower surface is impermeable, so that

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -h. \tag{15.3}$$



15.1.2 Boundary conditions at upper surface

The free surface is at $z = \eta(x,t)$.

Kinematic condition

A particle of fluid on the surface will remain on the surface, i.e.

$$\frac{D}{Dt}(z-\eta) = 0,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z},$$

is the material derivative following a fluid particle. Thus

$$\begin{aligned} \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)(z-\eta) &= 0, \\ -\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + w &= 0, \\ -\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z} &= 0. \end{aligned}$$

at the surface.

Linearization gives

$$-\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial z} = 0, \tag{15.4}$$

at the surface. Furthermore, instead of imposing the surface boundary conditions at $z = \eta(x,t)$, we can impose them at $z = 0$. Thus

$$\begin{aligned} \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} &= \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \eta \left.\frac{\partial^2 \phi}{\partial z^2}\right|_{z=0} + \dots, \\ &= \left.\frac{\partial \phi}{\partial z}\right|_{z=0}, \end{aligned}$$

so that Eq. (15.4) applies at $z = 0$.

Dynamic condition

Bernoulli's equation is

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\left(u^2 + w^2\right) + g\eta = C,$$

at the surface, where C is a constant. Linearizing, we have

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + g\eta = C, \tag{15.5}$$

at $z = 0$.

For a fluid with surface tension

$$p = p_{atm} - \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right),$$

where σ is the coefficient of surface tension, p_{atm} is the atmospheric pressure, and R_1 and R_2 are the radii of curvature in two orthogonal planes. For plane waves we can take $1/R_2 = 0$, and

$$\frac{1}{R_1} = \frac{\partial^2 \eta / \partial x^2}{\left[1 + (\partial \eta / \partial x)^2\right]^{3/2}},$$

which linearizes to

$$\frac{1}{R_1} = \frac{\partial^2 \eta}{\partial x^2}.$$

Thus

$$p = p_{atm} - \sigma \frac{\partial^2 \eta}{\partial x^2}.$$

Eq. (15.5) becomes

$$\frac{\partial \phi}{\partial t} + \left(\frac{p_{atm}}{\rho} - \frac{\sigma}{\rho} \frac{\partial^2 \eta}{\partial x^2} \right) + g\eta = C,$$

at $z = 0$. The time derivative is

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\sigma}{\rho} \frac{\partial^3 \phi}{\partial x^2 \partial z} + g \frac{\partial \phi}{\partial z} = 0, \tag{15.6}$$

using Eq. (15.4).

15.2 Wave solution

Consider a traveling wave of the form

$$\eta(x, t) = A e^{i(kx - \omega t)}. \tag{15.7}$$

Letting

$$\phi(x, z, t) = \Phi(z) e^{i(kx - \omega t)},$$

Eq. (15.1) becomes

$$-k^2 \Phi + \frac{d^2 \Phi}{dz^2} = 0. \tag{15.8}$$

The solution is

$$\Phi = a e^{kz} + b e^{-kz},$$

so that

$$\phi = \left(ae^{kz} + be^{-kz}\right) e^{i(kx-\omega t)}.$$

Since

$$\frac{\partial \phi}{\partial z} = k\left(ae^{kz} - be^{-kz}\right) e^{i(kx-\omega t)},$$

the boundary condition at the lower surface, Eq. (15.3) becomes

$$k\left(ae^{-kh} - be^{kh}\right) e^{i(kx-\omega t)} = 0.$$

This is true for all x and t, so that we must have

$$ae^{-kh} - be^{kh} = 0,$$

from which

$$b = ae^{-2kh}.$$

Thus

$$\phi = ae^{-kh}\left\{e^{k(z+h)} + e^{-k(z+h)}\right\} e^{i(kx-\omega t)},$$

or

$$\phi = c\ \cosh k(z+h) e^{i(kx-\omega t)},$$

where $c = 2ae^{-kh}$.

From Eqs. (15.4) and (15.7), we have

$$-Ai\omega e^{i(kx-\omega t)} = c\ k \sinh kh e^{i(kx-\omega t)},$$

from which

$$c = -\frac{iA\omega}{k\sinh kh},$$

so that

$$\phi(x,z,t) = -\frac{iA\omega}{k\sinh kh}\cosh k(z+h) e^{i(kx-\omega t)}$$

From Eqs. (15.2), the velocity components are

$$u(x,z,t) = \frac{A\omega}{\sinh kh}\ \cosh k(z+h) e^{i(kx-\omega t)},$$
$$w(x,z,t) = -\frac{iA\omega}{\sinh kh}\ \sinh k(z+h) e^{i(kx-\omega t)}.$$

Using the boundary condition Eq. (15.6), we get

$$\left(\frac{i^2 A\omega^3}{k\sinh kh}\right)\cosh kh\ e^{i(kx-\omega t)} - \left(\frac{i\sigma A\omega k^2}{\rho\sinh kh}\right)\sinh kh\ e^{i(kx-\omega t)} - \left(\frac{iA\omega g}{\sinh kh}\right)\sinh kh\ e^{i(kx-\omega t)} = 0.$$

This reduces to

$$\frac{\omega^2}{k}\coth kh - \frac{\sigma k^2}{\rho} = g,$$

or

$$\omega = \sqrt{\left(gk + \frac{\sigma}{\rho}k^3\right)\tanh kh}.$$

The phase velocity is

$$\begin{aligned} c &= \frac{\omega}{k}, \\ &= \sqrt{\left(\frac{g}{k} + \frac{\sigma}{\rho}k\right)\tanh kh}. \end{aligned}$$

15.3 Special cases

15.3.1 Shallow-water waves

Taking $kh \to 0$, $\tanh kh \to kh$, and neglecting σ, we have

$$\begin{aligned} \omega &= k\sqrt{gh}, \\ c &= \sqrt{gh}. \end{aligned}$$

These are non-dispersive.

15.3.2 Deep-water waves

For $kh \to \infty$, and $\tanh kh \to 1$, we get

$$\begin{aligned} \omega &= \sqrt{gk + \frac{\sigma}{\rho}k^3}, \\ c &= \sqrt{\frac{g}{k} + \frac{\sigma}{\rho}k}. \end{aligned}$$

These are dispersive.

Gravity waves are deep-water waves dominated by gravity for which

$$\begin{aligned} \omega &= \sqrt{gk}, \\ c &= \sqrt{\frac{g}{k}}, \\ c_g &= \frac{1}{2}\sqrt{\frac{g}{k}}, \\ &= \frac{c}{2}. \end{aligned}$$

Capillary waves, on the other hand, are dominated by surface tension, and for these

$$\begin{aligned}
\omega &= \sqrt{\frac{\sigma}{\rho}k^3}, \\
c &= \sqrt{\frac{\sigma}{\rho}k}, \\
c_g &= \frac{3}{2}\sqrt{\frac{\sigma}{\rho}k}., \\
&= \frac{3}{2}c.
\end{aligned}$$

http://en.wikipedia.org/wiki/Capillary_wave
http://en.wikipedia.org/wiki/Dispersion_(water_waves)

Exercises

1.

Chapter 16

Internal gravity waves

Exercises

1.

Chapter 17

Hydraulic jump

Exercises

1.

Chapter 18

Instabilities in fluids

18.1 Parallel flow

Substituting a two-dimensional perturbation of the steady-state velocity field

$$\mathbf{u} = (U(z) + u'(x,z,t))\mathbf{i} + v'(x,z,t)\mathbf{j},$$

and

$$\mathbf{u}' = \mathbf{a}e^{i\alpha(x-ct)}$$

into the Navier-Stokes equation an linearizing, we get the Orr-Sommerfeld equation

$$\frac{\mu}{i\alpha\rho}\left(\frac{d^2}{dz^2} - \alpha^2\right)^2 \varphi = (U-c)\left(\frac{d^2}{dz^2} - \alpha^2\right)\varphi - U''\varphi.$$

where φ is the streamfunction.

18.2 Kelvin-Helmholtz instability

18.3 Stratified flow

Exercises

1.

Chapter 19

Acoustics in gases

Consider the propagation of small pressure, density, temperature and velocity disturbances in an otherwise quiescent gas. The gas is assumed to obey the perfect gas law

$$p = \rho R T. \tag{19.1}$$

The pressure, density, temperature and velocity are

$$p = p_0 + p', \tag{19.2a}$$
$$\rho = \rho_0 + \rho', \tag{19.2b}$$
$$T = T_0 + T', \tag{19.2c}$$
$$u = u_0 + u', \tag{19.2d}$$

respectively, where p_0, ρ_0, T_0 and u_0 are the background values, and $p'(x,t)$, $\rho'(x,t)$, $T'(x,t)$ and $u'(x,t)$ are small perturbations around them. We will assume that $u_0 = 0$.

The one-dimensional equations of mass and momentum conservation are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

Using Eqs. (19.2), these become

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0, \tag{19.3a}$$
$$\frac{\partial u'}{\partial t} = -u'\frac{\partial u'}{\partial x} - \frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right)^{-1}\frac{\partial p'}{\partial x}, \tag{19.3b}$$
$$= -\frac{1}{\rho_0}\left(1 + \frac{\rho'}{\rho_0} + \dots\right)\frac{\partial p'}{\partial x}, \tag{19.3c}$$
$$= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}. \tag{19.3d}$$

where the non-linear terms have been neglected.

We also assume that the fluctuations are fast enough for the heat conduction to be negligible, so that the process is isentropic. So, in addition to the above, we have

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma,$$

where γ is the ratio of specific heats. From Eqs. (19.2), this becomes

$$\begin{aligned} 1 + \frac{p'}{p_0} &= \left(1 + \frac{\rho'}{\rho_0}\right)^\gamma, \\ &= 1 + \gamma\frac{\rho'}{\rho_0} + \dots \end{aligned}$$

Thus

$$p' = \gamma\frac{p_0}{\rho_0}\rho'. \tag{19.4}$$

From Eqs. (19.3), we get

$$\begin{aligned} \frac{\partial^2\rho'}{\partial t^2} &= -\rho_0\frac{\partial^2 u'}{\partial t\,\partial x}, \\ &= \frac{\partial^2 p'}{\partial x^2}. \end{aligned}$$

Using Eq. (19.4),

$$\frac{\partial^2\rho'}{\partial t^2} = c^2\frac{\partial^2\rho'}{\partial x^2}. \tag{19.5}$$

where

$$c = \sqrt{\gamma\frac{p_0}{\rho_0}}.$$

Since $p_0 = \rho_0 R T_0$

$$c = \sqrt{\gamma R T_0}.$$

For air at 20°C, $c = 343.2$ m/s.

■ **Example**

Q: Show that similar second-order wave equations hold for p', u' and T'.
A: Eq. (19.4) directly gives

$$\frac{\partial^2 p'}{\partial t^2} = c^2\frac{\partial^2 p'}{\partial x^2}. \tag{19.6}$$

Furthermore, from Eq. (19.1),

$$\begin{aligned} p_0 + p' &= (\rho_0 + \rho')R(T_0 + T'), \\ &= \rho_0 R T_0 + \rho' R T_0 + \rho_0 R T'. \end{aligned}$$

so that

$$T' = \frac{1}{\rho_0 R}p' - \frac{T_0}{\rho_0}\rho'.$$

Multiply Eq. (19.5) by $-T_0/\rho_0$ and add the result to Eq. (19.6) multipled by $1/\rho_0 R$ to get a wave equation in T'.

Exercises

1.

Chapter 20

Shocks in gases

Exercises

1.

Appendix A

Appendix

A.1 Complex numbers

The unit of the *imaginary* numbers is i, defined by $i^2 = -1$. A *complex* number is one that has a real and an imaginary part. Thus, a complex number z can be represented as $z = x + iy$, where x and y are real numbers. The real part of z is x, written as $\Re(z) = x$, and the imaginary part is $\Im(z) = y$. For two complex numbers to be equal their real and imaginary parts must both be equal. Complex numbers can be added, subtracted, multiplied, divided using the same rules as for real numbers.

It is useful to show complex numbers as points on a plane; so Fig. A.1 shows the point P to be the complex number $x + iy$[1]. One must, however, remember that this representation is just a matter of convenience, and that the complex number is *not* a two-dimensional vector.

Definitions: The *complex conjugate* of $z = x + iy$ is $z^* = x - iy$. The *absolute value* (or *modulus*) of z is $|z| = +\sqrt{x^2 + y^2}$; it is the length r is Fig. A.1. The *argument* of z is the angle it makes with the abscissa, which is $\theta = \tan^{-1}(y/x)$ in the figure.

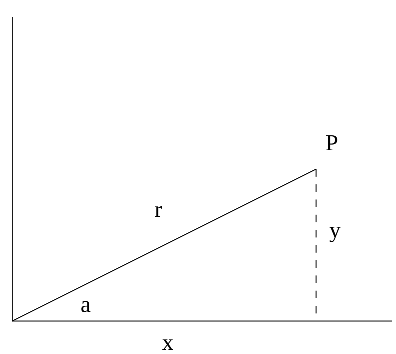


Figure A.1: Planar representation of a complex number.

[1]Called an Argand diagram after Jean-Robert Argand (1768-1822).

A.1.1 Euler's formula

If the Taylor series expansion for the exponential of an imaginary number is assumed to be valid, then

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \\ &= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right] \end{aligned}$$

so that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{A.1}$$

Furthermore

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{A.2}$$

From Eq. (A.1) and Fig. A.1, we can show that

$$\begin{aligned} re^{i\theta} &= r(\cos\theta + i\sin\theta) \\ &= x + iy. \end{aligned}$$

This a complex number can be represented in its Cartesian form $x + iy$, or its equivalent polar form $re^{i\theta}$.

Also, from Eqs. (A.1) and (A.2)

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned}$$

Complex numbers can be used to find particular solutions of linear ordinary and partial differential equations. The method is equivalent to using real functions, but with much easier algebra. We will illustrate with an example. Let us find the particular solution of

$$\ddot{y} + \dot{y} + y = F(t), \tag{A.3}$$

using the method of undetermined coefficients, where

(a)

$$F(t) = \cos\omega t, \tag{A.4}$$

(b)

$$F(t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right), \tag{A.5}$$

(c)

$$F(t) = e^{i\omega t}. \tag{A.6}$$

A.1.2 Using real numbers

Using Eq. (A.4), we propose a particular solution of the form

$$y = A\cos(\omega t + \phi),$$

where A and ϕ are real numbers representing the amplitude and the phase angle, respectively. Substituting in Eq. (A.3), we get

$$-\omega^2 A\cos(\omega t + \phi) - \omega A\sin(\omega t + \phi) + A\cos(\omega t + \phi) = \cos\omega t.$$

Expanding the equation

$$(1-\omega^2)A\left(\cos\omega t\cos\phi - \sin\omega t\sin\phi\right) - \omega A\left(\sin\omega t\cos\phi + \cos\omega t\sin\phi\right) = \cos\omega t.$$

Taking the inner product of the above with respect to $\cos\omega t$ and $\sin\omega t$, respectively, we get

$$\begin{aligned}(1-\omega^2)A\cos\phi - \omega A\sin\phi &= 1,\\ -(1-\omega^2)A\sin\phi - \omega A\cos\phi &= 0.\end{aligned}$$

From this, we have

$$\begin{aligned}\tan\phi &= \frac{-\omega}{1-\omega^2},\\ A &= \frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}}.\end{aligned}$$

Therefore, the final solution is

$$y = \frac{\cos(\omega t + \phi)}{\sqrt{(1-\omega^2)^2+\omega^2}}.$$

A.1.3 Complex form of real numbers

With Eq. (A.5), the particular solution is

$$y = Ce^{i\omega t} + \overline{C}e^{-i\omega t},$$

where $\overline{C}$ is the complex conjugate of C. Substituting in Eq. (A.3), we get

$$-\omega^2(Ce^{i\omega t} + \overline{C}e^{-i\omega t}) + i\omega(Ce^{i\omega t} - \overline{C}e^{-i\omega t}) + (Ce^{i\omega t} + \overline{C}e^{-i\omega t}) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right).$$

Expanding the $e^{i\omega t}$ and $e^{-i\omega t}$ terms in the above equation using Euler formula, and collecting the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$, we have

$$\begin{aligned}-\omega^2(C+\overline{C}) + i\omega(C-\overline{C}) + C + \overline{C} &= 1,\\ -\omega^2(C-\overline{C}) + i\omega(C+\overline{C}) + C - \overline{C} &= 0.\end{aligned}$$

Adding the two equations together,

$$-2\omega^2 C + 2i\omega C + 2C = 1.$$

The expression for C is

$$C = \frac{1}{2}\frac{1-\omega^2-i\omega}{(1-\omega^2)^2+\omega^2},$$

And

$$\overline{C} = \frac{1}{2}\frac{1-\omega^2+i\omega}{(1-\omega^2)^2+\omega^2}.$$

Therefore, the expression for the particular solution is

$$\begin{aligned} y &= \frac{1}{2}\frac{1-\omega^2-i\omega}{(1-\omega^2)^2+\omega^2}e^{i\omega t} + \frac{1}{2}\frac{1-\omega^2+i\omega}{(1-\omega^2)^2+\omega^2}e^{-i\omega t}, \\ &= \frac{1-\omega^2}{(1-\omega^2)^2+\omega^2}\cos\omega t + \frac{\omega}{(1-\omega^2)^2+\omega^2}\sin\omega t, \\ &= \frac{\cos(\omega t+\phi)}{\sqrt{(1-\omega^2)^2+\omega^2}}. \end{aligned}$$

where,

$$\tan\phi = \frac{-\omega}{1-\omega^2}.$$

Note: If we notice the linear independence of $e^{i\omega t}$ and $e^{-i\omega t}$ in Eq. (**??**), we can actually collect the coefficients of these terms directly,

$$\begin{aligned} -\omega^2 C + i\omega C + C &= \frac{1}{2}, \\ -\omega^2\overline{C} - i\omega\overline{C} + \overline{C} &= \frac{1}{2}. \end{aligned}$$

And the results are the same as the previous ones.

A.1.4 Using complex numbers

In the case of Eq. (A.6), we can take

$$y = Be^{i\omega t}.$$

Substituting into Eq. (A.3), we get

$$-\omega^2 B + i\omega B + B = 1,$$

From which, we can solve for B

$$B = \frac{1-\omega^2-i\omega}{(1-\omega^2)^2+\omega^2}.$$

And the solution is

$$\begin{aligned} y &= \frac{1-\omega^2-i\omega}{(1-\omega^2)^2+\omega^2}e^{i\omega t}, \\ &= \frac{e^{i\phi}}{\sqrt{(1-\omega^2)^2+\omega^2}}e^{i\omega t}, \\ &= \frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}}e^{i(\omega t+\phi)}, \end{aligned}$$

where

$$\tan\phi = \frac{-\omega}{1-\omega^2}.$$

The real part of the solution is

$$\frac{\cos(\omega t+\phi)}{\sqrt{(1-\omega^2)^2+\omega^2}}.$$

which is exactly the same as the results from the previous two methods.

A.2 Solution of first-order PDE

The procedure for the solution of

$$P(x,y)\frac{\partial u}{\partial x} + Q(x,y)\frac{\partial u}{\partial y} = R(x,y)$$

is the following. We write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}.$$

If the two ordinary differential equations equations can be solved, their solutions are

$$C_1 = C_1(x,t),$$
$$C_2 = C_2(x,t).$$

The solution is then

$$C_1 = f(C_2).$$

A.3 Classification of second-order PDEs

A linear second-order PDE in the unknown $u(x,y)$ is of the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial x} + f = g.$$

where a through g are functions of x and y. It is classified depending on whether the *discriminant* $D = b^2 - 4ac$ is locally zero, positive or negative. The three canonical cases are the following.

$D = 0$: Parabolic (heat equation)

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}.$$

$D < 0$: Elliptical (Laplace's equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$D > 0$: Hyperbolic (wave equation)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

A.4 Electromagnetic waves

Electromagnetic waves is the where much of the theoretical material has been developed. This is introduced even though the rest of the manuscript deals mainly with mechanical waves.

Maxwell's equations of electromagnetic theory are

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$

where $\mathbf{H}$, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{D}$, $\mathbf{J}$, and ρ are the magnetic intensity, magnetic induction, electric field, electric displacement, current density, and charge density, respectively. For linear materials $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{J} = g\mathbf{E}$ (Ohm's law), and $\mathbf{B} = \mu\mathbf{H}$, where ϵ is the permittivity, g is the electrical conductivity, and μ is the permeability. For free space $\epsilon = 8.8542 \times 10^{-12}$ $\mathrm{C^2N^{-1}m^{-2}}$, and $\mu = 1.2566 \times 10^{-6}$ $\mathrm{NC^{-2}s^2}$,

For $\rho = 0$ and constant ϵ, g and μ, it can be shown that

$$\nabla^2\mathbf{H} - \epsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} - g\mu\frac{\partial \mathbf{H}}{\partial t} = 0$$
$$\nabla^2\mathbf{E} - \epsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} - g\mu\frac{\partial \mathbf{E}}{\partial t} = 0$$

The speed of an electromagnetic wave in free space is $c = 1/\sqrt{\mu\epsilon}$. The directional energy flux density is given by the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

Exercises

1.

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