Some properties of the Riemann integral

Here are proofs of Theorems 3.3.3-3.3.5, Corollary 3.3.6 and Theorem 3.3.7 for any Riemann integrable functions on $[a, b]$. Because the statements in the book are for continuous functions I added ′ to the number of the theorem or corollary to distinguish it from the corresponding one in the book.

**Theorem 3.3.3′:** If $f$ and $g$ are Riemann integrable on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g$ is Riemann integrable on $[a, b]$ and

$$\int_a^b (\alpha f(x) + \beta g(x)) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx. \quad (1)$$

**Proof:** (i) If $\alpha \geq 0$, by §2.5 #8

$$\sup_{[c,d]} \alpha f = \alpha \sup_{[c,d]} f$$

for any subinterval $[c, d] \subset [a, b]$. Hence for any partition $P$ of $[a, b]$, $U_P(\alpha f) = \alpha U_P(f)$.

Also §2.5 #8 holds for the infinimum; for any $S \subset \mathbb{R}$

$$\inf\{\alpha x : x \in S\} = \alpha \inf S \quad \text{if} \ \alpha \geq 0.$$  

Hence

$$\inf_P\{U_P(\alpha f)\} = \inf_P\{\alpha U_P(f)\} = \alpha \inf_P\{U_P(f)\} = \alpha \int_a^b f(x) \, dx. \quad (2)$$

Similarly $L_P(\alpha f) = \alpha L_P(f)$ so

$$\sup_P\{L_P(\alpha f)\} = \alpha \int_a^b f(x) \, dx. \quad (3)$$

By (2), (3) and the definition of the Riemann integral, $\alpha f$ is Riemann integrable on $[a, b]$ and

$$\int_a^b \alpha f(x) \, dx = \alpha \int_a^b f(x) \, dx. \quad (4)$$

(ii) For any $S \subset \mathbb{R}$,

$$\sup_S (-f) = -\inf_S f$$

Hence, $U_P(-f) = -L_P(f)$ so $\inf_P\{U_P(-f)\} = -\sup_P\{L_P(f)\} = -\int_a^b f(x) \, dx$. Similarly, $\sup_P\{L_P(-f)\} = -\int_a^b f(x) \, dx$ so $-f$ is Riemann integrable on $[a, b]$ and

$$\int_a^b -f(x) \, dx = -\int_a^b f(x) \, dx. \quad (5)$$

Combining (4) and (5) shows that (4) holds for any $\alpha \in \mathbb{R}$. 

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(iii) Because \( f \) and \( g \) are Riemann integrable on \([a, b]\), for any \( \epsilon > 0 \) we can find partitions \( P_1 \) and \( P_2 \) such that
\[
\int_a^b f(x) \, dx - \epsilon \leq L_{P_1}(f) \leq U_{P_1}(f) \leq \int_a^b f(x) \, dx + \epsilon \tag{6}
\]
and
\[
\int_a^b g(x) \, dx - \epsilon \leq L_{P_2}(g) \leq U_{P_2}(g) \leq \int_a^b g(x) \, dx + \epsilon. \tag{7}
\]
Also, for any interval \([c, d]\) by §2.5 #9
\[
\sup_{[c,d]} (f + g) \leq \sup_{[c,d]} f + \sup_{[c,d]} g
\]
so for any partition \( P \)
\[
U_P(f + g) \leq U_P(f) + U_P(g) \tag{8}
\]
and similarly
\[
L_P(f) + L_P(g) \leq L_P(f + g). \tag{9}
\]
Adding (6) and (7) and using (8), (9) and Lemma 1 shows that if \( Q = P_1 \cup P_2 \),
\[
\int_a^b f(x) \, dx + \int_a^b g(x) \, dx - 2\epsilon \leq L_{P_1}(f) + L_{P_2}(g) \leq L_Q(f + g) \leq U_Q(f + g) \leq U_P(f) + U_P(g) \leq \int_a^b f(x) \, dx + \int_a^b g(x) \, dx + 2\epsilon.
\]
This holds for every \( \epsilon > 0 \). Hence
\[
\sup_P \{L_P(f + g)\} = \inf_P \{U_P(f + g)\} = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
\]
Thus \( f + g \) is Riemann integrable on \([a, b]\) and
\[
\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx. \tag{10}
\]
The theorem follows from (4), (5) and (10).

**Remark:** This result says that the Riemann integrable functions on \([a, b]\) form a vector space and integration is a linear operator (transformation) from this vector space to \( \mathbb{R} \).
Theorem 3.3.4': If $f$ and $g$ are Riemann integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$.

Proof: Because $f(x) \leq g(x)$, for any partition $P$ of $[a, b]$, $U_P(f) \leq U_P(g)$. Hence any lower bound for $\{U_P(f)\}$ is a lower bound for $\{U_P(g)\}$. In particular,

$$\int_a^b f(x) \, dx = \inf_P \{U_P(f)\} \leq \inf_P \{U_P(g)\} = \int_a^b g(x) \, dx.$$

Theorem 3.3.5': If $f$ is Riemann integrable on $[a, b]$ then so is $|f|$ and

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$  \hspace{1cm} (11)

Proof: Let $\epsilon > 0$ and let $P$ be a partition of $[a, b]$ such that $U_P(f) - L_P(f) \leq \epsilon$. Let $m_i = \inf_{[x_{i-1}, x_i]} f$, $m'_i = \inf_{[x_{i-1}, x_i]} |f|$, $M_i = \sup_{[x_{i-1}, x_i]} f$, $M'_i = \sup_{[x_{i-1}, x_i]} |f|$. There are three cases.

Case (i): If $m_i \geq 0$, then $M'_i = M_i$, $m'_i = m_i$ so

$$M'_i - m'_i = M_i - m_i.$$

Case (ii): If $M_i < 0$ then $M'_i = -M_i$, $m'_i = -M_i$ so

$$M'_i - m'_i = M_i - m_i.$$

Case (iii): If $M_i > 0$, $m_i < 0$ then $M'_i = \max\{M_i, -m_i\}$ and $m'_i \geq 0$ so

$$M'_i - m'_i \leq \max\{M_i, -m_i\} < M_i - m_i.$$

In each case

$$M'_i - m'_i \leq M_i - m_i$$

so

$$U_P(|f|) - L_P(|f|) \leq U_P(f) - L_P(f) \leq \epsilon$$  \hspace{1cm} (12)

and, by Lemma 3, $|f|$ is integrable. Now (11) follows from Theorems 3.3.3' and 3.3.4' since $f(x), -f(x) \leq |f(x)|$.

Corollary 3.3.6': If $f$ is Riemann integrable on $[a, b]$ then

$$\left| \int_a^b f(x) \, dx \right| \leq (b - a) \sup_{[a,b]} |f(x)|.$$  \hspace{1cm} (13)

Proof: By Theorem 3.3.5'

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$
Now apply Theorem 3.3.4' to the right side with $g(x)$ the constant function $\sup_{[a,b]} |f|$.

**Theorem 3.3.7**: If $f$ is Riemann integrable on $[a, b]$ and $a < c < b$ then

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$  \hspace{1cm} (14)

**Proof**: For any partition $P$ of $[a, b]$, let $P_c$ be $P$ if $c$ is a point of $P$ and the partition obtained from $P$ by adding the point $c$ otherwise. Let $P_1$ be the points in $P_c$ which are less than or equal to $c$, so $P_1$ is a partition of $[a, c]$, and let $P_2$ be the points that are greater than or equal to $c$ so $P_2$ is a partition of $[c, b]$. Then

$$L_P(f) \leq L_{P_1}(f) + L_{P_2}(f) \leq U_{P_1}(f) + U_{P_2}(f) = U_{P_c}(f) \leq U_P(f)$$

Hence

$$\sup_P \{L_P(f)\} \leq \sup_{P_1} \{L_{P_1}(f)\} + \sup_{P_2} \{L_{P_2}(f)\} \leq \inf_{P_1} \{U_{P_1}(f)\} + \inf_{P_2} \{U_{P_2}(f)\} \leq \inf_P \{U_P(f)\}.$$  

Since the right and left ends are equal to $\int_a^b f(x) \, dx$,

$$\int_a^b f(x) \, dx = \sup_{P_1} \{L_{P_1}(f)\} + \sup_{P_2} \{L_{P_2}(f)\} = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$