Math 40750 Overview for Final

Topics since the midterm are marked with a †.

• Main focus of course
  – Linear PDE, especially the classic three
    * Heat equation
    * Wave equation
    * Laplace’s equation and Poisson’s equation
  – Why these?
    * Important for many applications
    * Models for other linear problems
    * Many nonlinear problems can be approximated by these

• Linearity
  – Solutions of a homogeneous linear problem form a vector space
  – Solve a nonhomogeneous problem by finding a particular solution, adding a solution of corresponding homogeneous problem

• Techniques
  – Separation of variables
    * Build solution out of simple solutions
    * Simple solutions — separated solutions (products of functions on one variable)
  – †Integration against a kernel
  – †Fourier transform

• Tools
  – Fourier series
    * Orthogonal functions
      · Projection onto orthogonal set
      · Bessel’s inequality
    * Fourier series
      · Definition, orthogonality relations
      · Sine series, cosine series
    * Convergence Theorem
      · Dirichlet kernel
      · Riemann Lemma
      · Gibbs phenomenon
      · Uniform convergence
Bessel’s inequality and Parseval’s Theorem for Fourier series

†Fourier transform

* Definition
* Properties
  · Convolution \( \rightarrow \) multiplication: the transform of \( f_1 \ast f_2 \) is \( 2\pi F_1 F_2 \).
  · Differentiation \( \rightarrow \) multiplication by transform variable: the transform of \( f' \) is \( i\mu F \).
* Inversion formula (convergence theorem)
* Plancherel’s Theorem (Parseval’s theorem)

• Specific equations, problems

  – Initial value problem for heat equation on rod (or in a slab) with homogeneous boundary conditions
    * Find separated solutions satisfying boundary conditions by separation of variables
    * Ignore initial value when finding separated solutions
    * Try to write initial value as (finite) linear combination of initial values of separated solutions
      · If you can, the corresponding linear combination of separated solutions solves
    * †Otherwise, try to write initial value as infinite series with each term the initial value of a separated solution
      · Leads to Fourier series
      · Solution of initial value problem should be infinite series with terms the corresponding separated solutions
    * †Proving you really have a solution in series case requires
      · Justifying differentiating term–by–term
      · Show initial value is right using Convergence Theorem
    * †Uniqueness for solution of initial value problem

  – Steady state solution of heat equation on rod (or in a slab) with nonhomogeneous boundary conditions
    * Often a simple ODE to solve
    * Will need this to get particular solution when we solve initial value problem with nonhomogeneous boundary conditions

  – †Heat equation on a rod with nonhomogeneous boundary conditions
    * Stage 1: Find steady state solution
    * Stage 2: Transform to a homogeneous problem
    * Stage 3: Solve the homogeneous problem by separation of variables
    * Stage 4: Verification
    * Stage 5: Asymptotic behavior
- †Wave equation, 1 space variable
  * D’Alembert’s formula for the solution
  * Vibrating string: solution by separation of variables
- †Applications of multiple Fourier series
  * Heat equation in a rectangular column
  * Laplace’s equation
  * 2 dimensional wave equation—vibrating rectangular membrane
- †Laplace’s equation
  * Separated solutions, series solutions in polar coordinate
  * Poisson’s formula for finding a harmonic function in a disk with given boundary values
  * Mean value property of harmonic functions (Prop. 3.1.1.1)
  * Maximum principle for harmonic functions (Prop. 3.1.1.2-3)
- †Solution of initial value problem for heat equation on the line
  * Solution is given by convolution of initial value with Gauss–Weierstrass kernel
  * This solution was obtained using Fourier transform
  * Solution on half line with Dirichlet or Neumann boundary conditions using method of images