Sets of Measure Zero

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In other words, A has measure 0 if for every $\epsilon > 0$ there are open intervals $I_1, I_2, \ldots, I_n, \ldots$ such that $A \subset \cup I_n$ and $\sum \ell(I_n) \leq \epsilon$.

Almost everywhere means except on a set of measure 0.

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  Obtain A_{n+1} from A_n by removing the open middle third of each remaining interval.
  Then
  $$A = \bigcap_{n=1}^{\infty} A_n.$$

If you expand numbers in a ternary expansion, so if $x \in [0, 1]$ you write

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \text{ where } a_j = 0, 1 \text{ or } 2,$$

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The map

$$\sum_{j=1}^{\infty} \frac{a_j}{3^j} -> \sum_{j=1}^{\infty} \frac{a_j/2}{2^j}$$

maps the Cantor set onto the unit interval.

So the Cantor set is uncountable.

To see that it has measure 0, notice that

$$A_n = \bigcup_{k=1}^{2^n} B_{n,k}, \;\; B_{n,k} = [a_{n,k}, b_{n,k}], \;\; \ell(B_{n,k}) = \frac{1}{3^n}.$$

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Let

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Now

$$A \subset A_n \subset \bigcup_{k=1}^{2^n} I_{n,k}, \qquad \sum_{k=1}^{2^n} \ell(I_{n,k}) = \frac{2^n}{3^n} + \frac{2^{n+1}}{10^n} \to 0 \text{ as } n \to \infty$$

so A has measure 0.