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WAVELET TRANSFORMS FOR SYSTEM IDENTIFICATION AND ASSOCIATED PROCESSING CONCERNS

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ABSTRACT

The time-frequency character of wavelet transforms allows increased flexibility – as both traditional time and frequency domain system identification approaches can be adapted to examine non-linear and non-stationary response data. However, a number of additional processing concerns must be understood to fully exploit the power of the multi-resolution, dual-domain transform, particularly for the popular Morlet wavelet. Unfortunately, in prior applications of wavelet transforms for system identification, the implications of the aforementioned concerns were often negligible, as these studies considered mechanical systems characterized by higher frequency, broader-band signals. It was the subsequent analysis of Civil Engineering structures that highlighted the need to understand more fully these processing concerns in order to insure the successful identification of dynamic system properties. This study identifies a number of these considerations that are a direct consequence of the wavelet's multi-resolution character, including considerations for selection of wavelet central frequencies and strategies to resolve end effects errors. In total, this study serves as an overview of these processing considerations for Civil Engineering structures, helping to lessen the challenges associated with the transition into the time-frequency domain.

Keywords: wavelet transforms, Morlet wavelet, system identification, time-frequency analysis

INTRODUCTION

While the Fourier transform has reshaped the manner in which engineers interpret signals, it becomes evident that by breaking a signal down into a series of trigonometric basis functions, time-varying features cannot be captured. The realization that non-stationary features often characterize processes of interest led to the definition of alternative transforms that rely on bases with compact support , one of the most popular of which is the wavelet transform.

A host of discrete and continuous wavelets have been applied to a variety of problems ranging from image and acoustic processing to fractal analysis, though only recently have they been extended to Civil Engineering applications (Gurley & Kareem, 1999). Their specific application for system identification is still advancing, but shows great promise, particularly

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since the wavelet transform produces a representation in frequency and time -- permitting the adaptation of a number of system identification schemes derived for either domain.

This flexibility is conceptualized in Figure 1 for a simple harmonic signal. The squared modulus of the wavelet transform, or *scalogram*, is shown three dimensionally in the time-frequency domain. As discussed in greater detail in Carmona *et al.* (1998), the wavelet coefficients take on maximum values at the *instantaneous frequency* , corresponding to the dominant frequency component in the signal at that instant in time. This defines a *ridge* in the time-frequency plane. Extracting the values of the wavelet coefficients along this ridge yields the wavelet *skeleton*, whose real and imaginary components approximate the signal and its Hilbert transform at that ridge frequency (see left inset of Fig. 1). A time-domain based system identification scheme using the amplitude and phase of the asymptotic signal can then take advantage of these skeleton components (e.g. Ruzzene *et al*., 1997; Staszewski, 1998). Similarly, a slice taken at a given time, across the range of frequencies, yields the *instantaneous spectrum* of the signal (see right inset of Fig. 1), indicating the frequency content at that instant in time. Much like Fourier spectra, the peak of this spectrum corresponds to the instantaneous frequency defining the ridge and the bandwidth of the spectrum provides an indication of the spread of frequencies present in the signal at each instant in time. This spectral information can be utilized in more traditional frameworks for system identification via frequency response functions, for coherence analysis (Gurley *et al*., 2002) and for time-frequency signal analysis (Kijewski & Kareem, 2002a). As discussed further in Kijewski and Kareem (2002b), researchers have now begun to exploit this dual identification potential in Civil Engineering, revealing that, for the analysis of more narrowbanded signals, certain processing concerns emerge, which shall be summarized herein for a popular class of wavelets – the Morlet.

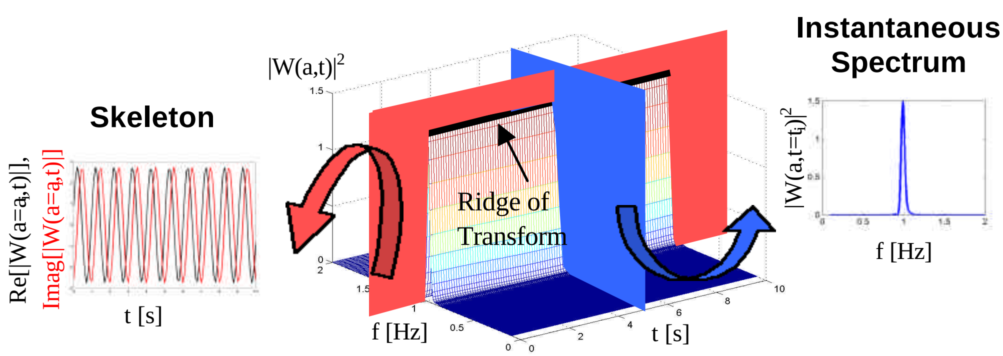

Fig. 1. Conceptualization of dual potential of wavelet transform

WAVELET TRANSFORM THEORY

The wavelet is a linear transform that decomposes an arbitrary signal *x(t)* via basis functions that are simply dilations and translations of the parent wavelet *g(t)* through convolution

$$W(a,t)=\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} x(\tau)g^*\left(\frac{\tau-t}{a}\right)d\tau\,. \tag{1}$$

Dilation by the scale *a*, inversely proportional to frequency, represents the periodic nature of the signal. By this approach, time-frequency localization is possible, since the parent wavelet serves as a window function. Since it is quite natural to view information in terms of harmonics instead of scales, the Morlet wavelet (Grossman & Morlet, 1985)

$$g(t) = e^{i2pf_o t} e^{-t^2/2} = e^{-t^2/2}\left(\cos(2pf_o t) + i\sin(2pf_o t)\right) \tag{2}$$

has become a popular choice for analysis. As a result of obvious analogs, the wavelet scale is uniquely related to the Fourier frequency f for this parent wavelet, according to $a = f_o / f$. The dilations of this temporally-localized wave allow the effective frequency of this sine-cosine pair, oscillating at central frequency f_o, to change in order to match the signal's harmonic components.

RESOLUTIONS

The accuracy of Morlet wavelet-based system identification is dependent upon the time and frequency resolutions, which are merely scaled versions of the parent wavelet resolutions. However, for the Morlet wavelet, the use of a Gaussian window on the Fourier basis functions makes the precise definition of temporal duration impractical. Instead Gabor (1946) proposed a mean square definition to establish effective durations in time and frequency. Using this approach, an effective temporal duration Dt_i and frequency duration Df_i for a scaled Morlet wavelet at frequency f_i can be defined as the product of that scale and the mean square duration of the Morlet's Gaussian window, given by Chui (1992) as

$$\Delta t_i = \frac{1}{\sqrt{2}}\frac{f_o}{f_i} \quad \text{and} \quad \Delta f_i = \frac{1}{2p\sqrt{2}}\frac{f_i}{f_o}. \tag{3 a,b}$$

Physically, the measures provided in Eq. (3) indicate that two pulses in time cannot be identified unless they are more than Dt apart. Similarly, two distinct frequency contributions cannot be discerned unless they are more than Df apart. It is clear from Eq. (3) that the central frequency f_o is a critical parameter in defining the resolution capability of the Morlet wavelet and should be adapted in a given analysis to achieve desired performance. The virtues of progressive adaptation of this parameter are discussed more fully in Kijewski and Kareem (2002a).

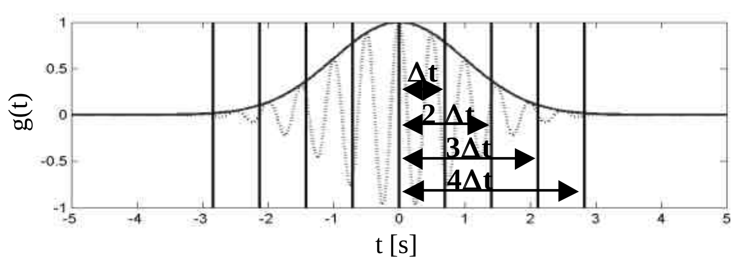

FIG. 2. Real component of Morlet wavelet enveloped by Gaussian window with temporal duration measures marked by vertical bars

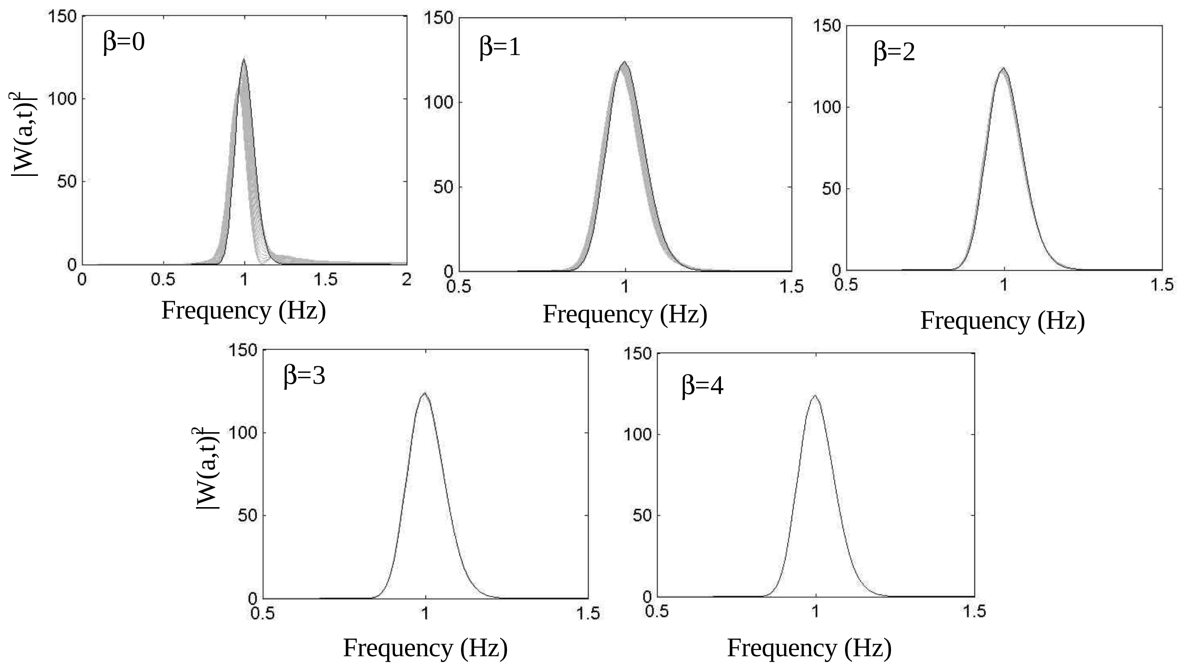

FIG. 3. Deviations of simulated instantaneous spectra (gray) from theoretical result (black) as end effects regions are progressively neglected

END EFFECTS THEORY

The resolutions of the resulting wavelet analysis also have direct bearing on the significance of end effects, which have been noted in a number of applications, e.g. Staszewski (1998). In many cases, the a priori knowledge of the signal characteristics allows anomalies to be qualitatively distinguished and neglected in subsequent analyses. However, this is in general not possible, requiring a quantitative guideline to establish what portions of the wavelet-transformed signal are accurate. By examining the convolution operation in Eq. (1) in light of the parent wavelet in Eq. (2), it is evident that, although the wavelet is focused at a given time and represents the signal content in that vicinity, the window extends equally into the past and future. As further demonstrated by Fig. 2, Eq. (3a) assumes the Morlet wavelet effectively spans $2Dt_i$ in the time domain, or one standard deviation of the Guassian window. However , there is a considerable portion of the window beyond one standard deviation from $t=0$. A stricter interpretation would define the effective temporal duration of this wavelet as several standard deviations of the Gaussian window. Dependent on the desired level of accuracy, an integer multiple b of the measure in Eq. (3a) can be imposed to quantify the reliable region within a set of wavelet-transformed data of length T, according to

$$b\Delta t_i \leq t_j \leq T - b\Delta t_i \,. \tag{4}$$

Though the implications of end effects are discussed in more detail in Kijewski and Kareem (2002c), the end effects can have substantial influence on the quality of wavelet coefficients. An illustration of the implications of end effects on spectral amplitude is provided in Fig. 3 where the calculated instantaneous spectra at each time are plotted one atop the other, essentially collapsing the scalogram in time. In comparison to the theoretical prediction, the improvement in instantaneous spectral amplitude and shape is obvious as the result of progressively neglecting more of the end effects region, requiring a value of b = 4 in Eq. (4) to sufficiently negate the end effects phenomenon. However, in the case of more sensitive spectral measures such as the bandwidth, b=6 is necessary to eliminate any significant deviation between the theoretical prediction and the wavelet result (Kijewski & Kareem, 2002c). Though such deviations are easily explained by the end effects phenomenon, simply neglecting these regions in analysis yields to a considerable loss of data.

END EFFECTS MELIORATION: SIGNAL PADDING

The loss of considerable regions of a signal is the unfortunate consequence of end effects. One possible solution to this problem would be to pad the beginning and end of the signal with surrogate values, placing the true signal of interest at the center of an elongated vector and leaving the virtual values at the tails to be corrupted by end effects. In the padding operation, the signal's characteristics are locally preserved by reflecting a portion of the signal about its beginning and end (Kijewski & Kareem 2002c). As the lowest frequency being considered in the analysis f_1 will yield the largest duration Dt_1, it dictates the maximum end effects anticipated. b is then selected based on the desired accuracy level, and the time ordinates of the sampled time vector $t = [t_1 \ldots t_N]$ closest to the termination of the end effects regions are then identified by

$$t_n = \min[t > b\Delta t_1] \;\; \text{and} \;\; t_m = \max[t < (t_N - b\Delta t_1)]. \tag{5}$$

The modified signal x_{MOD} is constructed by reflecting the signal x (for even functions) or its negative (for odd functions) for the duration of bDt_1 about t_1 and t_N, according to

$$x_{MOD} = \left[\pm x_n \quad \pm x_{n-1} \quad \ldots \quad x_1 \quad \ldots \quad x_N \quad \pm x_{N-1} \quad \ldots \quad \pm x_m\right] \tag{6}$$

where x_n and x_m are the values of the sampled signal x at t_n and t_m. x_{MOD} is then wavelet transformed and the coefficients calculated from the padded regions are simply neglected, retaining only the coefficients of the true signal for meaningful analysis.

End Effects Melioration for Spectral Measures

While the influence of end effects in the frequency domain is visualized in Fig. 3, they may be further quantified via time-varying spectral measures. The signal under consideration, shown in Fig. 4a, is the free vibration response of a SDOF oscillator with natural frequency f_n of 0.15 Hz and critical damping ratio x of 0.01. In the case of a f_o=1 Hz wavelet analysis, the end effects have no influence on the estimate of instantaneous frequency from the ridge of the transform, but have a considerable effect on the amplitude of the wavelet skeleton for the first and last few cycles of oscillation, as shown in Fig. 4b,e. The vertical dotted lines mark increments of Dt for this analysis. It is not until $3Dt$ that the end effects on the wavelet skeleton amplitude diminish. This is rectified by applying the padding operation for $b=4$, as shown in Fig. 4c,f. After doing so, the wavelet skeleton can hardly be discerned from the actual signal, though as Fig. 4d,g reveals, a slight deviation of the amplitude is still present at the very beginning and end of the signal, though it is small relative to the signal's amplitude at these points in time. The halfpower

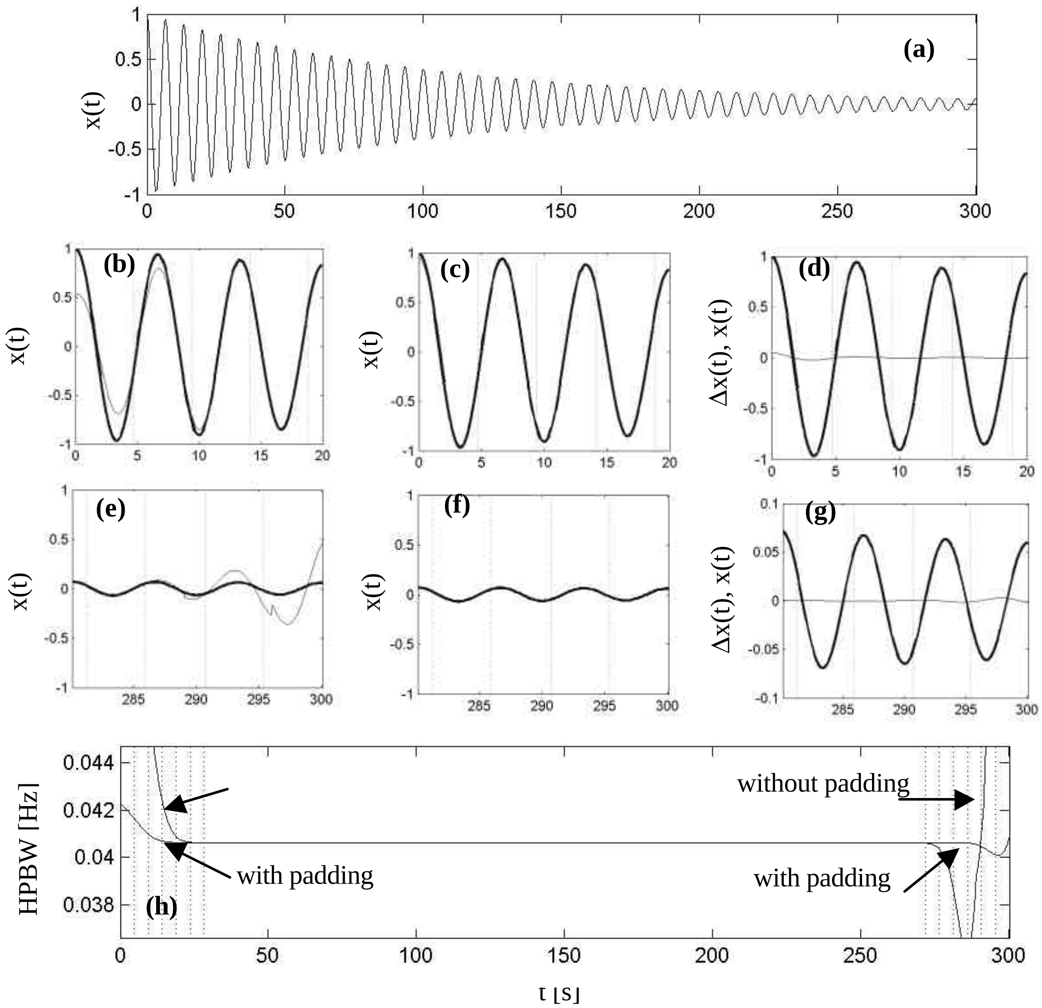

FIG. 4. (a) Signal; (b) & (e) signal (dark) and wavelet skeleton (light) at ends without padding; (c) & (f) signal (dark) and wavelet skeleton (light) at ends with padding; (d) & (g) signal (dark) and difference between signal and wavelet skeleton (light) at ends with padding; (e) bandwidth estimate with & without padding

bandwidth (HPBW) identified from the wavelet instantaneous spectra is constant, as expected for this linear oscillator, with the exception of the end effects region. As shown in Fig. 4h, the bandwidth measure, being more sensitive, is significantly influenced by the end effects. When padded with $b=4$, the bandwidth accuracy is vastly improved, though Fig. 4h demonstrates that within the first *3Dt*, the bandwidth is still deviating, a result that cannot be fully improved with larger values of b. This is due to the fact that the remaining slight inaccuracies in the amplitude lead to a more marked inaccuracy in the sensitive bandwidth measure. Note also that the bandwidth of the resulting wavelet instantaneous spectra are larger than their Fourier equivalent, as a result of the windowing applied by the Gaussian function of the Morlet wavelet.

TABLE 1. Estimate of damping from wavelet skeleton of SDOF system

	f_o=1 Hz, β=0	f_o=1 Hz, β=4	f_o=0.5 Hz, β =4	f_o=0.25 Hz, β =4
	5Δt =23.57 s	3Δt =14.14 s	3Δt =7.07 s	3Δt =3.53 s
1st Cycle (0-6.67 s)	-0.0678	0.0034	0.0054	0.0097
2nd Cycle (6.67-13.33 s)	-0.0123	0.0084	0.0099	0.0102
3rd Cycle (13.33-20.0 s)	0.0067	0.0098	0.0099	0.0101
4th Cycle (20.0-26.67 s)	0.0099	0.0099	0.0099	0.0101

End Effects Melioration for Time Domain System Identification

Time domain system identification on the system in Fig. 4a may proceed as discussed in the introduction, with the full procedure provided in Ruzzene *et al*. (1997) and Staszewski (1998). As a result of the padding operation, the estimation of damping in the time domain is also enhanced , though not completely rectified. As shown in Fig. 4e, the deviations in the amplitude with padding are slight and diminish with each cycle of oscillation, but still have marked impact for the more sensitive estimation of damping in this system. The error in the wavelet skeleton's amplitude is most significant at $t=0$, though reduced from 50% to about 5% with the addition of padding. This does not affect the estimation of instantaneous frequency, which is consistently within 1%. Table 1 lists the identified damping from each cycle of the decay. The time span of each cycle is provided in parenthesis. As shown in column one, without padding, the damping can only be reliably estimated beyond about *5Dt*. Therefore, direct application of the techniques discussed in Ruzzene *et al*. (1997) and Staszewski (1998) for such narrowband systems should proceed using only the wavelet data beyond *5Dt.* However, the padding operation leads to a vast improvement in the estimates from the first three cycles and produces highly accurate estimates after only *3Dt*, allowing more of the signal to be used in system identification.

Table 2: Wavelet identification of MDOF system with closely spaced modes

	WT Resolutions		Actual		Identified	
	Δt	Δf	f_n	ξ	f_n	ξ
Mode 1	7.49 s	0.011 Hz	0.567 Hz	0.01	0.567 Hz	0.010
Mode 2	4.21 s	0.019 Hz	1.007 Hz	0.01	1.007 Hz	0.010
Mode 3	3.87 s	0.020 Hz	1.095 Hz	0.01	1.095 Hz	0.010

Note that since the identification of this system is proceeding in the time domain, a reduced value of f_o can be used to diminish end effects, though sacrificing the frequency resolution. When doing so, it becomes possible to identify the damping within 1% accuracy using virtually the full length of padded data, as demonstrated in the last two columns of Table 1. While this is attractive, it is demonstrated in Kijewski and Kareem (2002b) that this may not be plausible for MDOF systems, particularly those with closely spaced modes. Results from this study, shown in Table 2, show a high level of accuracy is only achievable for sufficiently large values of f_o. The results in Table 2 were achieved using f_o=8 Hz. The poor temporal resolution listed in column one is a necessary sacrifice to sufficiently refine the frequency resolutions and fully separate the modes.

CONCLUSIONS

The wavelet transform, by virtue of its multi-resolution capabilities, is gaining popularity, not only for time-frequency analysis, but also for identification of mechanical systems. In particular, the Morlet wavelet has become a popular choice by virtue of its direct relationship to the Fourier transform. It was emphasized that application of this parent wavelet requires judicious selection of central frequency in light of the resulting time and frequency resolutions. These considerations become significant for Civil Engineering structures, whose dynamics are often more narrowbanded than traditional mechanical systems. For such systems, the presence of end effects can compromise the accuracy of wavelet skeletons and have even more marked effects on bandwidth measures. The span of end effects regions was quantified through a flexible criterion, and, recognizing the significant losses possible for low frequency systems, a simple padding scheme was proposed to extend the length of the signal at both ends. While the padding operation is demonstrated to improve the scalogram amplitudes and enhance the accuracy of wavelet-based system identification, the reliability of damping estimation is still diminished within *3Dt* of the beginning and end of the signal. Though an improvement over the results without padding, to minimize this effect, the central frequency should be kept to the smallest value possible without compromising the ability to separate closely spaced modes.

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