Recursive Modeling of Dynamic Systems

By Yousun Li[1] and Ahsan Kareem[2]

Abstract: The dynamic response analysis of structural systems to a variety of random excitations using recursive models is presented. The methodology permits analysis of stationary, nonstationary, or transient response of structures under stochastic loads, e.g., correlated multi-input loading due to fluctuations in wind or wave surface profile, or seismic excitation. For the nonstationary, or transient, excitation, the analysis involves a direct computation of the output covariance from a given nonstationary correlation structure of the input. The accuracy and stability of the simulated time histories is assessed. A detailed example is presented to illustrate the proposed recursive model. The concept of an ARMA (auto-regressive moving average) system is presented in which the ARMA representation of the response is obtained in terms of the ARMA description of the stationary excitation. The usefulness of the recursive approach for nonlinear systems is demonstrated by means of an example involving an elasto-plastic beam subjected to a suddenly applied load.

Introduction

Recursive models have been used for numerical integration of uncoupled dynamic equations of motion in the literature. The classical recursive formulation relies on Z-transform methods (Jury 1964; Stagner and Hart 1970). Cronin (1973) implemented Duhamel's integral in a recursive format to evaluate the response of uncoupled systems. Recently, the response time histories of a coupled six-degree-of-freedom model of an offshore tension leg platform under random wind and wave fields have been simulated numerically using a combination of recursive modeling techniques (Li 1988). Hoshiya et al. (1984) presented a method of obtaining covariance response in recursive form for a multi-degree-of-freedom linear structural system subjected to nonstationary random excitation. The covariance response matrix was derived from the state space matrix. The formulation involves decoupling of the equations of motion by means of a modal superposition technique with implicit assumption of proportional damping. The recursive modeling approach has been applied in a variety of other problems, e.g., simulation of wave surface profile or wind velocity fluctuations (e.g., Samaras and Shinozuka 1985; Spanos and Mignolet 1986; Li and Kareem 1987).

In this paper, dynamic response analysis of structural systems to a correlated multi-input random excitation and seismic loading using recursive models is presented. The response covariance is expressed in a recursive form, which permits evaluation of the response statistics under nonstationary stochastic loading. Other applications include transient response statistics. The accuracy and stability of the recursive model are investigated. A detailed example is presented that highlights the analysis of response statistics and

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accuracy and stability of the recursive model. The concept of an ARMA system is presented in which the ARMA representation of the response is obtained in terms of the ARMA description of the stationary input. Finally, the usefulness of the recursive technique for nonlinear systems is demonstrated by means of an example of an elasto-plastic system subjected to a suddenly applied load. The computationally efficient procedure presented here holds the promise of becoming a convenient design office tool.

Formulation of Recursive Model

The input-output relationship of a multi-degree-of-freedom (MDOF) system is described by the following matrix equation:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{M}^*\ddot{\mathbf{Y}} + \mathbf{C}^*\dot{\mathbf{Y}} + \mathbf{K}^*\mathbf{Y} = \mathbf{F}(t) \quad (1)$$

in which $\mathbf{M}$ = mass matrix ($n \times n$); $\mathbf{C}$ = damping matrix ($n \times n$); $\mathbf{K}$ = stiffness matrix ($n \times n$); $\mathbf{X}$, $\dot{\mathbf{X}}$, and $\ddot{\mathbf{X}}$ = response displacement, velocity, and acceleration vectors; $\mathbf{M}^*$, $\mathbf{C}^*$, and $\mathbf{K}^*$ = coefficient matrices associated with the type of loading; and $\mathbf{Y}$, $\dot{\mathbf{Y}}$, and $\ddot{\mathbf{Y}}$ = input loading vectors. The preceding matrix equation may be solved by the following recursive model:

$$\mathbf{X}_n + \sum_{r=1}^{P} \mathbf{A}_r\mathbf{X}_{n-r} = \sum_{r=-Q^-}^{Q^+} \mathbf{B}_r\mathbf{Y}_{n-r} \quad (2)$$

in which $\mathbf{Y}_n$ = the input vector at time $n\Delta t$; $\mathbf{X}_n$ = output vector at time $n\Delta t$; Δt = time increment; and $\mathbf{A}_r$ and $\mathbf{B}_r$ = coefficient matrices. The output at time $n\Delta t$ is a weighted sum of the past, present, and future inputs and the past outputs. Introducing the backward shift operator $B[B\mathbf{X}_n = \mathbf{X}_{n-1}$ and $(1 - B)\mathbf{X}_n = \mathbf{X}_n - \mathbf{X}_{n-1}]$ and the following Newmark-β time-integration scheme in Eq. 1, we obtain

$$\mathbf{X}_n = \mathbf{X}_{n-1} + \Delta t\dot{\mathbf{X}}_{n-1} + 0.5(1 - 2\beta)\Delta t^2\ddot{\mathbf{X}}_{n-1} + \beta\Delta t^2\ddot{\mathbf{X}}_n$$

and

$$\dot{\mathbf{X}}_n = \dot{\mathbf{X}}_{n-1} + (1 - \delta)\Delta t\ddot{\mathbf{X}}_{n-1} + \delta\Delta t\ddot{\mathbf{X}}_n \quad (3)$$

The preceding substitution provides the following relationship for structures initially at rest.

$$(\mathbf{E}_2B^2 + \mathbf{E}_1B + \mathbf{E}_0)\mathbf{X}_n = (\mathbf{E}_2^*B^2 + \mathbf{E}_1^*B + \mathbf{E}_0^*)Y_n \quad (4)$$

The parameters in Eq. 4 are

$$\mathbf{E}_0 = -\mathbf{M} - \delta\Delta t\mathbf{C} - \beta\Delta t^2\mathbf{K} \quad (5a)$$

$$\mathbf{E}_1 = 2\mathbf{M} + (-1 + 2\delta)\Delta t\mathbf{C} + (-0.5 - \delta + 2\beta)\Delta t^2\mathbf{K} \quad (5b)$$

$$\mathbf{E}_2 = -\mathbf{M} + (1 - \delta)\Delta t\mathbf{C} + (-0.5 + \delta - \beta)\Delta t^2\mathbf{K} \quad (5c)$$

$$\mathbf{E}_0^* = -\mathbf{M}^* - \delta\Delta t\mathbf{C}^* - \beta\Delta t^2\mathbf{K}^* \quad (5d)$$

$$\mathbf{E}_1^* = 2\mathbf{M}^* + (-1 + 2\delta)\Delta t\mathbf{C}^* + (-0.5 - \delta + 2\beta)\Delta t^2\mathbf{K}^* \quad (5e)$$

and

$$\mathbf{E}_2^* = -\mathbf{M}^* + (1 - \delta)\Delta t\mathbf{C}^* + (-0.5 + \delta - \beta)\Delta t^2\mathbf{K}^* \quad (5f)$$

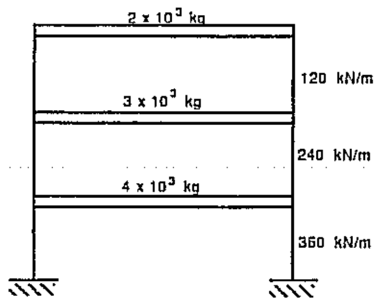

FIG. 1. Lumped-Mass Model of Example Building

The displacement response vector is given by

$$\mathbf{X}_n = \mathbf{E}_0^{-1}(-\mathbf{E}_1\mathbf{X}_{n-1} - \mathbf{E}_2\mathbf{X}_{n-2} + \mathbf{E}_0^*\mathbf{Y}_n + \mathbf{E}_1^*\mathbf{Y}_{n-1} + \mathbf{E}_2^*\mathbf{Y}_{n-2}) \quad (6)$$

The preceding matrix equation describes Eq. 2 and the coefficient matrices are given by

$$\mathbf{A}_1 = \mathbf{E}_0^{-1}\mathbf{E}_1; \quad \mathbf{A}_2 = \mathbf{E}_0^{-1}\mathbf{E}_2 \quad (7a)$$

$$\mathbf{B}_0 = \mathbf{E}_0^{-1}\mathbf{E}_0^*; \quad \mathbf{B}_1 = \mathbf{E}_0^{-1}\mathbf{E}_1^*; \quad \mathbf{B}_2 = \mathbf{E}_0^{-1}\mathbf{E}_2^* \quad (7b)$$

for

$$P = 2; \quad Q^- = 0; \quad Q^+ = 2 \quad (7c)$$

A recursive formulation for the response covariance may be derived from Eq. 2. Multiplying Eq. 2 with $\mathbf{Y}_m^T$ and $\mathbf{X}_m^T$ and taking expectations of both sides provides the following recursive covariance matrices:

$$\mathbf{C}_{xy}(n,m) + \sum_{r=1}^{P} \mathbf{A}_r\mathbf{C}_{xy}(n-r,m) = \sum_{r=-Q^-}^{Q^+} \mathbf{B}_r\mathbf{C}_{yy}(n-r,m) \quad (8)$$

and

$$\mathbf{C}_{xx}(n,m) + \sum_{r=1}^{P} \mathbf{A}_r\mathbf{C}_{xx}(n-r,m) = \sum_{r=-Q^-}^{Q^+} \mathbf{B}_r\mathbf{C}_{xy}(n-r,m) \quad (9)$$

in which $\mathbf{C}_{xy}(n,m)$ = cross-correlation between $\mathbf{X}_n$ and $\mathbf{Y}_m$. These equations can be solved recursively for a given set of initial conditions.

Numerical Example

The preceding recursive formulation is illustrated by means of an example building subjected to a transient ground acceleration. The transient response of this building under stationary time-limited earthquake excitation is given by Madsen and Krenk (1982) using a random-vibration-based approach. The example building is shown in Fig. 1. The applied loading is of 10 sec duration. The frequency contents are described by the Kanai-Tajimi spectral density function given by

$$S_{FF}(\omega) = S_0\left(\frac{\omega_g^4 + 4\xi_g^2\omega_g^2\omega^2}{(\omega_g^2 - \omega^2)^2 + 4\xi_g^2\omega_g^2\omega^2}\right) \quad (10)$$

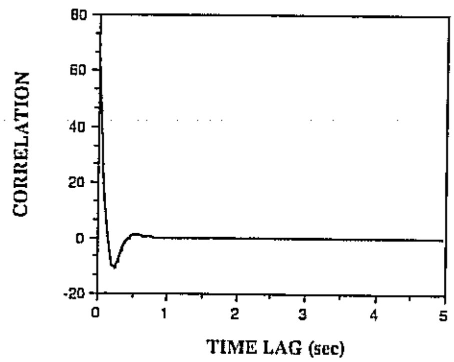

FIG. 2. Correlation Function of Input Ground Acceleration

in which $\omega_g = 4\pi$ rad/sec; and $\xi_g = 0.6$. The intensity of ground acceleration S_0 is taken as 1.0 m^2/sec^3. For this loading, the terms on the right-hand side of Eq. 1 take the following form: $\mathbf{M}^* = 0$; $\mathbf{C}^* = 0$; and $\mathbf{K}^* = -\mathbf{M}$ and $\mathbf{Y}$ = ground acceleration. The correlation function obtained from the spectral description of the input is presented in Fig. 2. The parameters for this recursive analysis are chosen to be equal to: $\delta = 0.5$; $\beta = 0.0833$, and $\Delta t = 0.05$ sec. The coefficient matrices are evaluated for this problem based on Eqs. *5a–f* and *7a–c* and are

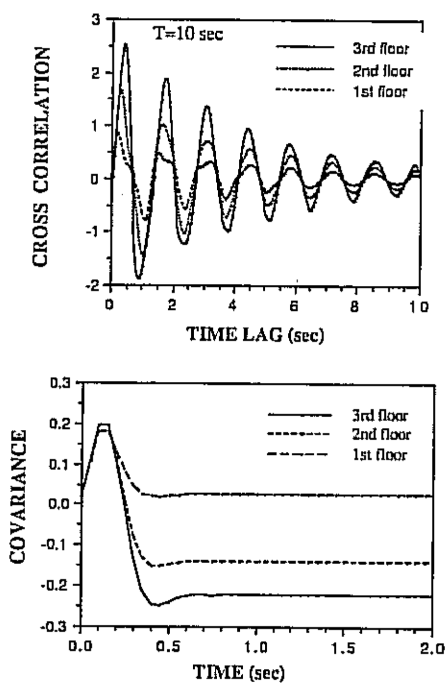

FIG. 3. Cross Correlation and Covariance of Input-Output

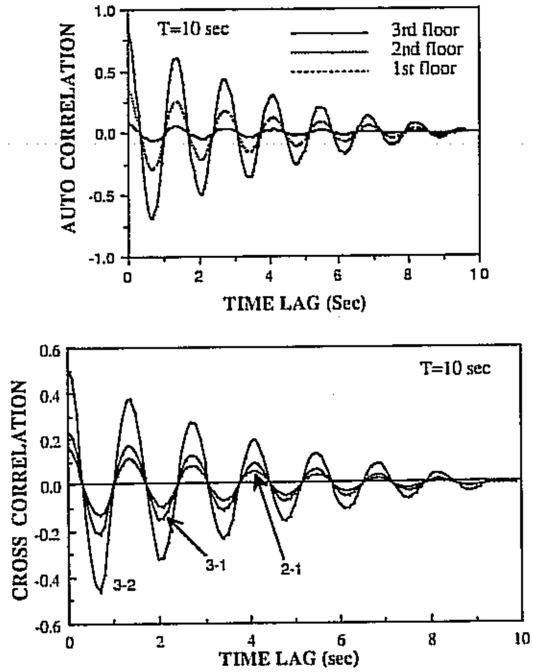

FIG. 4. Auto and Cross-Correlation Functions of Response

$$\mathbf{A}_1 = \begin{bmatrix} -1.82E+00 & -1.56E-01 & -6.90E-03 \\ -1.04E-01 & -1.67E+00 & -1.98E-01 \\ -3.45E-03 & -1.49E-01 & -1.59E+00 \end{bmatrix};$$

$$\mathbf{A}_2 = \begin{bmatrix} 9.66E-01 & 1.65E-02 & 3.71E-03 \\ 1.10E-02 & 9.53E-01 & 1.65E-02 \\ 1.85E-03 & 1.24E-02 & 9.45E-01 \end{bmatrix}$$

$$\mathbf{B}_0 = \begin{bmatrix} 2.02E-04 & 4.15E-06 & 4.41E-07 \\ 2.77E-06 & 1.99E-04 & 4.87E-06 \\ 2.21E-07 & 3.66E-06 & 1.96E-04 \end{bmatrix};$$

$$\mathbf{B}_1 = \begin{bmatrix} 2.02E-03 & 4.15E-05 & 4.41E-06 \\ 2.77E-05 & 1.99E-03 & 4.87E-05 \\ 2.21E-06 & 3.66E-05 & 1.96E-03 \end{bmatrix}$$

$$\mathbf{B}_2 = \begin{bmatrix} 2.02E-04 & 4.15E-06 & 4.41E-07 \\ 2.77E-06 & 1.99E-04 & 4.87E-06 \\ 2.21E-07 & 3.66E-06 & 1.96E-04 \end{bmatrix} \quad \text{.........................} \quad (11)$$

By solving Eqs. 8 and 9 recursively, the covariance of structural response is obtained. In Fig. 3, the cross correlation and covariance of response at each floor level with the excitation are presented. The autocorrelation of response at each floor and cross correlation of response between different

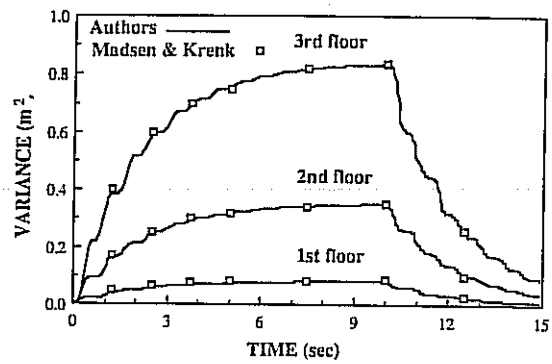

FIG. 5. Variance of Response at Each Floor

floors are presented in Fig. 4. A direct comparison of these results to the figures provided in Madsen and Krenk (1982) is not possible, since the related results in their paper are expressed in the modal coordinates, whereas the present analysis is accomplished in the physical coordinates. The variance of the transient floor displacements in the physical coordinates is presented in Fig. 5 with Madsen and Krenk's (1982) data. The results are almost coinciding.

The method presented here is computationally convenient and accurate for stationary, transient, or nonstationary response of MDOF systems. It does not require eigenvalue analysis. Furthermore, a proportional damping condition is not a prerequisite for the analysis, which makes it very attractive for the treatment of soil-structure or structure-appendage systems.

Accuracy and Stability Analysis

One of the merits of using a recursive model is the convenience of evaluating the stability and accuracy of the computational procedure. Consequently, this approach facilitates the selection of the most suitable model and time increment that offers a stable response with least computational effort and accuracy to the prescribed limits.

Numerical stability of time-integration schemes implies that the integrated output of a system will be bounded for bounded inputs. For time integration schemes, e.g., the Newmark-β method, the stability is ensured by limiting the integration time step to a fraction of the minimum natural period of the MDOF system. In the case of Newmark-β method

$$\Delta t < T_{\min} \frac{\left\{ \xi(\delta - 0.5) + \left[\frac{\delta}{2} - \beta + \xi^2(\delta - 0.5)^2 \right]^{1/2} \right\}}{2\pi\left(\frac{\delta}{2} - \beta\right)} \quad \text{(12)}$$

in which $T_{\min}$ = the minimum time period associated with the system (Belytschko and Hughes 1983). For $\delta = 0.5$, $\beta = 0.25$, and ξ (damping factor) $= 0$, the numerical scheme becomes unconditionally stable.

TABLE 1. Error Associated with Time Integration Scheme (Newmark-β)

Δt (1)	β (2)	e (%) (3)	f_n (4)
0.5	0	2	0.204
0.5	0.0833	0.5	0.2
0.5	0.25	6	0.192
1.0	0	23	0.216
1.0	0.0833	10	0.2
1.0	0.25	12	0.18
1.677	0	38	0.27
1.677	0.0833	77	0.202
1.677	0.25	16	0.144

Note: $\delta = 0.5$; f_n = resonant frequency = 0.2 Hz.

A linear discrete time system is stable provided all the poles of the recursive model lie outside the unit circle in the **Z**-plane. For a second-order recursive model, the absolute values of the eigenvalues of the matrix **Z** in the following matrix equation larger than 1 will ensure a stable system:

$$\mathbf{E}_2\mathbf{Z}\mathbf{Z} + \mathbf{E}_1\mathbf{Z} + \mathbf{E}_0 = 0 \quad (13)$$

and

$$\mathbf{Z} = \frac{1}{2}\mathbf{E}_2^{-1}[\mathbf{E}_1 \pm (\mathbf{E}_1\mathbf{E}_1 - 4\mathbf{E}_2\mathbf{E}_0)^{1/2}] \quad (14)$$

It can be shown that the preceding condition for stability also provides the stability condition previously stated for Newmark-β method (Eq. 12), thus confirming that the stability of a recursive model represents the stability of the associated time-integration scheme. In this manner, the stability of the present recursive numerical scheme can be ascertained by checking its stationarity. This is accomplished by assessing the location of the poles of the system with respect to the unit circle.

A comparison of the transfer function of the recursive model to the exact transfer function provides an assessment of the numerical error given by

$$\mathbf{e} = \int_{f_{min}}^{f_{max}} |[\hat{\mathbf{H}}(f)\hat{\mathbf{H}}(f)^* - \mathbf{H}(f)\mathbf{H}(f)^*][\mathbf{H}(f)\mathbf{H}(f)^*]^{-1}|df \quad (15a)$$

in which

$$\hat{\mathbf{H}}(f) = \left[I^+ \sum_{r=1}^{P} \mathbf{A}_r \exp(jr2\pi\Delta t f)\right]^{-1}\left[\sum_{r=-Q^-}^{Q^+} \mathbf{B}_r \exp(jr2\pi\Delta t f)\right] \quad (15b)$$

$$\mathbf{H}(f) = [-(2\pi f)^2\mathbf{M} - j(2\pi f)\mathbf{C} + \mathbf{K}]^{-1}[-(2\pi f)^2\mathbf{M}^* - j(2\pi f)\mathbf{C}^* + \mathbf{K}^*] \quad (15c)$$

An example of a discretized five-degree-of-freedom system with each mass equal to 4.5×10^5 kg, inter-story stiffness equal to 8,770 kN/m and the modal damping ratio of 2% in the fundamental mode was employed to study the accuracy of the recursive scheme. The results are described in Table 1 considering the response in the first mode only. The results delineate error associated with the different values of the model parameter β and integration time step. For $\beta = 0.25$, the numerical scheme is unconditional and the convergence is of the order $O(\Delta t^2)$. In the case of $\beta = 0.0833$ and zero damping, the convergence order is $O(\Delta t^4)$. The conventional convergence expressed in terms of $O(\Delta t^i)$ cannot explicitly represent the error in a manner listed in Table 1. Rather, it indicates how fast the numerical results approach the exact values by decreasing Δt. For large time-step-to-period ratio $O(\Delta t^4)$ may not be better than $O(\Delta t^2)$, as noted in Table 1. The changes in the period are also included in Table 1 in terms of the response frequency. As Δt increases, the numerical error and period also magnify. The best results are for $\Delta t = 0.5$, in which the time-step-to-period ratio is 0.1.

ARMA Representation of Response

The parametric time series modeling, as popularized by Box and Jenkins (1976) for time series forecasting, have been found to provide good simulation of random processes. An ARMA (auto-regressive moving average) model combines the features of AR (auto-regressive) and MA (moving average) processes. In this model, an input driving sequence $\mathbf{W}_n$ and the output sequence $\mathbf{X}_n$ are related by a linear recursive operator

$$\mathbf{X}_n + \sum_{r=1}^{P} \mathbf{\Phi}_r \mathbf{X}_{n-r} = \sum_{r=0}^{Q} \mathbf{\Psi}_r \mathbf{W}_{n-r} \qquad (16)$$

The vector $\mathbf{X}_n$ is said to be an ARMA process of order (p,q). The excitation sequence $\mathbf{W}_n$ is composed of a zero mean and unit variance, uncorrelated random process (i.e., white noise), and $\mathbf{\Phi}_r$ and $\mathbf{\Psi}_r$ are auto-regressive and moving average coefficient matrices, respectively, which are determined from exact knowledge of the correlation structure of the time series. The significance of ARMA processes lies in the fact that a stationary time series often can be described by an ARMA model involving fewer parameters than an MA or AR process by itself. Finding an optimal ARMA model with the lowest orders is the desired objective.

The purpose of this section is to use a recursive model to express the system response in terms of an ARMA model given the ARMA representation of the input. In this manner both the time histories of the response vector and the associated cross-spectral density matrix may be obtained directly. Considering that the inputs are expressed in terms of an ARMA model

$$\mathbf{Y}_n + \sum_{r=1}^{p} \mathbf{\Phi}_r \mathbf{Y}_{n-r} = \sum_{r=0}^{q} \mathbf{\Psi}_r \mathbf{W}_{n-r} \qquad (17)$$

in which $\mathbf{Y}_n$ = the input excitation vector; and $\mathbf{W}_n$ = a white noise vector. The preceding equations can also be expressed as

$$\mathbf{\Phi}(B)\mathbf{Y}_n = \mathbf{\Psi}(B)\mathbf{W}_n \qquad (18)$$

To obtain an ARMA model describing the system response, substitute Eq. 18 in Eq. 4 and after subsequent rearrangement

$$\mathbf{E}(B)\mathbf{X}_n = \mathbf{E}^*(B)\mathbf{\Phi}^{-1}(B)\mathbf{\Psi}(B)\mathbf{W}_n \qquad (19)$$

in which matrix operator $\mathbf{E}(B) = \mathbf{E}_2B + \mathbf{E}_1B + \mathbf{E}_0$. The desired ARMA format for the output is

$$\sum_{r=0}^{p+2} \mathbf{\Phi}_r\mathbf{X}_{n-r} = \sum_{r=0}^{q+2} \mathbf{\Psi}_r\mathbf{W}_{n-r} \qquad (20)$$

In this manner, the input-output relationship is given by

$$\sum_{r=0}^{p+2} \begin{bmatrix} \mathbf{\Phi}_r & 0 \\ 0 & \mathbf{A}_r \end{bmatrix} \begin{Bmatrix} \mathbf{Y}_{n-r} \\ \mathbf{X}_{n-r} \end{Bmatrix} = \sum_{r=0}^{q+2} \begin{bmatrix} \mathbf{\Psi}_r & 0 \\ \mathbf{B}_r & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{W}_{n-r} \\ 0 \end{Bmatrix} \qquad (21)$$

in which $\mathbf{A}_r$ and $\mathbf{B}_r$ are defined in Eq. 2.

In the preceding equations, the evaluation of the coefficient matrices is difficult for the general loading representation expressed in Eq. 1. However, it is possible to obtain these coefficient matrices for special class of excitations acting individually such as wind, waves, earthquakes, or out-of-balance machine components. This is possible if $\mathbf{E}^*(B)$ can be expressed in the form of $\mathbf{E}^*(\alpha_0 + \alpha_1B + \alpha_2B^2)$, in which $\mathbf{E}^*$ is an invertible matrix independent of operator B. This reduces Eq. 19 to an ARMA format

$$\Phi(B)E^{*-1}E(B)X_n = (\alpha_0 + \alpha_1B + \alpha_2B^2)\Psi(B)W_n \qquad (22)$$

In the following, the coefficient matrices are derived for typical environmental loadings. A related application of this concept may be found in the simulation of stochastic processes, e.g., ocean surface profile. In such an approach, a time series is obtained as an output of a white-noise-excited dynamic system whose power spectral density is represented by a differential equation (Spanos 1983). The application to the simulation of multiply correlated random fields is a possible extension.

Typical Applications

The first application concerns external excitation applied at each degree of freedom, which results in $\mathbf{M}^* = 0$, $\mathbf{C}^* = 0$, and $\mathbf{K}^* = 1$, and $\mathbf{Y}(t)$ is the applied load. This case is a typical example of wind or wave loads acting on structures. Eq. 4 reduces to

$$(\mathbf{E}_2B^2 + \mathbf{E}_1B + \mathbf{E}_0)\mathbf{X}_n = (\alpha_2B^2 + \alpha B + \alpha_0)\mathbf{Y}_n \qquad (23)$$

in which $\alpha_0' = -\beta\Delta t^2$; $\alpha_1' = (-0.5 + 2\beta - \delta)\Delta t^2$; and $\alpha_2' = (-0.5 - \beta + \delta)\Delta t^2$. The ARMA model for the output displacement vector is given by

$$\sum_{r=0}^{p+2} \mathbf{\Phi}_r\mathbf{X}_{n-r} = \sum_{r=0}^{q+2} \mathbf{\Psi}_r\mathbf{W}_{n-r} \qquad (24)$$

in which $\mathbf{\Phi}_0 = \mathbf{I}$; $\mathbf{\Phi}_r = \mathbf{E}_0^{-1}(\mathbf{\Phi}_r\mathbf{E}_0 + \mathbf{\Phi}_{r-1}\mathbf{E}_1 + \mathbf{\Phi}_{r-2} + \Phi_{r-2}\mathbf{E}_2)$; $\mathbf{\Psi}_r = \mathbf{E}_0^{-1}(\alpha_0\mathbf{\Psi}_r + \alpha_1\mathbf{\Psi}_{r-1} + \alpha_2\mathbf{\Psi}_{r-2})$; and $\mathbf{\Phi}_r = \mathbf{\Psi}_r = 0$ for $r < 0$.

A numerical example is presented in the following section for this case in which the dynamic response of a wind excited tall building is examined.

Another related application is an extension of the previous case in which the input is fluctuation in either wind velocity or wave surface profile rather than the wind or wave loads directly. In this case $\mathbf{M}^* = 0$, $\mathbf{C}^* = 0$, and $\mathbf{E}^* = \mathbf{K}^*$, and $\alpha_0 = -0.5\ \beta\Delta t^2$; $\alpha_1 = (-0.5 - \delta + B)\Delta t^2$ and $\alpha_2 = (-0.5 + \delta - \beta/2)\Delta t^2$ in Eq. 23. A typical element in the $\mathbf{K}^*$ matrix for the wind loading, is given by $K_{ij}^* = \rho C_D \Sigma_{j=1}^{n} a_j n_{ij} U_j$, in which ρ and C_D are the air density and the drag coefficient, respectively, a_j and U_j are the jth element area and mean wind speed at the centroid of the element, and n_{ij} denotes the transfer matrix that relates the force on the jth element to the force being transferred to the ith degree of freedom.

For structures subjected to ground acceleration, $\mathbf{M}^* = 0$, $\mathbf{C}^* = 0$, and $\mathbf{E}^* = \mathbf{K}^* = -\mathbf{M}$. The values of α_0, α_1, and α_2 are the same as given in the previous example. The ground acceleration is described by the $\mathbf{Y}$ vector.

Vibrations induced by an unbalanced machine can be expressed in an ARMA format. In this case, $\mathbf{C}^* = \mathbf{K}^* = 0$; and $\mathbf{E}^* = \mathbf{M}^*$. A typical element of $\mathbf{M}^*$ represents the mass of the unbalanced component related to a given degree of freedom. The displacement vector $\mathbf{Y}$ represents eccentricity of the respective unbalanced mass. In this situation, the value of α_i in Eq. 22 is equal to -1, 2, and -1 for $i = 0$, 1, and 2, respectively.

Numerical Example

A numerical example of an ARMA system is presented to illustrate the salient features of this approach using a high-rise building subjected to the along-wind aerodynamic excitation. The building dimensions are 100 ft square (31 m × 31 m) in plan and 600 ft (183 m) tall. The structural system is lumped at five levels and the associated mass and stiffness matrices are described in Appendix I. The five eigen frequencies of this building in cycles/sec (Hz) are 0.20, 0.583, 0.921, 1.182, and 1.348, respectively, and the modal damping ratios corresponding to these five modes in the ascending order are 1%, 1.57%, 2.14%, 2.52%, and 2.9%, respectively (Kareem 1981).

The along-wind aerodynamic loading is based on the quasi-steady and strip theories

$$S_{F_{ii}}(n) = \frac{4F_0^2}{V_0^2} S_u(n)L(n) \qquad (25)$$

$$S_{F_{ij}}(n) = \sqrt{S_{F_{ii}}(n)S_{F_{jj}}(n)}Coh_{ij}(n) \qquad (26)$$

$$Coh_{ij}(n) = \exp\left[\frac{-nC_z|z_i - z_j|}{\frac{1}{2}(U_i + U_j)}\right] \qquad (27)$$

$$F_0 = \frac{1}{2}\rho C_D V_0^2 BD \qquad (28)$$

$$L(n) = J_H J_V \qquad (29)$$

$$J_H = \frac{2}{(\gamma_y C_y)^2}[\exp(-\gamma_y C_y) + \gamma_y C_y - 1] \qquad (30)$$

$$J_V = \frac{2}{(\gamma_z C_z)^2}[\exp(-\gamma_z C_z) + \gamma_z C_z - 1] \qquad (31)$$

$$\gamma_z = \frac{2n\phi D}{V_0}; \quad \gamma_y = \frac{2n\phi B}{V_0}; \quad \phi = \frac{\sqrt{1 + r^2}}{(1 + r)}; \quad r = \frac{C_y B}{C_z D} \qquad (32)$$

in which $S_{F_{ii}}(n)$ = power spectral density (PSD) of wind loading at the ith level; $S_{F_{ij}}(n)$ = cross-PSD between loads at the ith and jth levels, J_H and J_V

account for the spatial-temporal correlations; C_y and C_z = decay constants equal to 16 and 10, respectively; $S_u(n)$ = wind velocity spectrum; B = building width; H = building height; $D = H/5$; V_0 = wind velocity at the centroid of a segment; ρ = air density; and C_D = drag coefficient. Details concerning the preceding model and theoretical background are available in Kareem (1985) and Davenport (1977).

The preceding multi-level correlated description of the wind load effects was used to develop an ARMA model. In this case, the development of an ARMA model poses difficulty to the high Niquist frequency associated with a small time increment required by the time-integration of the dynamic system. Generally, this problem may be eliminated by simulating the load time history at larger time increments. A subsequent interpolation of the simulated time histories for obtaining smaller time increments helps to meet the requirement of the time-integration models (Li 1988). However, the ARMA system being sought in this example may become more complicated by the introduction of an interpolation scheme. Therefore, an alternate approach is introduced to fit an accurate ARMA model. This is accomplished by introducing a two stage approach in which the load time history, e.g., $\mathbf{Y}(t)$ is transformed to $\mathbf{F}(t)$

$$\mathbf{F}_n = (1 - \alpha B)\mathbf{Y}_n \qquad (33)$$

in which α = a constant; and B = backward shift operator. The spectral description of $\mathbf{F}(t)$ is obtained in terms of the transfer function corresponding to Eq. 33 and the PSD of $\mathbf{Y}(t)$. An ARMA model consistent with the spectral description of $\mathbf{F}(t)$ is conveniently obtained as a result of a change in the shape of the target function

$$\mathbf{F}_n + \sum_{r=1}^{p'} \mathbf{\Phi}'_r \mathbf{F}_{n-r} = \sum_{r=1}^{q} \mathbf{\Psi}_r \mathbf{W}_{n-r} \qquad (34)$$

Using Eq. 33, it is possible to obtain

$$\mathbf{Y}_n + \sum_{r=1}^{p'+1} \mathbf{\Phi}_r \mathbf{Y}_{n-r} = \sum_{r=1}^{q} \mathbf{\Psi}_r \mathbf{W}_{n-r} \qquad (35)$$

in which the coefficient matrices $\mathbf{\Phi}_r$ are determined from Eqs. 33 and 34. Based on this approach an ARMA model with orders $p = 4$ and $q = 3$ was obtained that best describes the loading under consideration.

The coefficient matrices concerning the loading and response ARMA models are given in Appendix I. The simulated description of the loading time history provided excellent agreement with the target spectral description of the wind loading. A time history of the along-wind load acting on the 5th level of the building derived from the ARMA model is presented in Fig. 6.

Following the procedure outlined in this section, an ARMA model (6 × 5) for the building displacement response at five levels was obtained. The spectral matrix of the building response is obtained from the ARMA model by the following expression:

$$\mathbf{G}_x(f) = \tilde{\mathbf{H}}(f)\tilde{\mathbf{H}}^{*T}(f) \qquad (36a)$$

$$\tilde{\mathbf{H}}(f) = \left[\mathbf{I} + \sum_{r=1}^{p+2} \Phi_r \exp(jr2\pi\Delta tf)\right]^{-1} \left[\sum_{r=0}^{q+2} \Psi_r \exp(jr2\pi\Delta tf)\right] \qquad (36b)$$

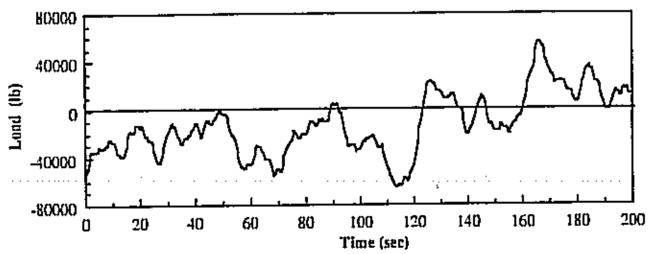

FIG. 6. Time History of Wind Load on 5th Level

in which $\bar{\mathbf{H}}(f)$ = the transfer function of the ARMA model; and * and T = conjugate and matrix transpose, respectively. The diagonal elements of the spectral matrix $\mathbf{G}_x(f)$ represent the power spectral density (PSD) functions and the off-diagonal elements describe the cross-spectral relationships. The PSD of the building response at the 5th level is presented in Fig. 7. The off-diagonal terms provide the cross-spectral density function, cross correlation or coherence, and phase information. For the sake of illustration, in Fig. 8 the coherence and phase of the displacement response at the 5th and

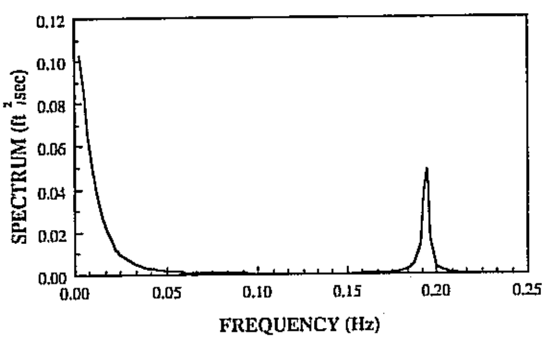

FIG. 7. PSD of 5th Level Displacement

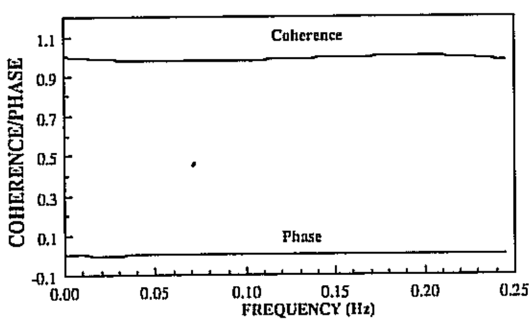

FIG. 8. Coherence and Phase for Response at 5th and 3rd Levels

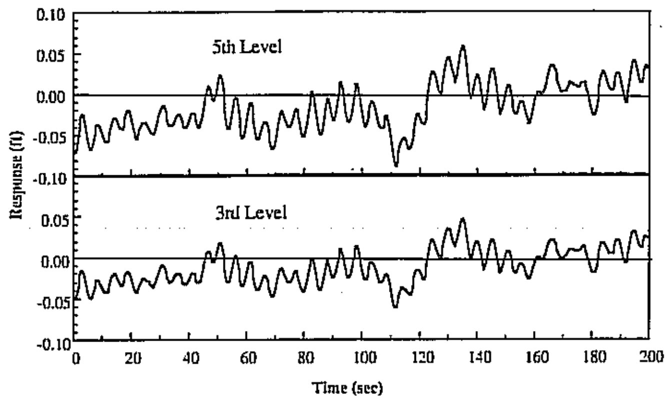

FIG. 9. Time Histories of Displacement Response

3rd levels are presented. The associated response time histories were generated from the ARMA model and are given in Fig. 9. Both the time histories in Fig. 9 and the coherence and phase relationships given in Fig. 8 suggest that the responses at the 5th and 3rd levels are almost fully coherent. This is quite obvious, since the displacement response is dominated by the fundamental mode. The response time histories clearly reflect the composition of the spectral contents shown in Fig. 7; i.e., a low frequency signature, contributed by the background response, superimposed by the resonant component. The ARMA system presented here enhances the versatility of ARMA modeling and provides a convenient computational tool to describe the response of a system in terms of an ARMA model, given the ARMA description of a correlated vector-valued excitation.

RESPONSE OF NONLINEAR SYSTEMS

A system behavior becomes nonlinear if the elements of the stiffness or damping matrix depend on the system displacement vector or its derivatives, or the system input becomes a function of the response. Generally, in these cases an iterative numerical scheme is used to estimate the response. However, for the cases in which the system stiffness is a function of displacement, **K**(**X**), setting $\delta = 0.5$ and $\beta = 0$ (which is a central difference scheme) leads to a recursive model in which the coefficient matrices depend only on the past time histories of response, thus eliminating the iterative solution.

Example

The response of a simply supported beam with elasto-plastic properties given in Biggs (1964) is used to illustrate the current technique and to compare the results of the proposed method to the given solution. In Fig. 10, the beam is shown as an idealized single-degree-of-freedom system following Biggs (1964). Details of the elastoplastic characteristics and loading are also included in Fig. 10. The structural mass = 0.0259 kip-sq sec/in., and stiffness = 83.4 kip/in. Following the proposed procedure, the recursive coefficients are evaluated to be equal to $A_1 = -0.0259$; $A_2 = 0.0515$; $A_3 =$

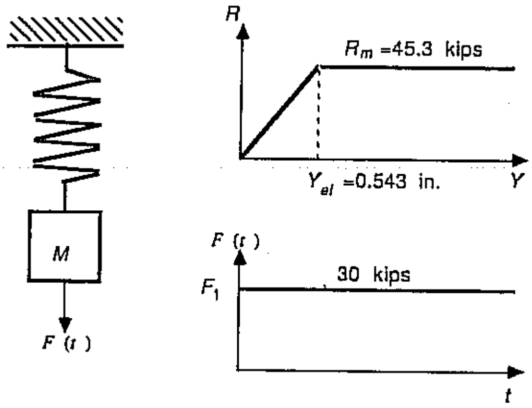

FIG. 10. Elastoplastic System Subjected to Suddenly Applied Load

-0.0259; $B_1 = 0$; $B_2 = -400 \times 10^{-6}$; $B_3 = 0$. The response at various Δt values are presented in Table 2 with the results from Biggs (1964), which are indicated as the analytical values. The results show excellent agreement for both maximum and minimum displacements and their time of occurrence. For larger Δt, the results exhibit a departure from the analytical results. The displacement response is plotted as a function of time in Fig. 11. The given yield displacement and the maximum displacement given in Biggs

TABLE 2. Response of Elasto-Plastic System

Method (1)	Yield time (2)	Maximum displacement (at time) (3)	Minimum displacement (at time) (4)
Analytical	0.0371	0.806 (0.0669)	0.438 (0.122)
$\Delta t = 0.02$	0.0248	0.886 (0.06)	0.548 (0.10)
$\Delta t = 0.01$	0.0314	0.819 (0.06)	0.455 (0.12)
$\Delta t = 0.002$	0.361	0.804 (0.066)	0.438 (0.122)

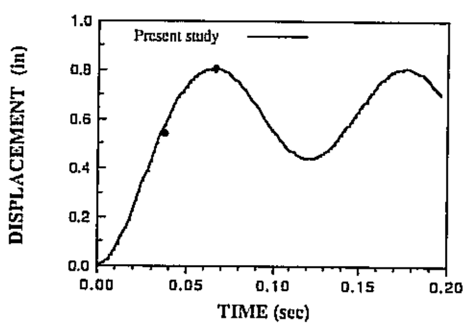

FIG. 11. Response of Elastoplastic System

(1964) are indicated in the figure by solid points. As stated earlier, the results are almost coincident.

Concluding Remarks

A computationally efficient recursive procedure for evaluating the response statistics of dynamic systems subjected to stationary, transient, or nonstationary correlated multi-input random excitation was presented. A proportional damping matrix is not a prerequisite for the application of this technique. The computations are carried out in physical coordinates, which precludes the need for evaluating the system mode shapes. The procedure also offers a convenient means of evaluating the stability and accuracy of the numerical procedure. The response may be expressed in terms of an ARMA model based on the given ARMA representation of a stationary input. In this manner, both the time histories and the cross-spectral density matrix of the response may be obtained from the derived ARMA model. The computational procedure also permits response analysis of nonlinear systems. A number of examples have been presented here that demonstrate excellent agreement with the exact solutions.

Acknowledgment

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Appendix I. Building Mass, Stiffness, and Damping Matrices, and Input/Output ARMA Coefficient Matrices

1. Number of D.O.F. = 5.
2. Input order of AR Parameters = 4.
3. Input order of MA Parameters = 3.
4. $\delta = 0.50$.
5. $\beta = 0.25$.
6. Time step = 0.5 sec.
7. Reduced wind velocity.
8. At top of the building = 4.
9. Wind spectrum—Harris.
10. Boundary layer—urban.

Input Data

Mass Matrix (Slugs)

$$\begin{bmatrix} 4.50E+05 & 0.00E+00 & 0.00E+00 & 0.00E+00 & 0.00E+00 \\ 0.00E+00 & 4.50E+05 & 0.00E+00 & 0.00E+00 & 0.00E+00 \\ 0.00E+00 & 0.00E+00 & 4.50E+05 & 0.00E+00 & 0.00E+00 \\ 0.00E+00 & 0.00E+00 & 0.00E+00 & 4.50E+05 & 0.00E+00 \\ 0.00E+00 & 0.00E+00 & 0.00E+00 & 0.00E+00 & 4.50E+05 \end{bmatrix} \quad \ldots\ldots\ldots (37)$$

Damping Matrix (Slug/sec)

$$\begin{bmatrix} 3.41E+04 & -1.99E+04 & -4.40E+03 & -1.75E+03 & -6.80E+02 \\ -1.99E+04 & 4.96E+04 & -1.72E+04 & -3.33E+03 & -1.07E+03 \\ -4.40E+03 & -1.72E+04 & 5.05E+04 & -1.66E+04 & -2.65E+03 \\ -1.75E+03 & -3.33E+03 & -1.66E+04 & 5.15E+04 & -1.55E+04 \\ -6.80E+02 & -1.07E+03 & -2.65E+03 & -1.55E+04 & 5.40E+04 \end{bmatrix} \quad \ldots\ldots (38)$$

Stiffness Matrix (lb/ft)

$$\begin{bmatrix} 8.77E+06 & -8.77E+06 & 0.00E+00 & 0.00E+00 & 0.00E+00 \\ -8.77E+06 & 1.75E+07 & -8.77E+06 & 0.00E+00 & 0.00E+00 \\ 0.00E+00 & -8.77E+06 & 1.75E+07 & -8.77E+06 & 0.00E+00 \\ 0.00E+00 & 0.00E+00 & -8.77E+06 & 1.75E+07 & -8.77E+06 \\ 0.00E+00 & 0.00E+00 & 0.00E+00 & -8.77E+06 & 1.75E+07 \end{bmatrix} \quad \ldots\ldots (39)$$

Input ARMA Parameters

MA Parameter Matrix:

0th

$$\begin{bmatrix} 1.6873E-04 & 5.9501E-05 & 2.0016E-05 & 5.9058E-06 & 1.1150E-06 \\ 7.4139E-05 & 1.0817E-04 & 3.6230E-05 & 1.0648E-05 & 2.0038E-06 \\ 3.1786E-05 & 4.7410E-05 & 8.2115E-05 & 2.4032E-05 & 4.5084E-06 \\ 1.3288E-05 & 1.9778E-05 & 3.4823E-05 & 5.7141E-05 & 1.0687E-05 \\ 4.8278E-06 & 7.1412E-06 & 1.2543E-05 & 2.0781E-05 & 2.5670E-05 \end{bmatrix} \quad \ldots (40)$$

1st

$$\begin{bmatrix} -1.3384E+02 & 1.7540E+02 & 1.2235E+00 & -9.6152E+01 & -1.2698E+01 \\ 2.4083E+02 & -3.4807E+02 & 1.0369E+02 & 1.3480E+01 & -3.3117E+01 \\ 1.4668E+01 & 7.4353E+01 & -2.1282E+02 & 1.8725E+02 & -4.6267E+01 \\ -6.5659E+01 & 8.2267E+00 & 1.1947E+02 & -1.0439E+02 & 1.2956E+00 \\ -4.0914E+01 & -3.2059E+01 & -6.6420E+01 & -8.7660E+01 & 1.9017E+02 \end{bmatrix} \quad \ldots (41)$$

2nd

$$\begin{bmatrix} -1.0609E+03 & 6.5478E+01 & -4.4587E+01 & -3.7248E+01 & -2.2777E+01 \\ 6.0331E+01 & -9.9193E+02 & 6.8963E+01 & -2.2565E+01 & -4.3577E+01 \\ -4.9998E+01 & 2.3667E+01 & -7.3722E+02 & 6.4458E+01 & -5.6490E+01 \\ -3.6142E+01 & -1.5933E+01 & 2.5713E+01 & -5.0359E+02 & -8.2646E+00 \\ -8.7736E+00 & 8.6816E+00 & -9.7908E+00 & -6.5239E+01 & -1.8746E+02 \end{bmatrix} \quad \ldots (42)$$

3rd

$$\begin{bmatrix} -1.2023E+02 & -1.0321E+01 & -1.4151E+01 & 7.2652E+00 & -7.4217E-01 \\ -1.7109E+01 & -8.8784E+01 & 4.4650E+00 & -9.4139E+00 & -3.4965E+00 \\ -1.7722E+01 & -4.0951E-01 & -6.2811E+01 & -1.3648E+01 & -7.1436E+00 \\ 1.5542E+00 & -5.5673E+00 & -1.0925E+01 & -4.4643E+01 & -3.8775E+00 \\ 5.0800E+00 & 9.5983E+00 & 1.0329E+01 & -3.8203E+00 & -6.7485E+01 \end{bmatrix} \quad \ldots (43)$$

AR Parameter Matrix

1st

$$\begin{bmatrix} -1.9035E+00 & 5.1950E-02 & -1.2891E-02 & -1.0334E-01 & -3.6223E-02 \\ 8.7920E-02 & -2.0425E+00 & 2.9296E-02 & -1.0429E-03 & -7.4344E-02 \\ -1.3129E-03 & 1.7714E-02 & -2.0186E+00 & 1.4060E-01 & -1.0506E-01 \\ -3.4934E-02 & -2.2364E-03 & 6.4543E-02 & -1.9999E+00 & -4.5216E-02 \\ -1.9818E-02 & -1.8210E-02 & -4.8678E-02 & -1.0267E-01 & -1.4859E+00 \end{bmatrix} \quad \ldots\ldots (44)$$

2nd

$$\begin{bmatrix} 7.2102E-01 & -9.9673E-02 & -2.2142E-02 & 1.5044E-01 & 1.6127E-02 \\ -1.3956E-01 & 9.2101E-01 & -4.0662E-02 & -3.3403E-02 & 4.2378E-02 \\ -2.6557E-02 & -2.7977E-02 & 9.1386E-01 & -2.3747E-01 & 6.3293E-02 \\ 4.5119E-02 & -8.0089E-03 & -1.0962E-01 & 8.9222E-01 & 2.3709E-02 \\ 3.2500E-02 & 4.0533E-02 & 8.5708E-02 & 1.2150E-01 & -6.8574E-02 \end{bmatrix} \quad \ldots\ldots (45)$$

3rd

$$\begin{bmatrix} 4.0090E-01 & 4.7184E-02 & 5.4260E-02 & -3.1988E-02 & 5.8485E-02 \\ 4.5598E-02 & 3.5428E-01 & 2.7336E-03 & 5.0287E-02 & 9.2002E-02 \\ 3.9019E-02 & 6.2609E-03 & 3.2577E-01 & 8.8858E-02 & 1.1106E-01 \\ -4.4003E-03 & 1.5146E-02 & 4.1819E-02 & 3.2643E-01 & 3.9944E-02 \\ -1.2306E-02 & -2.5775E-02 & -3.7249E-02 & 5.6033E-03 & 8.0717E-01 \end{bmatrix} \quad \ldots\ldots (46)$$

4th

$$\begin{bmatrix} -2.1342E-01 & 5.3954E-04 & -2.0943E-02 & -1.7246E-02 & -3.9203E-02 \\ 7.5833E-03 & -2.3124E-01 & 9.0621E-03 & -1.7038E-02 & -6.2416E-02 \\ -1.2262E-02 & 4.1267E-03 & -2.1773E-01 & 9.6560E-03 & -7.3520E-02 \\ -6.8318E-03 & -5.3599E-03 & 4.0013E-03 & -2.1465E-01 & -2.1284E-02 \\ -7.2418E-04 & 3.5606E-03 & -3.3307E-04 & -2.7720E-02 & -2.4418E-01 \end{bmatrix} \quad \ldots\ldots (47)$$

Output ARMA Parameters

MA Parameter Matrix

0th

$$\begin{bmatrix} 1.6873E-04 & 5.9501E-05 & 2.0016E-05 & 5.9058E-06 & 1.1150E-06 \\ 7.4139E-05 & 1.0817E-04 & 3.6230E-05 & 1.0648E-05 & 2.0038E-06 \\ 3.1786E-05 & 4.7410E-05 & 8.2115E-05 & 2.4032E-05 & 4.5084E-06 \\ 1.3288E-05 & 1.9778E-05 & 3.4823E-05 & 5.7141E-05 & 1.0687E-05 \\ 4.8278E-06 & 7.1412E-06 & 1.2543E-05 & 2.0781E-05 & 2.5670E-05 \end{bmatrix} \quad \ldots (48)$$

1st

$$\begin{bmatrix} 3.3450E-04 & 1.2247E-04 & 4.1181E-05 & 6.3610E-06 & -1.5318E-07 \\ 1.5802E-04 & 2.0285E-04 & 7.4429E-05 & 2.2231E-05 & 1.1049E-06 \\ 6.6767E-05 & 9.2658E-05 & 1.5690E-04 & 5.4640E-05 & 6.9515E-06 \\ 2.4204E-05 & 3.8530E-05 & 7.1077E-05 & 1.1074E-04 & 2.3678E-05 \\ 7.1797E-06 & 1.2632E-05 & 2.2929E-05 & 3.6799E-05 & 5.9773E-05 \end{bmatrix} \quad \ldots (49)$$

2nd

$$\begin{bmatrix} 7.8493E-05 & 3.8719E-05 & 1.0871E-05 & -1.0859E-05 & -8.1650E-06 \\ 6.0053E-05 & 2.3389E-05 & 2.4472E-05 & 6.0985E-06 & -9.4221E-06 \\ 2.1112E-05 & 1.9986E-05 & 2.6866E-05 & 2.7548E-05 & -6.0167E-06 \\ -2.2134E-07 & 7.7267E-06 & 2.2240E-05 & 2.2015E-05 & 9.2799E-06 \\ -3.6000E-06 & 6.2654E-07 & 2.3492E-06 & -1.2897E-06 & 3.2892E-05 \end{bmatrix} \cdots\cdots\cdots (50)$$

3rd

$$\begin{bmatrix} -1.8208E-04 & -5.5808E-05 & -2.3647E-05 & -1.7363E-05 & -1.1847E-05 \\ -6.2869E-05 & -1.3488E-04 & -3.1287E-05 & -1.3051E-05 & -1.4849E-05 \\ -3.3982E-05 & -5.0823E-05 & -9.2370E-05 & -1.4683E-05 & -1.6010E-05 \\ -2.0993E-05 & -2.2124E-05 & -3.1352E-05 & -6.2542E-05 & -1.1467E-05 \\ -9.6134E-06 & -8.0854E-06 & -1.4175E-05 & -3.1038E-05 & -1.4174E-05 \end{bmatrix} \cdots\cdots (51)$$

4th

$$\begin{bmatrix} -1.0529E-04 & -3.5379E-05 & -1.5261E-05 & -6.2318E-06 & -5.3868E-06 \\ -4.4534E-05 & -6.9382E-05 & -1.9424E-05 & -8.7160E-06 & -7.0324E-06 \\ -2.3161E-05 & -2.8023E-05 & -4.8299E-05 & -1.3610E-05 & -8.7047E-06 \\ -1.0944E-05 & -1.2202E-05 & -1.9235E-05 & -3.3870E-05 & -9.4965E-06 \\ -3.8467E-06 & -3.2294E-06 & -6.3977E-06 & -1.4915E-05 & -1.6284E-05 \end{bmatrix} \cdots\cdots (52)$$

5th

$$\begin{bmatrix} -1.0483E-05 & -3.8257E-06 & -1.9081E-06 & -1.8387E-07 & -4.3662E-07 \\ -5.4872E-06 & -5.7903E-06 & -1.8644E-06 & -1.1487E-06 & -7.0603E-07 \\ -3.0484E-06 & -2.4617E-06 & -3.8516E-06 & -1.9870E-06 & -1.1550E-06 \\ -1.0890E-06 & -1.1022E-06 & -1.8956E-06 & -2.9134E-06 & -1.7407E-06 \\ -1.8530E-07 & -8.0114E-09 & -2.5922E-07 & -1.1853E-06 & -3.3204E-06 \end{bmatrix} \cdots\cdots (53)$$

AR Parameter Matrix

1st

$$\begin{bmatrix} -2.1926E+00 & -9.8233E-01 & -3.6859E-01 & -2.7333E-01 & -1.5155E-01 \\ -8.2871E-01 & -1.8668E+00 & -6.8562E-01 & -2.5404E-01 & -2.7591E-01 \\ -3.6310E-01 & -7.4315E-01 & -1.7059E+00 & -4.8522E-01 & -4.1767E-01 \\ -1.7523E-01 & -3.1345E-01 & -6.1309E-01 & -1.6493E+00 & -4.6682E-01 \\ -6.8661E-02 & -1.0530E-01 & -2.2307E-01 & -8.4638E-01 & -7.8615E-01 \end{bmatrix} \cdots\cdots (54)$$

2nd

$$\begin{bmatrix} 2.2946E+00 & 1.8160E+00 & 7.7825E-01 & 5.0948E-01 & 8.1118E-02 \\ 1.7526E+00 & 1.3437E+00 & 1.4535E+00 & 6.3713E-01 & 1.3069E-01 \\ 7.2168E-01 & 1.4287E+00 & 1.1821E+00 & 1.2939E+00 & 3.0499E-01 \\ 3.3471E-01 & 5.8441E-01 & 1.2983E+00 & 1.1002E+00 & 8.3873E-01 \\ 1.3624E-01 & 2.3966E-01 & 5.3366E-01 & 1.1501E+00 & 8.5539E-03 \end{bmatrix} \cdots\cdots\cdots\cdots (55)$$

3rd

$$\begin{bmatrix} -1.6425E+00 & -6.4292E-01 & -3.3936E-01 & -1.9163E-01 & 1.2756E-02 \\ -6.6792E-01 & -1.2542E+00 & -6.1010E-01 & -2.6755E-01 & 8.6721E-02 \\ -2.5638E-01 & -5.7231E-01 & -1.2200E+00 & -5.6480E-01 & 6.4180E-02 \\ -1.3861E-01 & -2.2345E-01 & -5.6113E-01 & -1.1267E+00 & -3.1600E-01 \\ -7.9175E-02 & -1.3178E-01 & -2.7887E-01 & -1.4270E-01 & -8.7326E-01 \end{bmatrix} \quad \ldots\ldots (56)$$

4th

$$\begin{bmatrix} 2.7182E-01 & -4.2750E-01 & -2.2388E-01 & -8.0163E-02 & 2.3729E-02 \\ -5.5274E-01 & 7.4113E-01 & -3.5355E-01 & -2.8200E-01 & 3.9801E-02 \\ -2.3432E-01 & -2.7942E-01 & 7.3666E-01 & -5.5838E-01 & -1.8260E-02 \\ -6.6009E-02 & 1.0960E-01 & -3.1124E-01 & 6.6894E-01 & -3.6895E-01 \\ -2.0661E-03 & -2.2662E-02 & -9.0544E-02 & -1.2772E-01 & 1.9043E-01 \end{bmatrix} \quad \ldots\ldots (57)$$

5th

$$\begin{bmatrix} 4.8685E-01 & 2.2901E-01 & 1.6707E-01 & 1.1394E-02 & 1.6792E-01 \\ 2.9157E-01 & 2.6684E-01 & 1.8290E-01 & 1.4035E-01 & 1.6832E-01 \\ 1.4010E-01 & 1.6136E-01 & 2.2504E-01 & 2.7144E-01 & 2.1213E-01 \\ 5.0831E-02 & 6.3437E-02 & 1.7773E-01 & 2.0863E-01 & 3.9937E-01 \\ 1.6792E-02 & 1.8354E-02 & 5.0494E-02 & -4.0326E-02 & 7.4979E-01 \end{bmatrix} \quad \ldots\ldots\ldots\ldots (58)$$

6th

$$\begin{bmatrix} -2.0486E-01 & -1.1510E-03 & -1.4521E-02 & 2.1909E-02 & -1.4220E-01 \\ 9.6247E-03 & -2.3373E-01 & 1.6087E-02 & 2.8715E-02 & -1.6653E-01 \\ -8.6556E-03 & 1.8579E-04 & -2.0961E-01 & 5.4426E-02 & -1.7070E-01 \\ -7.1677E-03 & -3.5787E-03 & 1.0566E-02 & -1.8887E-01 & -1.0079E-01 \\ -4.2952E-03 & 2.5064E-03 & 1.1647E-02 & -8.7350E-03 & -2.6607E-01 \end{bmatrix} \quad \ldots\ldots (59)$$

Response Variance (sq ft)

$$[1.6637E-03 \quad 1.4049E-03 \quad 9.6998E-04 \quad 4.9782E-04 \quad 1.3568E-04] \ldots\ldots (60)$$

Appendix II. References

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APPENDIX III. NOTATION

The following symbols are used in this paper:

$\mathbf{A}_i, \mathbf{B}_i$ = coefficient matrices of filter in Eq. 2;
B = backward shift operator;
$\mathbf{C}$ = damping matrix ($n \times n$);
$\mathbf{C}^*$ = loading matrix, Eq. 1 ($n \times n$);
$\mathbf{C}_{xx}(n,m)$ = correlation matrix of displacement response;
$\mathbf{C}_{xy}(n,m)$ = cross-correlation matrix of displacement response and input;
f = natural frequency;
$\mathbf{H}(f), \hat{\mathbf{H}}(f)$ = transfer functions defined in Eq. 15;
$\mathbf{K}_t$ = stiffness matrix ($n \times n$);
$\mathbf{K}^*$ = loading matrix, Eq. 1 ($n \times n$);
$\mathbf{M}$ = mass matrix ($n \times n$);
$\mathbf{M}^*$ = loading matrix, Eq. 1 ($n \times n$);
$\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}$ = displacement velocity and acceleration, vectors;
$\mathbf{Y}, \dot{\mathbf{Y}}, \ddot{\mathbf{Y}}$ = loading vectors, Eq. 1;
$\mathbf{W}_n$ = vector of white noises;
α_i = constants;
β, δ = constants in Newmark-β method;
Δt = discrete time interval;
$\Phi(B)$ = AR operator; and
$\Psi(B)$ = MA operator.