The Beta Risk model
Text pp 586-588

Problem 1 1. *Finance people like to talk about betas and factors (common and risk).*

2. *Factor is a systematic component driving the cross-section of returns over time.*

3. *Returns for any security are driven by common and idiosyncratic factors.*

4. *Systematic risk is exposure to the common factor(s). The market doesn't compensate you for bearing idiosyncratic risk (exposure to idiosyncratic factor).*

5. *These issues are analyzed within what is called the beta-risk framework.*

The beta-risk framework is designed to answer the question: Over long periods of time, WHY do some assets pay high returns and others pay low returns?

Answer: Market compensates investors for bearing systematic risk. Those that have high exposure to the risk factor.

In finance, all asset pricing models can be represented as:

$$\bar{r}_i^e = \beta_{1i}\lambda_1 + \beta_{2i}\lambda_2 + \cdots + \beta_{ki}\lambda_k$$

in the case of a $k-$factor model, where

$$r_{t,i}^e = \alpha_i + \beta_{1i}f_{1t} + \beta_{2i}f_{2t} + \cdots + \beta_{ki}f_{kt} + \epsilon_{t,i}$$

We call the λ_j risk prices. The β_{ki} are called betas.

Let's start simple and think about a single-factor model.

$$\begin{aligned}
\bar{r}_i^e &= \lambda\beta_i \\
r_{t,i}^e &= \alpha_i + \beta_i f_t + \epsilon_{t,i} \\
E\left(r_{t,i}^e\right) &= \alpha_i + \beta_i E\left(f_t\right) \\
\bar{r}_i^e &= \alpha_i + \beta\bar{f} = \lambda\beta \\
\alpha_i &= \beta\left(\lambda - \bar{f}\right) = 0 \\
\lambda &= \bar{f}
\end{aligned}$$

What if α_i .ne. 0. α_i is known as Jensen's alpha. In context of mutual funds, α_i tells us that manager i has superior (or lousy) talent.

What is factor? IN the CAPM, f_t is the market excess return. IN the Fama-French world, they are the market, high-minus low and big minus small.

In economics, people like quantities, like GDP growth, inflation, consumption growth.

All asset pricing models have the beta-risk form.

Let $p_{t,i}$ be the price of some stock with payoff $x_{t+1,i} = p_{t+1,i} + d_{t,i}$.Then the rule of rational life says for the marginal investor,

$$\begin{aligned}
p_{t,i}u'(c_t) &= E_t\left(\beta u'(c_{t+1})\, x_{t+1,i}\right) \\
1 &= E_t\left(\underbrace{\frac{\beta u'(c_{t+1})}{u'(c_t)}}_{m_{t+1}} \underbrace{\frac{x_{t+1,i}}{p_{t,i}}}_{(1+r_{t+1,i})}\right) = E_t m_{t+1}\left(1 + r_{t+1,i}\right) \\
1 &= E_t m_{t+1}\left(1 + r_t^f\right) \\
0 &= E_t\left(m_{t+1}\left(r_{t+1,i} - r_t^f\right)\right) = E_t m_{t+1} r_{t+1,i}^e \\
0 &= E\left(m_{t+1} r_{t+1,i}^e\right) = E\left(m_t r_{t,i}^e\right)
\end{aligned}$$

Assume

$$m_t = 1 - b\left(f_t - \bar{f}\right)$$

Substitute this into Euler equation

$$\begin{aligned}
0 &= E\left(1 - b\left(f_t - \bar{f}\right)\right) r_t^e \\
&= E\left(r_t^e\right) - bE\left(f_t - f\right) r_t^e \\
&= \bar{r}^e - \underbrace{b\mathrm{Var}\left(f_t\right)}_{\lambda} \underbrace{\frac{\mathrm{Cov}\left(f_t, r_t^e\right)}{\mathrm{Var}\left(f_t\right)}}_{\beta}
\end{aligned}$$

i.e.,

$$\bar{r}_i^e = \lambda \beta_i$$

Let's estimate and test the CAPM with the time-series method.

The time-series method can be used when factors are excess returns. Here $f_t = r_{mt} - r_t^f$ is the market excess return.

1. Ask if λ is significant. Regress f_t on a constant, test if it's zero

$$f_t = \lambda + \epsilon_t$$

2. Run the regressions

$$r_{t,i}^e = \alpha_i + \beta_i f_t + \epsilon_{t,i}$$

   Do individual $t-$tests on the α_i (they should all be 0).

3. Plot $\bar{r}_i^e$ against β_i

4. Cheap (but incorrect) joint test of the α_i $H_0 : \alpha_1 = \cdots = \alpha_n = 0$.If the α_i are independent,
$$t_1^2 + t_2^2 + \cdots + t_N^2 \sim \chi_N^2$$

5. If the α_i are independent,
$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}' \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{\alpha_1^2}{\sigma_{11}} + \frac{\alpha_2^2}{\sigma_{22}}$$
But if not independent, :
$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}' \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \sim \chi_2^2$$
$: \alpha_2 \left(\alpha_1 \frac{\sigma_{12}}{-\sigma_{11}\sigma_{22}+\sigma_{12}^2} - \alpha_2 \frac{\sigma_{11}}{-\sigma_{11}\sigma_{22}+\sigma_{12}^2} \right) + \alpha_1 \left(-\alpha_1 \frac{\sigma_{22}}{-\sigma_{11}\sigma_{22}+\sigma_{12}^2} + \alpha_2 \frac{\sigma_{12}}{-\sigma_{11}\sigma_{22}+\sigma_{12}^2} \right)$

6. Next time, I show you how to do this on eviews.