Matrix Operations

1. Matrix addition: Two matrices A and B can be added together if they have the same dimensions. Add element to element

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$
$$C = A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

2. Matrix subtraction: Same dimensions, subtract elements

$$C = A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}$$

3. Multiplying a matrix by a scalar. So if c is a scalar, and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$cA = c \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} c = Ac$$

4. Matrix multiplication. If $A : (m \times n)$ and $B : (n \times k)$, you can multiply AB, but you can't multiply BA.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$
$$C = AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ b_{11}a_{31} + b_{21}a_{32} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$$

5.

$$\begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 b_1 + a_2 b_2$$
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

6. The determinant of a 2 by 2 matrix. (square). It's the product of the diagonal elements minus the product of the off-diagonal elements.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\det(A) = |A| = ad - cb$$

7. Matrix Inverse. Can be computed only for square matrices. If A^{-1} exists, then

$$AA^{-1} = A^{-1}A = I$$
$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Let's check

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} &= \frac{1}{ad-cb}\begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -cb+ad \end{pmatrix} \\
&= \frac{1}{ad-cb}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-cb \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I
\end{aligned}$$

8. The inverse won't exist if there is linear dependence across rows or columns. e.g.,

$$A = \begin{pmatrix} a & 2a \\ c & 2c \end{pmatrix}$$
$$|A| = 2ac - 2ac = 0$$
$$\frac{1}{|A|} \text{ undefined}$$

9. Why we need matrix algebra. consider the regression

$$y_t = \alpha + \beta x_t + \epsilon_t$$

We represent the regression this way. Stack all the observations.

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_T \end{pmatrix}}_{X}\underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{b} + \underbrace{\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_T \end{pmatrix}}_{\epsilon}$$
$$Y = Xb + \epsilon$$

Let's multiply the last equation by X'

$$E(X'Y) = (X'X)\,b + \underbrace{E(X'\epsilon)}_{0}$$
$$b = \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{2\times 1} = \underbrace{(X'X)^{-1}}_{2\times 2}\underbrace{E(X'Y)}_{2\times 1}$$

$$
\begin{aligned}
y_t &= \alpha + \beta_1 x_t + \beta_2 z_t + \epsilon_t \\
X &= \begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ \vdots & \vdots & \vdots \\ 1 & x_T & z_T \end{pmatrix} \\
b &= \begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{pmatrix} \\
Y &= Xb + \epsilon
\end{aligned}
$$