AR(1 to 3) MA(1 to 3)

AR(3)
We were studying the GARCH(1,1) model

$$\begin{aligned} r_t &= c + bx_t + \epsilon_t \\ \epsilon_t &\sim N\left(0, \sigma_t^2\right) \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned}$$

Parsimonious (lightly parameterized). Usually GARCH(1,1) is sufficient. But, there's no necessary reason to stop here. For example, GARCH(2,1) is

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \beta \sigma_{t-1}^2$$

or a GARCH(1,2)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2$$

Suppose you want to see if returns vary systematically with volatility.

$$\begin{aligned} r_t &= a + b\sigma_t^2 + \epsilon_t \\ \epsilon_t &\sim N\left(0, \sigma_t^2\right) \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \\ \sigma_{t+1}^2 &= \alpha_0 + \alpha_1 \epsilon_t^2 + \beta \sigma_t^2 \end{aligned}$$

$$\begin{aligned} r_{t+1} &= a + b\sigma_{t+1}^2 + \epsilon_{t+1} \\ E_t\left(r_{t+1}\right) &= a + bE_t\left(\sigma_{t+1}^2\right) \\ &= a + b\left(\alpha_0 + \alpha_1 \epsilon_t^2 + \beta \sigma_t^2\right) \end{aligned}$$

This is a GARCH in the mean model (GARCH-M).

$$\begin{aligned} r_t &= a + b\sigma_t + \epsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned}$$

Review: Impulse response is an experiment where we shock the error term one time (then shut it down forever) and see how the process (time-series) responds. Say you have an MA(1) model

$$y_t = \mu + \epsilon_t + \theta \epsilon_{t-1}$$

Impulse response experiment: $,,,.\epsilon_{-3}, \epsilon_{-2}, \epsilon_{-1} = 0, \epsilon_0 = 1, \epsilon_1 = \epsilon_2 = ... = 0$

$$\begin{aligned} y_{-1} &= \mu + \epsilon_{-1} + \theta\epsilon_{-2} = \mu \\ y_0 &= \mu + \epsilon_0 + \theta\epsilon_{-1} = \mu + 1 \\ y_1 &= \mu + \epsilon_1 + \theta\epsilon_0 = \mu + \theta \\ y_2 &= \mu + \epsilon_2 + \theta\epsilon_1 = \mu \end{aligned}$$

$$\begin{aligned} y_t &= \theta_1\epsilon_t + \theta_2\epsilon_{t-1} + \theta_3\epsilon_{t-2} \\ \theta_1 + \theta_2 + \theta_3 = 1.2 \\ \epsilon_t iid(0, \sigma^2) \end{aligned}$$

y_t has a unit root
y_t is nonstationary
y_t is stationary
y_t is iid.

$$\begin{aligned} y_t &= \theta_1 y_{t-1} + \epsilon_t \\ \theta_1 &= 1.2 \end{aligned}$$

$$\begin{aligned} y_t &= \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} \\ \epsilon_t iid(0, \sigma^2) \\ \theta_1 + \theta_2 = 0.8 \\ E_t\left(y_{t+3}\right) &= E_t\left(\epsilon_{t+3} + \theta_1\epsilon_{t+2} + \theta_2\epsilon_{t+1}\right) = 0 \end{aligned}$$