Let A be (1×2) and B be (2×1). A is a row vector and B is a column vector. $C = AB$ Multiplication is to do element by element multiplication, then sum the result.

$A = \begin{pmatrix} a_{11} & a_{12} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$,

$$\begin{aligned} C &= AB = \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \\ &= (a_{11}b_{11} + a_{12}b_{21}) = C, \text{(a scalar)}. \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \\ &= \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} \\ b_{21}a_{11} & b_{21}a_{12} \end{pmatrix} = D \ \text{(a matrix)} \end{aligned}$$

Next, let's do it with actual matrices: Let

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

$C = AB$, is formed by $c_{ij} = \sum a_{ij}b_{ji}$. The i, j element of C is formed from multiplying row i of A and column j of B.

$$\begin{aligned} C &= AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ b_{11}a_{31} + b_{21}a_{32} & a_{31}b_{12} + a_{32}b_{22} \end{pmatrix} \end{aligned}$$

note: Even if A and B are both square matrices, the order matters. $AB \neq BA$.

- **Determinant of a (2×2) matrix.** Subtract the product of the off-diagonal elements from the product of the diagonal elements.

  Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $|A| = \text{det}(A) = ad - bc$.

Note: You can only get a determinant from **square** matrices. Calculating the determinant by hand from anything bigger than a (2×2) is beyond the scope of this class. But that's okay because we'll be doing it by computer.

- **Matrix Inverse**. The inverse can only be computed for **square** matrices. It is the matrix when multiplied by itself gives the identity matrix. If $A^{-1}A = AA^{-1} = I$, then A^{-1} is the inverse of A. To get the inverse of a (2×2) matrix A (defined above), switch positions of the diagonal elements, multiply the off diagonal elements by -1, then divide everything by the determinant of A.

  $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Let's check:

  $$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{ad}{ad-bc} - \frac{bc}{ad-bc} & 0 \\ 0 & \frac{ad}{ad-bc} - \frac{bc}{ad-bc} \end{pmatrix} = I$$

  Again, computing the inverse of anything bigger than a (2×2) matrix by hand is beyond the scope of this class. We just ask the computer to do it.

  Sometimes the inverse doesn't exist. This happens if there is a (linear) dependence across rows or columns.If

  $$A = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}$$

  then

  $$|A| = 2ab - 2ab = 0$$

  and $1/|A|$ doesn't exist.

8.3 Why Matrix Algebra?

Well, one thing is we can write regression in matrix form. Begin with

$$y_t = \alpha + \beta x_t + \epsilon_t$$

stack the dependent variable observations in a column vector and independent variables Independent variables: constant (a vector of 1s) and x_t

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_T \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{b} + \underbrace{\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_T \end{pmatrix}}_{\epsilon}$$

$$y = Xb + \epsilon$$

Multiply through by X'

$$\begin{aligned} X'y &= X'Xb + X'\epsilon \\ X'Xb &= X'(y - \epsilon) \\ b &= \left(X'X\right)^{-1} X'y - \left(X'X\right)^{-1} X'\epsilon \end{aligned}$$

Least squares forces the residuals $\hat{\epsilon}$ to be uncorrelated with the regressors. $X'\hat{\epsilon} = 0$. Hence, in matrix form, the least squares formula is

$$\hat{b} = \left(X'X\right)^{-1} X'y$$