10 Time-Varying Volatility

Text: Chapter 9

10.1 Motivation

Financial returns are not normally distributed. They exhibit

1. Leptokurtosis (fat tails relative to the normal)
2. Volatility clusters
3. The **unconditional** distribution of short-horizon returns aren't normal. But their **conditional** distributions could be normal.

Why we might care

1. Suppose we want to estimate the value of market risk. (Sharpe ratios).

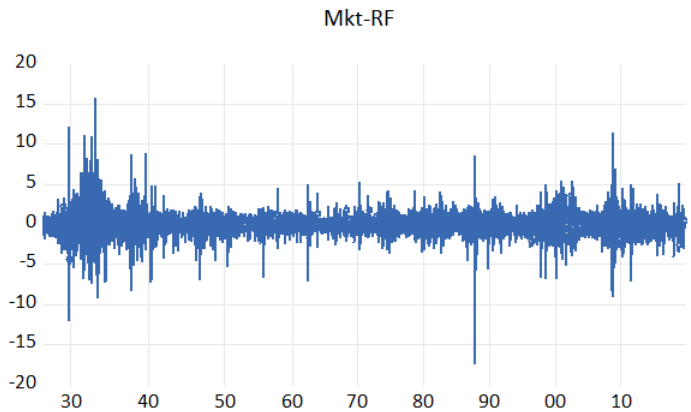
$$\text{Sharpe} = \frac{E\left(r_p - r_f\right)}{\sigma_p}$$

   where $E(r_p - r_f)$ is portfolio excess return and σ_p is portfolio volatility. Sharpe ratio is the average portfolio return per unit of volatility (a risk concept).

2. Volatility is a key parameter for pricing financial derivatives. All modern option pricing techniques rely on a volatility parameter for price evaluation.
3. Volatility is used for risk management assessment and in general portfolio management. Financial institutions want to know the current value of the volatility of the managed assets.
4. They also want to predict their future values. Volatility forecasting is important for institutions involved in options trading and portfolio management.
5. Volatility changes over time, which makes these pricing examples conditional on the current environment (high, low volatility). We want to model how volatility changes and what it depends on.

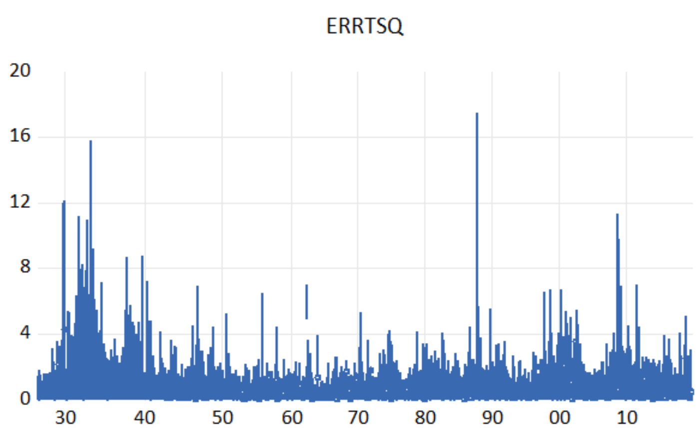
```
Dependent Variable:  ERRTSQ
Method:  Least Squares
Sample (adjusted):  7/02/1926 9/30/2019
Included observations:  24578 after adjustments
```

Figure 5: Market excess return



Let r_{mt}^e be the market excess return. Suppose we have only one observation. How would you form the sample variance? The sample standard deviation?

Figure 6: Square root of squared daily market excess returns



Does staring at this picture make you want to regress it on lags of itself?

```
Dependent Variable:  ERRTSQ
Sample (adjusted):  7/13/1926 9/30/2019
Included observations:  24571 after adjustments
```

10.2 The ARCH/GARCH class of models

- Popular way to model is with ARCH (autoregressive conditional heteroskedasticity) and GARCH (generalized ARCH).
- ARCH was invented by Robert Engle. The Nobel committee gave him the economics prize in part for this.
- GARCH was invented by Tim Bollerslev, who was Engle's student at UCSD.
- There's also,
  - EGARCH (exponential GARCH)
  - IGARCH (integrated GARCH)
  - STARCH (smooth-transition ARCH)
  - TARCH (threshold ARCH)
  - FIGARCH (fractionally integrated GARCH)
  - SWARCH (switching ARCH).
- Return on some asset

$$r_t = a + bx_t + \epsilon_t$$
$$\epsilon_t \sim N\left(0, \sigma_t^2\right)$$

  Notice t subscript on variance. σ_t^2 is the **conditional** variance of ϵ_t. Conditional on past observations of ϵ_t

$$\sigma_t^2 = E\left[\left(\epsilon_t - E_t\left(\epsilon_t\right)\right)^2 |\epsilon_{t-1}, \epsilon_{t-2}, \ldots\right] = \mathrm{Var}\left(\epsilon_t|\epsilon_{t-1}, \epsilon_{t-2}, \ldots\right)$$

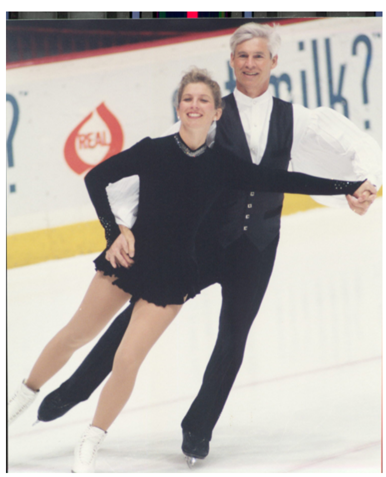
  This says the conditional variance changes over time. It is time-varying. It **moves around** over time. ARCH is a **parametric model** of the conditional variance.

- Intuition: remember how we want to think of conditional expectation as regression?
- Estimation done by **maximum likelihood**

Figure 7: Robert Engle Nobel Laureat and Heck of a Nice Guy

Nobel Prize citation: “for methods of analyzing economic time series with time- varying volatility (ARCH)”

Figure 8: Robert Engle Does Ice Dancing!



10.3 ARCH class of models

ARCH(1)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2$$

ARCH(2)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2$$

ARCH(q)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \cdots \alpha_q \epsilon_{t-q}^2$$

Test for ARCH effects

- Run the main regression

  $$r_t = \hat{a} + \hat{\beta} x_t + \hat{u}_t$$

  save the residuals $\hat{u}_t$

- Regress the squared residuals $\hat{u}_t^2$ on q lags of itself (to test for ARCH(q)).

  $$\hat{u}_t^2 = b_0 + b_1 \hat{u}_{t-1}^2 + \cdots b_q \hat{u}_{t-q}^2 + v_t$$

  where v_t is the error term. You can do an $F-$test on the coefficients.

- You can also do a Lagrange multiplier (LM) test. Get the R^2 from this regression.

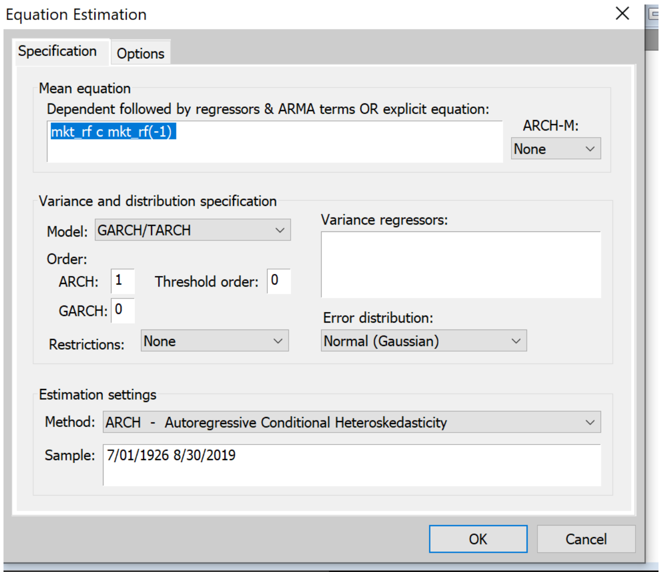
  $$TR^2 \sim \chi_q^2$$

- What does the F-test and LM test test?

  $$H_0 : (b_1 = 0) \cap (b_2 = 0) \cap \cdots (b_q = 0)$$

  The alternative is $H_A :$ NOT H_0.

- Test for and Estimate ARCH model in EViews, use `f-f_research_data_factors_daily.wf1`



Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/18/19 Time: 15:12
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 11 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.028996	0.004540	6.386850	0.0000
MKT_RF(-1)	0.246425	0.002153	114.4349	0.0000
		Variance Equation		
C	0.656324	0.003491	188.0134	0.0000
RESID(-1)^2	0.473991	0.008016	59.13110	0.0000

R-squared	-0.028109	Mean dependent var	0.029196
Adjusted R-squared	-0.028151	S.D. dependent var	1.062371
S.E. of regression	1.077221	Akaike info criterion	2.768679
Sum squared resid	28494.91	Schwarz criterion	2.770000
Log likelihood	-33992.61	Hannan-Quinn criter.	2.769107
Durbin-Watson stat	2.323311		

Equation: EQ02 Workfile: VOLATILITY::Untitled\

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/18/19 Time: 15:13
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 23 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*RESID(-2)^2 + C(6)*RESID(
-3)^2 + C(7)*RESID(-4)^2 + C(8)*RESID(-5)^2 + C(9)*RESID(-6)^2
+ C(10)*RESID(-7)^2 + C(11)*RESID(-8)^2 + C(12)*RESID(-9)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.053366	0.004332	12.31774	0.0000
MKT_RF(-1)	0.127875	0.006636	19.27105	0.0000
		Variance Equation		
C	0.160815	0.002808	57.26586	0.0000
RESID(-1)^2	0.122085	0.004743	25.74071	0.0000
RESID(-2)^2	0.113064	0.004983	22.69077	0.0000
RESID(-3)^2	0.110880	0.006016	18.43183	0.0000
RESID(-4)^2	0.120884	0.005092	23.74082	0.0000
RESID(-5)^2	0.094493	0.005543	17.04883	0.0000
RESID(-6)^2	0.095819	0.005632	17.01452	0.0000
RESID(-7)^2	0.057432	0.004902	11.71529	0.0000
RESID(-8)^2	0.091738	0.004974	18.44226	0.0000
RESID(-9)^2	0.078895	0.004899	16.10544	0.0000

R-squared	-0.000094	Mean dependent var	0.029196
Adjusted R-squared	0.000135	S.D. dependent var	1.062371

Oy! Too many parameters!

10.4 The GARCH class of models

GARCH(1,1)

$$\begin{aligned} r_t &= a + \beta x_t + \epsilon_t \\ \epsilon_t &\sim N\left(0, \sigma_t^2\right) \end{aligned}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

The (1,1) refers to number of lags of ϵ^2 and σ^2, and where

$$0 \leq \beta \leq 1$$

(class: why do we need this?)

GARCH(1,1) is constrained infinite ordered ARCH. Observe,

$$\begin{aligned} \sigma_{t-1}^2 &= \alpha_0 + \alpha_1 \epsilon_{t-2}^2 + \beta \sigma_{t-2}^2 \\ \sigma_{t-2}^2 &= \alpha_0 + \alpha_1 \epsilon_{t-3}^2 + \beta \sigma_{t-3}^2 \end{aligned}$$

substitute this into previous

$$\begin{aligned} \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \underbrace{\left(\alpha_0 + \alpha_1 \epsilon_{t-2}^2 + \beta \sigma_{t-2}^2\right)}_{\sigma_{t-1}^2} \\ &= \alpha_0 \left(1 + \beta\right) + \alpha_1 \epsilon_{t-1}^2 + \alpha_1 \beta \epsilon_{t-2}^2 + \beta^2 \sigma_{t-2}^2 \\ &= \alpha_0 \left(1 + \beta\right) + \alpha_1 \left(\epsilon_{t-1}^2 + \beta \epsilon_{t-2}^2\right) + \beta^2 \left(\alpha_0 + \alpha_1 \epsilon_{t-3}^2 + \beta \sigma_{t-3}^2\right) \\ &= \alpha_0 \left(1 + \beta + \beta^2\right) + \alpha_1 \left(\epsilon_{t-1}^2 + \beta \epsilon_{t-2}^2 + \beta^2 \epsilon_{t-3}^2\right) + \beta^3 \sigma_{t-3}^2 \end{aligned}$$

Keep going. $\beta^k = 0$ as $k \to \infty$.

$$\sigma_t^2 = \frac{\alpha_0}{1 - \beta} + \frac{\alpha_1}{\beta} \sum_{j=1}^{\infty} \beta^j \epsilon_{t-j}^2$$

GARCH(2,1)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \beta \sigma_{t-1}^2$$

GARCH(1,2)

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-1}^2$$

Usually, GARCH(1,1) does the job.

Equation: EQ02 Workfile: VOLATILITY::Untitled\

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/18/19 Time: 15:25
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 26 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.047571	0.004436	10.72388	0.0000
MKT_RF(-1)	0.126446	0.006698	18.87856	0.0000
		Variance Equation		
C	0.012641	0.000497	25.43512	0.0000
RESID(-1)^2	0.103414	0.002076	49.82175	0.0000
GARCH(-1)	0.886041	0.002373	373.4385	0.0000

R-squared	0.000338	Mean dependent var	0.029196
Adjusted R-squared	0.000298	S.D. dependent var	1.062371
S.E. of regression	1.062213	Akaike info criterion	2.428462
Sum squared resid	27706.47	Schwarz criterion	2.430113
Log likelihood	-29814.08	Hannan-Quinn criter.	2.428997
Durbin-Watson stat	2.107261		

`Recover the GARCH series, makes the conditional variance series`

`eqno.makegarch cvar where cvar is the series name`

`series csd = cvar`^{0}.5

10.5 ARCH-M, GARCH-M (in the mean)

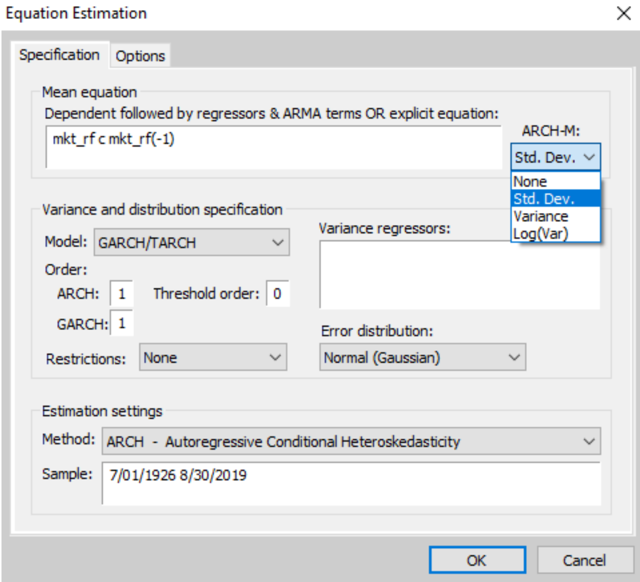
You might be interested in whether higher volatility associated with higher or lower returns? Here is a GARCH-M example.

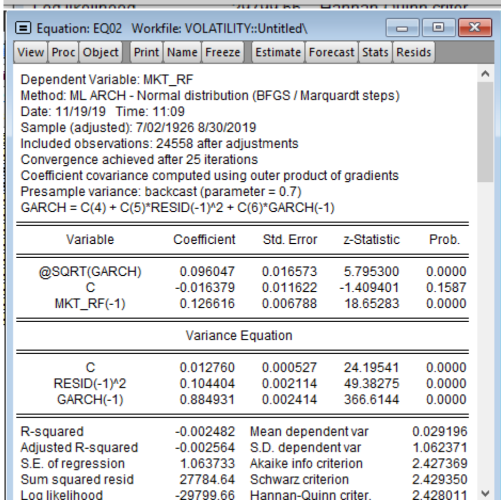
$$r_t^e = a + b\sigma_t + \epsilon_t$$

$$\epsilon_t \sim N\left(0, \sigma_t^2\right)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

- Use volatility as ‘regressor’ to preserve units.
- $b > 0$, high volatility, r^e expected to be large. $b < 0$, high volatility, r^e expected to be small.
- Estimation is by maximum likelihood.
- To implement, choose the option in EViews





Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/19/19 Time: 11:09
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 25 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(4) + C(5)*RESID(-1)^2 + C(6)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
@SQRT(GARCH)	0.096047	0.016573	5.795300	0.0000
C	-0.016379	0.011622	-1.409401	0.1587
MKT_RF(-1)	0.126616	0.006788	18.65283	0.0000
Variance Equation				
C	0.012760	0.000527	24.19541	0.0000
RESID(-1)^2	0.104404	0.002114	49.38275	0.0000
GARCH(-1)	0.884931	0.002414	366.6144	0.0000

R-squared	-0.002482	Mean dependent var	0.029196
Adjusted R-squared	-0.002564	S.D. dependent var	1.062371
S.E. of regression	1.063733	Akaike info criterion	2.427369
Sum squared resid	27784.64	Schwarz criterion	2.429350
Log likelihood	-29799.66	Hannan-Quinn criter.	2.428011