12 The Fama-MacBeth Method

Text: pp. 588-590.

12.1 Background

This is an alternative way to estimate and test factor models of asset pricing.

- The time-series method covered in the previous section can only be used if the factor(s) are excess returns. For time-series approach, factor must be a return because you estimate the factor risk premium by the mean return.

  $$\hat{\lambda} = \frac{1}{T}\sum_{t=1}^{T} f_t$$

  (If f_t isn't a return–say it's consumption growth, then mean consumption growth isn't the factor risk premium!)

- Fama-MacBeth method can be used when the factor(s) are excess returns or when they are not excess returns.

- In the single-factor model, the theory is about the cross section. If the theory is correct, $\alpha_i = 0$, λ is statistically significant.

  $$\bar{r}_i^e = \lambda\beta_i + \alpha_i$$

- We should be able to run a cross-sectional regression of mean excess returns on their betas and a constant.

  – Test if Jensen's alpha is zero.

  – Test to see if λ is significant.

- Estimation is straightforward. The challenge is computing standard errors on the lambdas (to form t-ratios)

  – Normally, the distribution theory we use to get standard errors (t-ratios) assumes the independent variables are data. Here, the regressors (the betas) are estimated. We call these things generated regressors.

  – Fama and MacBeth came up with a strategy for getting the standard errors on the lambdas.

12.2 Apply the Method to the CAPM

- We have n assets over T time periods with excess returns $r^e_{t,i}$. The CAPM is a single-factor model. $f_t = r^e_{t,m}$ is the excess return on the market portfolio.

- Stage 1: For each asset i, estimate its beta with a time-series regression.

$$\begin{aligned}
r^e_{t,1} &= \alpha_1 + \beta_1 f_t + \epsilon_{t,1} \longleftarrow \text{Asset 1} \\
r^e_{t,2} &= \alpha_2 + \beta_2 f_t + \epsilon_{t,2} \longleftarrow \text{Asset 2} \\
&\vdots \\
r^e_{t,n} &= \alpha_n + \beta_n f_t + \epsilon_{t,n} \longleftarrow \text{Asset n}
\end{aligned}$$

  This gives us n estimated betas $\hat{\beta}_i$. Sometimes people call the β's the 'factor loadings.'

- Stage 2 : Run a single cross-sectional regression of the (time-series) average excess returns on betas. Here we are regressing average excess returns on the betas. The betas are the independent variable.

  - Suppose you run the cross-sectional regression with a constant. It will look like this.

$$\bar{r}^e_i = \gamma + \lambda\beta_i + \alpha_i$$

    The estimated slope coefficient is λ, is called the price of risk, and the α_i are the regression residuals. The constant is γ and the α_i will be, by construction, zero on average. The theory says $\gamma = 0$, which we can test.

- We want to get the t-ratio on λ to see if the price of risk is significant. The problem here is the betas not data. They are estimated. We call them 'generated regressors.' Cannot use the t-ratio produced by the regression package. Fama-MacBeth does the following

- Let's say we ran the cross-sectional regression with a constant. For each time period, $t = 1, ...T$, run a cross-sectional regressions of returns (at time t) on the $\beta_i, i = 1, ..., n$.

$$\begin{aligned}
r^e_{1,i} &= \gamma_1 + \lambda_1\beta_i + \alpha_{1,i} \longleftarrow \text{Time period 1} \\
r^e_{2,i} &= \gamma_2 + \lambda_2\beta_i + \alpha_{2,i} \longleftarrow \text{Time period 2} \\
&\vdots \\
r^e_{T,i} &= \gamma_T + \lambda_T\beta_i + \alpha_{T,i} \longleftarrow \text{Time period T}
\end{aligned}$$

  - The regressors in every regression **are the same collection of** β_i. Only the dependent variable changes from one regression to the other.

– In each regression, γ_t is the regression constant. $\alpha_{t,i}$ is the error term. The λ_t are slope coefficients.

– We have a time-series of $\lambda's$. $(\lambda_1, ..., \lambda_T)$. Fama-MacBeth assumes the λ_t are i.i.d. The assumption is often justified by noting that returns are (almost) uncorrelated over time. So, to test if λ is significant, run a time-series regression of λ_t on a constant

$$\lambda_t = c + u_t$$

The t-ratio on the constant is the t-ratio for $\hat{\lambda}$.

– Run the time-series regression

$$\gamma_t = c + u_t$$

to test if constant is zero.

Suppose you ran the cross-sectional regression (above) without a constant (no $\gamma's$). Note: This is the only time you are allowed to run a regression without a constant. Then in addition to doing Fama-MacBeth above to test if λ is significant, we want to test if the α_i are jointly 0. These α_i are regression residuals and without the constant, there is nothing to constrain them to be 0. This is what we do.

- If you run the cross-sectional regression without constant, it will look like this.

$$\bar{r}_i^e = \lambda\beta_i + \alpha_i$$

This is the only time you are permitted to run a regression without constant. For each time period, $t = 1, ...T$, run a cross-sectional regressions of returns (at time t) on the $\beta_i, i = 1, ..., n$, without constant

$$\begin{aligned}
r_{1,i}^e &= \lambda_1\beta_i + \alpha_{1,i} \longleftarrow \text{Time period 1} \\
r_{2,i}^e &= \lambda_2\beta_i + \alpha_{2,i} \longleftarrow \text{Time period 2} \\
&\vdots \\
r_{T,i}^e &= \lambda_T\beta_i + \alpha_{T,i} \longleftarrow \text{Time period T}
\end{aligned}$$

Do Fama-MacBeth on the λ_t as above.

- To **testing** $\alpha = 0$**,** from the time-series regressions, compute the average α for each asset or portfolio i.

$$\bar{\alpha}_i = \frac{1}{T}\sum_{t=1}^{T}\hat{\alpha}_{t,i} \tag{40}$$

There will be n of these.

- Get covariance matrix of the residuals and call it Σ_α.It will be an $n \times n$ matrix.

$$\Sigma_\alpha = \begin{pmatrix} Var(\hat{\alpha}_1) & Cov\left(\hat{\alpha}_1, \hat{\alpha}_2\right) & \cdots & Cov\left(\hat{\alpha}_1, \hat{\alpha}_N\right) \\ \\ \vdots & & & \\ Cov\left(\hat{\alpha}_N, \hat{\alpha}_1\right) & Cov\left(\hat{\alpha}_N, \hat{\alpha}_2\right) & \cdots & Var\left(\hat{\alpha}_N\right) \end{pmatrix}$$

where

$$Var(\hat{\alpha}_i) = \frac{1}{T}\sum_{t=1}^{T}(\hat{\alpha}_{t,i} - \bar{\hat{\alpha}}_i)^2$$

$$Cov(\hat{\alpha}_i, \hat{\alpha}_j) = \frac{1}{T}\sum_{t=1}^{T}(\hat{\alpha}_{t,i} - \bar{\hat{\alpha}}_i)(\hat{\alpha}_{t,j} - \bar{\hat{\alpha}}_j)$$

- Form the test statistic

$$T\left(\bar{\alpha}_1, \bar{\alpha}_{2,} \cdots \bar{\alpha}_n\right)\Sigma_\alpha^{-1}\begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{pmatrix} \sim \chi^2_{n-1}$$

where the $\bar{\alpha}_i$ are the time-series means from (40).