6. The AR(1) also has an MA(∞) representation.

$$
\begin{aligned}
y_t &= a + \rho\underbrace{(a + \rho y_{t-2} + \epsilon_{t-1})}_{y_{t-1}} + \epsilon_t \\
&= a + \rho a + \rho^2\underbrace{(a + \rho y_{t-3} + \epsilon_{t-2})}_{y_{t-2}} + \rho\epsilon_{t-1} + \epsilon_t \\
&= a + \rho a + \rho^2 a + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^2 y_{t-3} \\
&\vdots \\
&= \underbrace{a\left(1 + \rho + \rho^2 + \rho^3 + \cdots\right)}_{a/(1-\rho)} + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots \\
&= \frac{a}{1-\rho} + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots
\end{aligned}
$$

and, the mean $\mathrm{E}(y_t)$ is,

$$
\begin{aligned}
\mathrm{E}\left(y_t\right) &= \mathrm{E}\left(\frac{a}{1-\rho} + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots\right) \\
&= \left(\frac{a}{1-\rho} + \mathrm{E}\epsilon_t + \rho\mathrm{E}\epsilon_{t-1} + \rho^2\mathrm{E}\epsilon_{t-2} + \rho^3\mathrm{E}\epsilon_{t-3} + \cdots\right) \\
&= \frac{a}{1-\rho}
\end{aligned}
$$

We can also do the algebra to get the variance and autocovariances, but let's not. Autoregressive models have infinite memory.

7. **Forecasting**

$$E_t(\tilde{y}_{t+1}) = \rho\tilde{y}_t$$

$$E_t(\tilde{y}_{t+2}) = \rho E_t(\tilde{y}_{t+1}) = \rho^2\tilde{y}_t$$

Hence,

$$E_t(\tilde{y}_{t+k}) = \rho^k\tilde{y}_t$$

5.4 AR(2) model

1. Let $\epsilon_t \overset{iid}{\sim} \left(0, \sigma_\epsilon^2\right)$. The second-order autoregressive model (AR(2)) is

$$
\begin{aligned}
y_t &= a + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t \\
\mu_y &= a + \rho_1\mu_y + \rho_2\mu_y
\end{aligned}
$$

This will be stationary if $|\rho_1 + \rho_2| < 1$. Second line comes from take expectation

$$a = \frac{\mu_y}{1 - \rho_1 - \rho_2}$$

2. Computing variance and autocovariances is too complicated to do by hand in this class, so we won't do it. It involves take variance and first-order covariance. I'll just show you the answer:

$$\sigma_y^2 = \rho_1^2 \sigma_y^2 + \rho_2^2 \sigma_y^2 + 2\rho_1 \rho_2 \gamma_1 + \sigma_\epsilon^2$$
$$\gamma_1 = \rho_1 \sigma_y^2 + \rho_2 \gamma_1 \rightarrow \gamma_1 = \frac{\rho_1 \sigma_y^2}{1 - \rho_2}$$

To finish off, need to solve these two equations for σ_y^2 and γ_1. Instead, we will rely on the computer.

3. Impulse response. Let $\tilde{y}_0 = \tilde{y}_1 = 0$, $\epsilon_1 = 1$.

$$\tilde{y}_1 = 1$$
$$\tilde{y}_2 = \rho_1$$
$$\tilde{y}_3 = \rho_1^2 + \rho_2$$
$$\tilde{y}_4 = \rho_1 \left(\rho_1^2 + \rho_2\right) + \rho_2 \rho_1$$

etc. It is possible to get **cyclical** impulse responses. $\rho_1 = 0.8, \rho_2 = -0.8$.

4. **Forecasting** Form the forecasts and input recursively.

$$E_t\left(\tilde{y}_{t+1}\right) = \rho_1 \tilde{y}_t + \rho_2 \tilde{y}_{t-1}$$
$$E_t\left(\tilde{y}_{t+2}\right) = \rho_1 \left(E_t\left(\tilde{y}_{t+1}\right)\right) + \rho_2 \tilde{y}_t$$
$$E_t\left(\tilde{y}_{t+3}\right) = \rho_1 \left(E_t\left(\tilde{y}_{t+2}\right)\right) + \rho_2 E_t\left(\tilde{y}_{t+1}\right)$$

5. The AR(2) is stationary if

$$|\rho_1 + \rho_2| < 1$$

5.5 Extensions

1. No need to stop at AR(2). Can add more and more lags.

2. In MA model, can add more and more lagged shocks.

3. Difference between MA and AR.

AR is dependence across time of observations.

MA is dependence across time of shocks.

4. MA memory is finite

5. AR memory is infinite (but diminishes exponentially)

6. Can combine MA and AR. Here's ARMA(1,1)

$$y_t = a + \rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1}$$

7. When the underlying data are non-stationary (has a trend), such as real GDP. We have to take logs of real GDP then take **first differences** to induce stationarity. In this case,

$$y_t = \ln(GDP_t) - \ln(GDP_{t-1})$$

5.6 How to select model?

1. Ensure variable is stationary

2. Estimate all the candidate models, keep variables with significant t-ratios. Not good

3. Estimate candidate models, compare their forecasting accuracy. Doesn't work well either.

   Underfitting: Omitted variables, produces bad forecasts

   Overfitting: Additional sampling variability in estimated coefficients produces bad forecasts

4. Can use **Information Criteria** to select the model.

6 Information Criteria for Model Selection: AIC, BIC, HPIC

Text: pp. 271-272.

- Ensure variable is stationary
- Estimate all the candidate models, keep variables with significant t-ratios. Not good
- Estimate candidate models, compare their forecasting accuracy. Doesn't work well either.
- Tradeoffs:
  - Underfitting: Omitted variables, produces bad forecasts
  - Overfitting: Additional sampling variability produces bad forecasts
  - Generally, lightly parameterized models produce better forecasts than heavily parameterized ones.
- **Information Criteria.** Let the data tell us how to specify the model. We use something called information criteria (IC).

6.1 A Little Background on Information Criteria

- Remember the log-likelihood function from eq.(12)?

$$\frac{LL}{T} = -\ln\left(\hat{\sigma}_{\epsilon}^{2}\right)^{\frac{1}{2}}$$

- Suppose we want to choose among ARMA(p,q), for $p = 0, ..., 5$, $q = 0, ..., 5$.
- You might be tempted to use the highest likelihood across models for selection. Can't do that because the maximized log likelihood (usually) continues to increase as you add parameters. Is like how R^2 keeps increasing when you add variables in regression.
- **Solution**: attach **penalty** for adding parameters. Different information criteria have different penalties.
- Maximizing the log likelihood, is to minimize $\ln(\sigma_{\epsilon}^{2})$. Information criteria: AIC, BIC, HPIC. The model that gives you the minimum IC is the one you want..
- First to do so was Akaike. False modesty to say A comes first in alphabet.

6.2 AIC, BIC, HPIC

1. Let k be number of parameters (count up the ρ_j, θ_j in ARMA model)

$$AIC = \ln\left(\hat{\sigma}_\epsilon^2\right) + \frac{2k}{T}$$
$$BIC = \ln\left(\hat{\sigma}_\epsilon^2\right) + \frac{k}{T}\ln(T)$$
$$HPIC = \ln\left(\hat{\sigma}_\epsilon^2\right) + \frac{2k}{T}\ln(\ln(T))$$

2. In subsequent studies, AIC usualy chooses too many parameters, BIC, too few, HPIC is sort of just right.

3. How to do this on Eviews?

4. Automatic ARMA (ARIMA) model selection.

   (a) Open the series of interest

   (b) Click Proc, Automatic ARIMA Forecasting

   (c) In Options tab, choose Model Selection and the Information Criterion you want to use.

6.3 The Random Walk model

If $\rho \to 1$, the AR(1) model becomes the **driftless random walk** model,

$$y_t = y_{t-1} + \sigma\epsilon_t \tag{17}$$

Terminology: There are some technical specifications about autoregressive models whereby ρ is called the **root** of the process. So when $\rho = 1$, people refer to this as a **unit root** process.

Properties

1. Backward substitution of
$$y_t = y_{t-1} + \epsilon_t$$
gives
$$y_t = (\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \cdots)$$

2. Shocks are permanent

3. Mean is zero

4. Variance doesn't exist. Infinite variance. What is the meaning of this?

5. The time series $\{y_t\}$ is non stationary. It is a unit-root process, and is unit-root nonstationary

6. The minimum mean square error (best) forecast of any future value is today's value.

7. The minimum mean square error (best) forecast of any future change is no change!

8. Asset prices (exchange rates) approximately follow a driftless random walk. Returns are nearly iid. Stock prices approximately follow a **random walk with drift**.

$$y_t = \mu + y_{t-1} + \sigma\epsilon_t \tag{18}$$

$$E_t\left(y_{t+1} - y_t\right) = \mu \tag{19}$$

$$E_t y_{t+k} = k\mu + y_t \tag{20}$$

Stochastic trend: By repeated backward substitution,

$$y_t = t\mu + y_0 + \sigma\left(\epsilon_t + \epsilon_{t-1} + \cdots + \epsilon_1\right) \tag{21}$$

- Here y_0 plays role of the constant, but it is not constant. It's a random variable. Hence, stochastic trend.
- Hence, cannot regress the random walk on a time trend to induce stationarity.
- Original Efficient Markets hypothesis based on this idea. Today's price "reflects all publically available information."
- **EviewsExam;les/arima_models_sims_randomwalks.prg and arima_models_sims_ar1s.prg**

Random walk with drift