5.3 Autoregressive models

These are models of more durable or persistent dependence over time. Pure AR models can be estimated by least squares (actually, least squares and maximum likelihood are the same for AR models). Combined AR and MA models (ARMA) need to be estimated by maximum likelihood.

5.3.1 Specification of the AR(1) model

$$
\begin{aligned}
y_t &= a + \rho y_{t-1} + \epsilon_t && (13)\\
&= \underbrace{\mu_y (1-\rho)}_{a} + \rho y_{t-1} + \epsilon_t && (14)\\
\epsilon_t &\overset{iid}{\sim} (0, \sigma_\epsilon^2) && (15)\\
|\rho| &< 1. && (16)
\end{aligned}
$$

1. The last two conditions together mean we are assuming that y_t is strictly stationary. Hence, take take expectations on both sides,

$$\mu_y = a + \rho\mu_y$$

$$\Rightarrow a = \mu_y(1-\rho)$$

2. Rewrite in deviations from mean form, $\tilde{y}_t \equiv y_t - \mu_y$. Constant is not interesting. Only dependence across time is interesting.

$$\tilde{y}_t = \rho\tilde{y}_{t-1} + \epsilon_t$$

3. Compute variance

$$
\begin{aligned}
y_t &= a + \rho y_{t-1} + \epsilon_t\\
\tilde{y}_t &= \rho\tilde{y}_{t-1} + \epsilon_t\\
\tilde{y}_t^2 &= (\rho\tilde{y}_{t-1} + \epsilon_t)(\rho\tilde{y}_{t-1} + \epsilon_t) = \rho^2\tilde{y}_{t-1}^2 + \epsilon_t^2 + 2\epsilon_t\rho\tilde{y}_{t-1}\\
E\left(\tilde{y}_t^2\right) &= \rho^2 E\left(\tilde{y}_{t-1}^2\right) + E\left(\epsilon_t^2\right) + 2\underbrace{E\left(\epsilon_t\rho\tilde{y}_{t-1}\right)}_{0}
\end{aligned}
$$

$$\sigma_y^2 = \rho^2\sigma_y^2 + \sigma_\epsilon^2$$

$$\Rightarrow \sigma_y^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}$$

- What happens if $\rho = 1$?
- This is why $|\rho| < 1$ is necessary for stationarity.
- Usually $0 < \rho < 1$ in economics and finance.
- Show some random walk realizations without drift. **arima_models.wf1, arima_models_sims_ran**

4. Compute autocovariances

$$\begin{aligned} \tilde{y}_t \tilde{y}_{t-1} &= \rho \tilde{y}_{t-1}^2 + \epsilon_t \tilde{y}_{t-1} \\ E\left(\tilde{y}_t \tilde{y}_{t-1}\right) &= \rho E\left(\tilde{y}_{t-1}^2\right) + \underbrace{E\left(\epsilon_t \tilde{y}_{t-1}\right)}_{0} \\ \gamma_1 &= \rho \sigma_y^2 \end{aligned}$$

First-order autocorrelation is ρ (why?)

Higher-ordered autocovariances

$$\gamma_1 = E(\tilde{y}_t \tilde{y}_{t-1}) = \rho \sigma_y^2$$

$$\gamma_2 = E(\tilde{y}_t \tilde{y}_{t-2}) = \rho \gamma_1 = \rho^2 \sigma_y^2$$

and so on. Hence,

$$Corr(y_t, y_{t-k}) = \rho^k$$

Notice how long lived is the dependence, as opposed to the MA models.

5. **Impulse response** function. Set all $\epsilon_{t-s} = 0$ for $s \neq 0$, $\epsilon_t = \sigma_\epsilon$, which we'll assume is $\sigma_\epsilon = 1$.

$$\begin{aligned} \tilde{y}_{t-1} &= 0 \\ \tilde{y}_t &= 1 \\ \tilde{y}_{t+1} &= \rho \\ \tilde{y}_{t+2} &= \rho^2 \\ \tilde{y}_{t+k} &= \rho^k \end{aligned}$$

arima_models.wf1, arima_models_sims_impulse.prg

6. The AR(1) also has an MA(∞) representation.

$$
\begin{aligned}
y_t &= a + \rho\underbrace{(a + \rho y_{t-2} + \epsilon_{t-1})}_{y_{t-1}} + \epsilon_t \\
&= a + \rho a + \rho^2\underbrace{(a + \rho y_{t-3} + \epsilon_{t-2})}_{y_{t-2}} + \rho\epsilon_{t-1} + \epsilon_t \\
&= a + \rho a + \rho^2 a + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^2 y_{t-3} \\
&\vdots \\
&= \underbrace{a\left(1 + \rho + \rho^2 + \rho^3 + \cdots\right)}_{a/(1-\rho)} + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots \\
&= \frac{a}{1-\rho} + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots
\end{aligned}
$$

and, the mean $\mathrm{E}(y_t)$ is,

$$
\begin{aligned}
\mathrm{E}\left(y_t\right) &= \mathrm{E}\left(\frac{a}{1-\rho} + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots\right) \\
&= \left(\frac{a}{1-\rho} + \mathrm{E}\epsilon_t + \rho\mathrm{E}\epsilon_{t-1} + \rho^2\mathrm{E}\epsilon_{t-2} + \rho^3\mathrm{E}\epsilon_{t-3} + \cdots\right) \\
&= \frac{a}{1-\rho}
\end{aligned}
$$

We can also do the algebra to get the variance and autocovariances, but let's not. Autoregressive models have infinite memory.

7. **Forecasting**

$$E_t(\tilde{y}_{t+1}) = \rho\tilde{y}_t$$

$$E_t(\tilde{y}_{t+2}) = \rho E_t(\tilde{y}_{t+1}) = \rho^2\tilde{y}_t$$

Hence,

$$E_t(\tilde{y}_{t+k}) = \rho^k\tilde{y}_t$$

5.4 AR(2) model

1. Let $\epsilon_t \overset{iid}{\sim} \left(0, \sigma_\epsilon^2\right)$. The second-order autoregressive model (AR(2)) is

$$
\begin{aligned}
y_t &= a + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t \\
\mu_y &= a + \rho_1\mu_y + \rho_2\mu_y
\end{aligned}
$$