Autoregressive models.

AR(1)

ϵ_t iid $\left(0,\sigma_\epsilon^2\right)$.For y_t to be stationary, $|\rho|<1$

$$y_t = a + \rho y_{t-1} + \epsilon_t$$

Find the interesting moments

$$\begin{aligned}
\underbrace{E\left(y_t\right)}_{\mu_y} &= a + \rho \underbrace{E\left(y_{t-1}\right)}_{\mu_y} + \underbrace{E\left(\epsilon_t\right)}_{0} \\
\mu_y\left(1-\rho\right) &= a
\end{aligned}$$

re-express as

$$y_t = \mu_y\left(1-\rho\right) + \rho y_{t-1} + \epsilon_t$$

Let's get the variance. It will be the same whether μ_y= 0 or not, so let's assume it's 0.

$$\begin{aligned}
y_t &= \rho y_{t-1} + \epsilon_t \\
\mathrm{Var}\left(y_t\right) &= E\left(y_t\right)^2 = E\left(\rho y_{t-1}+\epsilon_t\right)\left(\rho y_{t-1}+\epsilon_t\right) \\
&= E\left[\rho^2 y_{t-1}^2 + \epsilon_t^2 + 2\rho y_{t-1}\epsilon_t\right] \\
&= \left[\rho^2 E y_{t-1}^2 + E\epsilon_t^2 + 2\rho \underbrace{E\left(y_{t-1}\epsilon_t\right)}_{0}\right] \\
\mathrm{Var}\left(y_t\right) &= E\left(y_t\right)^2 = \sigma_y^2 = \rho^2\sigma_y^2 + \sigma_\epsilon^2
\end{aligned}$$

and finally,

$$\begin{aligned}
\sigma_y^2\left(1-\rho^2\right) &= \sigma_\epsilon^2 \\
\sigma_y^2 &= \frac{\sigma_\epsilon^2}{1-\rho^2}
\end{aligned}$$

What happens if $\rho = 1$? Variance goes to infinity (or is undefined)

Compute autocorrelations. First compute autocovariance, then divide by the variance.

Remember, we are assuming $\mu_y = 0$

$$\begin{aligned}
\gamma_1 &= E\left(y_t y_{t-1}\right) = E\left(\rho y_{t-1}+\epsilon_t\right)\left(y_{t-1}\right) \\
&= E\left(\rho y_{t-1}^2 + \epsilon_t y_{t-1}\right) \\
&= \left(\rho E y_{t-1}^2 + E\epsilon_t y_{t-1}\right) \\
&= \rho\sigma_y^2
\end{aligned}$$

And the first-order autocorrelation is

$$\operatorname{corr}(y_t, y_{t-1}) = \frac{\gamma_1}{\sigma_y \sigma_y} = \rho$$

The second-order autocorrelation:

$$\begin{aligned}
\gamma_2 &= E(y_t y_{t-2}) = E(\rho y_{t-1} + \epsilon_t) y_{t-2} \\
&= \left(\rho \underbrace{E(y_{t-1} y_{t-2})}_{\gamma_1} + \underbrace{E(\epsilon_t y_{t-2})}_{0} \right) \\
&= \rho\rho\sigma_y^2 = \rho^2 \sigma_y^2
\end{aligned}$$

$$\operatorname{corr}(y_t, y_{t-2}) = \frac{\gamma_2}{\sigma_y \sigma_y} = \rho^2$$

The $k - th$ order autocorrelation of the AR(1) model is ρ^k.

Autoregressive models have moving average representations.

$$\begin{aligned}
y_t &= \rho y_{t-1} + \epsilon_t \\
y_{t-1} &= \rho y_{t-2} + \epsilon_{t-1} \\
y_{t-2} &= \rho y_{t-3} + \epsilon_{t-2}
\end{aligned}$$

Substitute the second and third equations into the first.

$$\begin{aligned}
y_t &= \rho(\rho y_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
&= \rho^2 (y_{t-2}) + \rho\epsilon_{t-1} + \epsilon_t \\
&= \rho^2 (\rho y_{t-3} + \epsilon_{t-2}) + \rho\epsilon_{t-1} + \epsilon_t \\
&= \rho^3 y_{t-3} + \rho^2 \epsilon_{t-2} + \rho\epsilon_{t-1} + \epsilon_t \\
&\vdots \\
&= \rho^t \epsilon_0 + \rho^{t-1}\epsilon_1 + \cdots \rho^2 \epsilon_{t-2} + \rho\epsilon_{t-1} + \epsilon_t
\end{aligned}$$

This is useful for tracing out the impulse response function. Shock ϵ_t once, then shut it down and see how y responds. From the MA representation,

$$\begin{aligned}
y_1 &= \epsilon_1 = 1 \\
y_2 &= \underbrace{\epsilon_2}_{0} + \rho \underbrace{\epsilon_1}_{1} = \rho \\
y_3 &= \epsilon_3 + \rho\epsilon_2 + \rho^2 \epsilon_1 = \rho^2 \\
&\vdots \\
y_k &= \rho^{k-1}
\end{aligned}$$

Forecasting formula. Given what we know at t (I_t is the set of variables we know at t), what's the forecast of y_{t+1}?

$$y_{t+1} = \rho y_t + \epsilon_{t+1}$$

$$
\begin{aligned}
E\left(y_{t+1}|I_t\right) &= E\left(\rho y_t + \epsilon_{t+1}\right) = \rho y_t \\
E\left(y_{t+2}|I_t\right) &= \rho^2 y_t \\
E\left(y_{t+k}|I_t\right) &= \rho^k y_t \to 0 = \mu_y \text{ as } k \to \infty
\end{aligned}
$$

How can we estimate the AR(1)? Regression works. Regress y_t on y_{t-1}.

In Eviews, $y = c(1) + c(2)y(-1)$.

We could continue with an AR(2) model, but we'll do that later. Right now, consider an ARMA(1,1) model. This will help with the problem set, believe me.

$$
y_t = \rho y_{t-1} + \epsilon_t + \theta \epsilon_{t-1}
$$

Find the mean

$$
\begin{aligned}
E\left(y_t\right) &= E\left(\rho y_{t-1} + \epsilon_t + \theta \epsilon_{t-1}\right) \\
\mu_y &= \rho \mu_y = 0 \\
Var\left(y_t\right) &= E\left(y_t^2\right) \\
&= E\left(\rho y_{t-1} + \epsilon_t + \theta \epsilon_{t-1}\right)\left(\rho y_{t-1} + \epsilon_t + \theta \epsilon_{t-1}\right) \\
&= E(\rho^2 y_{t-1} + \epsilon_t^2 + \theta^2 \epsilon_{t-1}^2 + blahblahblah)
\end{aligned}
$$

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