Problem Set 2 Solution
Econ 40357 Financial Econometrics
University of Notre Dame
Professor Nelson Mark

Thursday 15 September 2022

Due 12:30 Thursday 22 September, electronic submission through Canvas. 100 points total on this problem set. Please read the problem set carefully, especially about **what you should report** as your answers. As before, submit **a single pdf** document. The first part of the document will have the group answers. The second part will be an appendix with individual Eviews output. The first page of the solutions needs to have your group number (or group name) with a list of your group members.

1. Consider the ARMA(1,2) model,

$$y_t = \rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$$

where ϵ_t iid $\left(0, \sigma_\epsilon^2\right)$.

(a) (15 points) Find the mean $E\left(y_t\right)$.
$\mu_y = E\left(y_t\right) = E\left(\rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}\right) = \rho E\left(y_t\right) \to \mu_y = 0$ if $|\rho| < 1$. If $\rho = 1$, then μ_y is undefined.

(b) (15 points) Find the variance $Var\left(y_t\right)$.First, we will need these covariances:

$$\sigma_{y_t,\epsilon_t} = \sigma_\epsilon^2$$

$$\sigma_{y_t,\epsilon_{t-1}} = \left(\rho + \theta_1\right)\sigma_\epsilon^2$$
$$\sigma_{y_t,\epsilon_{t-2}} = \left(\rho^2 + \rho\theta_1 + \theta_2\right)\sigma_\epsilon^2$$

I'm going to call $Var\left(y_t\right) = \sigma_y^2 = E\left(y_t^2\right)$

$$\sigma_y^2 = E\left(\rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}\right)\left(\rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}\right)$$

$$\begin{aligned}
&= \underbrace{E\left[\rho y_{t-1}\left(\rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}\right)\right]}_{\rho^2\sigma_y^2 + \rho\theta_1\sigma_{y_t,\epsilon_t} + \rho\theta_2\sigma_{y_t,\epsilon_{t-1}}} + \underbrace{E\left[\epsilon_t\left(\rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}\right)\right]}_{\sigma_\epsilon^2} \\
&+ \underbrace{E\left[\theta_1\epsilon_{t-1}\left(\rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}\right)\right]}_{\theta_1\rho\sigma_{y_t,\epsilon_t} + \theta_1^2\sigma_\epsilon^2} + \underbrace{E\left[\theta_2\epsilon_{t-2}\left(\rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}\right)\right]}_{\rho\theta_2\sigma_{y_t,\epsilon_{t-1}} + \theta_2^2\sigma_\epsilon^2}
\end{aligned}$$

Make substitutions

$$\begin{aligned}
\sigma_y^2 &= \rho^2\sigma_y^2 + \rho\theta_1\sigma_\epsilon^2 + \rho\theta_2\left(\rho + \theta_1\right)\sigma_\epsilon^2 + \sigma_\epsilon^2 + \theta_1\rho\sigma_\epsilon^2 + \theta_1^2\sigma_\epsilon^2 + \rho\theta_2\left(\rho + \theta_1\right)\sigma_\epsilon^2 + \theta_2^2\sigma_\epsilon^2 \\
\sigma_y^2\left(1 - \rho^2\right) &= \left(\left(1 + 2\rho\theta_1 + 2\rho\theta_2^2 + \theta_1^2 + \theta_2^2\right)\right)\sigma_\epsilon^2 \\
\sigma_y^2 &= \frac{\left(1 + 2\rho\theta_1 + 2\rho\theta_2^2 + \theta_1^2 + \theta_2^2\right)\sigma_\epsilon^2}{\left(1 - \rho^2\right)}
\end{aligned}$$

(c) (15 points) Find the first-order autocorrelation $\text{Corr}(y_t, y_{t-1})$.

$$\begin{aligned}
E\left(y_t y_{t-1}\right) &= \gamma_1 = E\left(\rho y_{t-1}+\epsilon_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2}\right)y_{t-1}\\
&= \rho\sigma_y^2+\theta_1\sigma_\epsilon^2+\theta_2\sigma_{y_t\epsilon_{t-1}}\\
&= \frac{\rho\left(2\rho\theta_1+2\rho\theta_1\theta_2+\theta_1^2+\theta_2^2+2\rho^2\theta_2+1\right)\sigma_\epsilon^2}{\left(1-\rho^2\right)}+\theta_1\sigma_\epsilon^2+\theta_2\left(\rho+\theta_1\right)\sigma_\epsilon^2\\
&= \left[\frac{\rho\left(2\rho\theta_1+2\rho\theta_1\theta_2+\theta_1^2+\theta_2^2+2\rho^2\theta_2+1\right)+\left(1-\rho^2\right)\theta_1+\left(1-\rho^2\right)\theta_2\left(\rho+\theta_1\right)}{\left(1-\rho^2\right)}\right]\sigma_\epsilon^2\\
&= \left[\frac{\rho\left(1+\theta_2+\theta_1^2+\theta_2^2\right)+\left(1+\rho^2\right)\left(\theta_1+\theta_2\right)+\rho^3\theta_2}{1-\rho^2}\right]\sigma_\epsilon^2
\end{aligned}$$

$$\text{Corr}\left(y_t, y_{t-1}\right)=\frac{\gamma_1}{\sigma_y^2}=\frac{\rho\left(1+\theta_2+\theta_1^2+\theta_2^2\right)+\left(1+\rho^2\right)\left(\theta_1+\theta_2\right)+\rho^3\theta_2}{\left(1+2\rho\theta_1+2\rho\theta_2^2+\theta_1^2+\theta_2^2\right)}$$

(d) (15 points) Find the second-order autocorrelation $\text{Corr}(y_t, y_{t-2})$.

$$\begin{aligned}
\gamma_2 &= E\left(y_t y_{t-2}\right)=E\left(\rho y_{t-1}+\epsilon_t+\theta_1\epsilon_{t-1}+\theta_2\epsilon_{t-2}\right)y_{t-2}\\
&= \rho\gamma_1+\theta_2\sigma_{y_t,\epsilon_t}\\
&= \rho\gamma_1+\theta_2\sigma_{y_t,\epsilon_t}\\
&= \left[\frac{\rho\left(\rho\left(1+\theta_2+\theta_1^2+\theta_2^2\right)+\left(1+\rho^2\right)\left(\theta_1+\theta_2\right)+\rho^3\theta_2\right)+\left(1-\rho^2\right)\theta_2}{1-\rho^2}\right]\sigma_\epsilon^2
\end{aligned}$$

and second-order autocorrelation is

$$\text{corr}\left(y_t, y_{t-2}\right)=\frac{\rho\left(\rho\left(1+\theta_2+\theta_1^2+\theta_2^2\right)+\left(1+\rho^2\right)\left(\theta_1+\theta_2\right)+\rho^3\theta_2\right)+\left(1-\rho^2\right)\theta_2}{\left(1+2\rho\theta_1+2\rho\theta_2^2+\theta_1^2+\theta_2^2\right)}$$

(e) (15 points) Find the forecasting rule $E\left(y_{t+1}|y_t,\epsilon_{t,}\epsilon_{t-1}\right)$

$$E\left(\rho y_t+\epsilon_{t+1}+\theta_1\epsilon_t+\theta_2\epsilon_{t-1}|y_t,\epsilon_t,\epsilon_{t-1}\right)=\rho y_t+\theta_1\epsilon_t+\theta_2\epsilon_{t-1}$$

(f) (15 points) Find the forecasting rule $E\left(y_{t+2}|y_t,\epsilon_{t,}\epsilon_{t-1}\right)$

$$\begin{aligned}
E\left(\rho y_{t+1}+\epsilon_{t+2}+\theta_1\epsilon_{t+1}+\theta_2\epsilon_t|y_t,\epsilon_t,\epsilon_{t-1}\right) &= \rho E\left(y_{t+1}|y_t,\epsilon_t,\epsilon_{t-1}\right)+\theta_2\epsilon_t\\
&= \rho\left(\rho y_t+\theta_1\epsilon_t+\theta_2\epsilon_{t-1}\right)+\theta_2\epsilon_t
\end{aligned}$$

2. (10 points) For doing the write-up entirely with word processing software (i.e., nothing handwritten) and listing your group number and group members at the beginning of your submission.