Asset Pricing
Econ 70427
Foundations

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Concepts to cover

- Euler equation approach
- Stories about the interest rate
- What is risk? Covariance is risk!
- Finance beta-risk approach
- The Arrow-Debreu approach
- Risk-neutral probabilities
- Incomplete markets

All Asset Pricing Follows from the Euler Equation

Begin with concrete assumptions

- Utility is time-separable and defined on consumption, $u(c_t)$, is twice-differentiable with positive but declining marginal utility.
- $0 < \beta < 1$ is subjective discount factor.
- Let $p_{t,i}$ be the price of traded asset $i = 1, \ldots, n$ with next period payoff $x_{t+1,i}$.
  - Equity: $x_{t+1} = p_{t+1} + d_t$
  - Discount bond: $x_{t+1} = 1$
  - Foreign exchange: $x_{t+1} = p_{t+1}$
- Euler equation (rule of rational life)

$$p_{t,i} u'(c_t) = \beta E\left(u'(c_{t+1}) x_{t+1,i} | I_t\right) \tag{1}$$

where I_t is the currently observable publically available information set. Will abbreviate $E(Y_{t+k}|I_t) \equiv E_t(Y_{t+k})$

- Divide both sides by $u'(c_t)$

$$p_{t,i} = E\left(\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1,i} | I_t\right) \tag{2}$$

Price is discounted value of future payoff.

- (Stochastic) discount factor is

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \tag{3}$$

- Now,

$$\begin{aligned} p_{t,i} &= E_t\left(m_{t+1} x_{t+1,i}\right) && (4) \\ 1 &= E_t\left(m_{t+1} \frac{x_{t+1,i}}{p_{t,i}}\right) && (5) \\ &= E_t\left(m_{t+1} R_{t+1,i}\right) && (6) \end{aligned}$$

where $R_{t+1,i} = 1 + r_{t+1,i}$ is the **gross** return, and r is the rate of return.

- (6) holds for every traded asset, including the risk-free asset

$$
\begin{aligned}
p_{t,f} &= 1/R_{t,f} = 1/(1 + r_{t,f}) && (7) \\
&= E_t\left(m_{t+1}\right) && (8) \\
1 &= E_t\left(m_{t+1} R_{t,f}\right) && (9)
\end{aligned}
$$

- Define the excess return as

$$
R^e_{t+1,i} = R_{t+1,i} - R_{t,f} = r^e_{t+1,i} = r_{t+1,i} - r_{t,f} \quad (10)
$$

Subtract (9) from (6)

$$
0 = E_t\left(m_{t+1} r^e_{t+1,i}\right) \quad (11)
$$

- This is a statement about risk, not about time. The interest rate is about time and consumption and saving. This is about paying an excess return to compensate for risk-bearing. What is the risk? That is what we want to understand.

Stories about the (risk-free) interest rate

$$p_{t,f} = \frac{1}{1 + r_{t,f}} = E_t\left(m_{t+1}\right) \tag{12}$$

- $\uparrow m_{t+1} \Rightarrow \downarrow r_{t,f}$
- $\uparrow m_{t+1} \Rightarrow$ future becomes more important. You want to provide more for the future. Why? Future becomes important because consumption will be scarce.
- Provide for future by saving. Lots of saving drives up $p_{t,f}$ and drives down $r_{t,f}$

Stories about the (risk-free) interest rate

- Let utility be CRRA, where $-1 < \gamma < \infty$.

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

- Also, let subjective rate of time preference be δ and write the discount factor as

$$\beta = e^{-\delta}$$

- Let's write $R_{t,f} \simeq e^{-r_{t,f}}$.From (7) and (8),

$$e^{-r_{t,f}} = E_t\left[e^{-\delta}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] \tag{13}$$

Assume a deterministic world, take logs of both sides, multiply by -1,

$$r_{f,t} = \delta + \gamma \Delta \ln \left(c_{t+1} \right) \tag{14}$$

- γ is the coefficient of relative risk aversion, but also the inverse of the intertemporal elasticity of substitution (IES).

$$\text{IES} = \frac{d \ln \left(c_{t+1} / c_t \right)}{d r_{f,t}} = \frac{1}{\gamma}$$

- Impatient (high δ) $\Rightarrow$ high $r_{f,t}$
- Low c_t (maybe saving), high c_{t+1} $\Rightarrow$ high $r_{f,t}$
- High c_t (maybe borrowing), low c_{t+1} $\Rightarrow$ low $r_{f,t}$

Interest Rates in Stochastic Log-Normal World

- Property of log-normal variates If $\ln(Y) \sim N(\mu, \sigma^2)$, then

$$E(Y) = e^{\mu + \frac{\sigma^2}{2}} \tag{15}$$

- Assume

$$\Delta \ln c_{t+1} \sim N_t \left(E_t \Delta ln \left(c_{t+1} \right), \text{Var}_t \left(\Delta \ln c_{t+1} \right) \right)$$

($N_t$is notation for conditionally normally distributed) Then

$$-\gamma \Delta \ln c_{t+1} \sim N_t \left(-\gamma E_t \Delta ln \left(c_{t+1} \right), \gamma^2 \text{Var}_t \left(\Delta \ln c_{t+1} \right) \right)$$

- From the Euler equation,

$$e^{-r_{t,f}} = e^{-\delta} E_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \quad (16)$$

$$= e^{-\delta} e^{-\gamma E_t \Delta ln(c_{t+1}) + \frac{\gamma^2 Var_t(\Delta \ln c_{t+1})}{2}} \quad (17)$$

Take logs of both sides, multiply by -1.

$$r_{t,f} = \delta + \gamma E_t \Delta \ln c_{t+1} - \frac{\gamma^2 Var_t \left(\Delta \ln c_{t+1} \right)}{2} \quad (18)$$

  - The second moment matters now.
  - The more volatile the economy, the lower is the interest rate.
  - Story: People like smooth consumption. A volatile economy is full of risk. This generates a stronger precautionary saving motive. Everybody saves, drives up the price of bonds and drives down the interest rate.
  - The less intertemporally substitutable people are (higher γ) the more this matters.

Time-Series Regression, in Population

Let y_t and x_t be stationary random variables, expressed as deviations from their means.

Consider the regression

$$y_t = \beta x_t + \epsilon_t \tag{19}$$

Multiply through by x_t, take expectations

$$\begin{aligned} y_t x_t &= \beta x_t^2 + \epsilon_t x_t & (20) \\ E\left(y_t x_t\right) &= \beta E\left(x_t^2\right) + E\left(\epsilon_t x_t\right) & (21) \end{aligned}$$

Solve for β,

$$\beta = \frac{Cov(x_t, y_t)}{Var(x_t)} \tag{22}$$

Covariance is Risk

- Exposure to risk is the covariation between an excess return and something people care about.
- In economics, we assume people care about consumption.
- Finance bros sometimes assume people care about other things (wealth, excess returns on other assets)

Covariance Risk

- Start with the Euler equation,

$$0 = E_t \left(m_{t+1} R^e_{t+1,i} \right) \tag{23}$$

- Decomposition of covariance,

$$Cov(X, Y) = E(XY) - E(X)E(Y) \tag{24}$$

- Let $X = m_{t+1}$, $Y = R^e_{t+1,i}$, then

$$0 = Cov \left(m_{t+1}, R^e_{t+1,i} \right) + E_t \left(m_{t+1} \right) E_t \left(R^e_{t+1,i} \right) \tag{25}$$

- Rearrange

$$E_t R^e_{t+1,i} = -\frac{Cov \left(m_{t+1}, R^e_{t+1,i} \right)}{E_t \left(m_{t+1} \right)} \tag{26}$$

Recall $E_t(m_{t+1}) = 1/R_{t,f}$. Substitute in

$$E_t R^e_{t+1,i} = -R_{t,f} Cov_t\left(m_{t+1}, R^e_{t+1,i}\right) \tag{27}$$

$$E_t R_{t+1,i} = R_{t,f} - R_{t,f} Cov_t\left(m_{t+1}, R^e_{t+1,i}\right) \tag{28}$$

- This is an explanation about the cross-section of returns.
- Those assets whose excess returns covary more negatively with m_{t+1} have higher average excess returns. They pay higher premiums.
- This is risk because m_{t+1} is negatively correlated with consumption growth $\Delta \ln(c_{t+1})$, so those returns covary positively with consumption growth.
- The risk is if you hold this asset, the excess return is low when consumption is low (which is exactly when you need it to be high).

The Beta-Risk Representation

- Finance bros talk about ‘betas’. The beta is the slope coefficient in regression of the return (excess return) on a ‘risk factor’.
- Our treatment thus far says the risk factor is the SDF, m_{t+1}.
- There can be more than one risk factor. For now, we stick to the SDF.
- If you’ve correctly identified the risk factor(s), average excess returns vary proportionately to their exposure to the risk factor (i.e., they vary proportionately to their betas).

Beta-Risk Representation

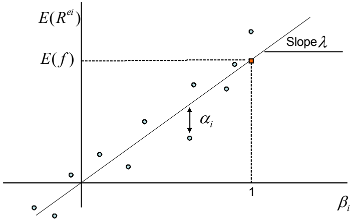
Look at (27). Condition down to unconditional moments, because we are interested in average returns over long-periods of time

$$ER^e_{t+1,i} = -R_{t,f}Cov\left(m_{t+1}, R^e_{t+1,i}\right) \quad (29)$$

$$= -R_{t,f}Var\left(m_{t+1}\right)\frac{Cov\left(m_{t+1}, R^e_{t+1,i}\right)}{Var\left(m_{t+1}\right)} \quad (30)$$

$$= \lambda\beta_i \quad (31)$$

- λ is called the 'price of risk'
- β_i, the beta, is the asset's exposure to the risk factor
- This is what finance bros call the 'consumption CAPM'



Finance Generalizations

Finance bros don't like the consumption-based model.

- Empirically, it doesn't work well
- Consumption observed quarterly. Asset returns observed (nearly) continuously.

Hypothesize that the SDF has the factor representation,

$$m_t = 1 - bf_t + m_t^0 \tag{32}$$

- f_t is called the 'common risk factor', and m_t^0 is an error term. The part of m_t not explained by the factor. We'll assume that m_t^0 is i.i.d.
- What is the factor? In the 'market model' or the CAPM, f_t is the excess return on the market portfolio.
- $b > 0$ says market return is high when consumption is high (when m is low). This happens to be true
- The famous Fama-French work uses 3 factors

$$m_t = 1 - b_1 f_{t,1} - b_2 f_{t,2} - b_3 f_{t,3} + m_t^0$$

  where the factors are the market excess return, the high minus low book to market portfolio returns and the small minus big firm portfolio returns. More on this later.

Substitute the single-factor for m_t in (29),

$$E\left(R^e_{t,i}\right) = -R_{t-1,f}Cov\left(m_{t+1}, R^e_{t,i}\right) \quad (33)$$

$$= -R_{t-1,f}Cov\left(1 - bf_t - m^0_t, R^e_{t,i}\right) \quad (34)$$

$$= \left(bVar\left(f_t\right)R_{t-1,f}\right)\frac{Cov(f_t, R^e_{t,i})}{Var(f_t)} \quad (35)$$

Now the beta is the covariance between the asset's excess return and the risk factor.

Existence and Uniqueness of the SDF

This next result justifies the previous analysis

Result:If the market is complete and the law of one price (no arbitrage condition) holds, a positive stochastic discount factor exists and it is unique.

- $s = \{1, 2\}$ states of nature. $\pi(s)$ is probability of state s
- $p_c(s)$: price of a state-s Arrow contingent claim security. It pays 1 unit (of consumption) if state s occurs.
- $p(x)$ is the price of some asset x, which pays off $x(s)$ in state s

The no-arbitrage condition, aka the law of one price is,

$$p(x) = p_c(1)x(1) + p_c(2)x(2) \tag{36}$$

- Cochrane calls this the Happy Meal assumption.
- Value of any asset representable as bundles of Arrow securities.
- Price of the bundle is the value of sum of individual parts.

Existence

$$
\begin{aligned}
p(x) &= p_c(1)x(1) + p_c(2)x(2) && (37)\\
&= \pi(1)\frac{p_c(1)}{\pi(1)}x(1) + \pi(2)\frac{p_c(2)}{\pi(2)}x(2) && (38)\\
&= \pi(1)m(1)x(1) + \pi(2)m(2)x(2) && (39)\\
&= E\left(mx\right) && (40)
\end{aligned}
$$

where

$$
m(s) = \frac{p_c(s)}{\pi(s)}. \qquad (41)
$$

This is existence and it gives back the Euler equation. Note that we've done this without any explicit assumptions about preferences or distributions about asset returns.

Uniqueness

From (41), take the price of the $s = 1$ Arrow security

$$p_c(1) = \pi(1)m(1)$$

This has to be true for any SDF. Suppose there is another SDF, $m^*(1)$

$$p_c(1) = \pi(1)m^*(1)$$

Then it must be the case that

$$m(1) = m^*(1)$$

This has to be true for all states s.

Risk-Neutral Probabilities

- Econ bros like to value assets using the **Arrow-Debreu** approach.
- An equivalent approach is the **risk-neutral probability** approach.
  - $\pi(s) \geq 0$, where $\sum_s \pi(s) = 1$, are the actual (physical) probabilities that state s occurs.
  - We going to call $\pi^*(s)$ the risk-neutral probabilities.

Let $p(x)$ be price of asset. $pc(s)$ is contingent claims price of Arrow security that pays off $x(s)$ in state (s). Start with (37),

$$
\begin{aligned}
p(x) &= \sum_s pc(s)\,x(s) = \sum_s \pi(s) \underbrace{\left(\frac{pc(s)}{\pi(s)}\right)}_{m(s)} x(s) \\
&= \sum_s \pi(s)\,m(s)\,x(s) = E(mx) = \frac{1}{R^f} \sum_s \underbrace{\left(\pi(s)\,m(s)\,R^f\right)}_{\pi^*(s)} x(s) \\
&= \frac{1}{R^f} \sum \pi^*(s)\,x(s)
\end{aligned}
$$

Hence,

$$
\pi^*(s) = \pi(s)\,m(s)\,R^f = \frac{m(s)}{E(m)}\pi(s) \tag{42}
$$

is called the risk-neutral probability.

What's the point?

- Think of asset pricing as if agents are risk neutral and use π^* in place of π. The π^* give more weight to states with higher than average *m*, which are states of low consumption.
- Risk aversion is like paying more attention to bad states relative to physical probabilities.
- Finance bros like the risk-neutral approach. Econ bros like the Arrow-Debreu approach, but as you can see, they are equivalent.

How to find risk-neutral probabilities

- Say you have $s = 1, \ldots, S$ states and $i = 1, \ldots n$ assets.
- Collect the $n + 1$ equations

$$p(x_i) = \frac{1}{R_{t,f}} \sum_s \pi^*(s) x_i(s) \tag{43}$$

$$\sum_s \pi^*(s) = 1 \tag{44}$$

and solve for the $\pi^*(s)$ subject to the conditions that for $s = 1, \ldots, S$

$$\pi^*(s) \geq 0$$

An example I stole

Notes from a corporation (call it Delta Airlines) today, due in 3 years trade at \$71.50. A zero-coupon Treasury Strip, due in 3 years sells at \$90.625. Use these prices to infer the risk-neutral probability of bankruptcy. Two assets (bonds). $p_D = \$71.50$ is the current price of the delta bond. $s = 1$ is non-bankruptcy state and bond pays off $x(1)$. $s = 2$ is the bankruptcy state and bond pays $x(2)$. p_T is the price of the Treasury Strip. It pays off $y(1) = y(2) = \$100$ in either state of the world. It is claimed that the historical default frequency of junk-rated bonds is 42 percent.

$$p_D = x(1)\frac{\pi^*(1)}{R^f} + x(2)\frac{\pi^*(2)}{R^f}$$
$$p_T = y(1)\frac{\pi^*(1)}{R^f} + y(2)\frac{\pi^*(2)}{R^f}$$

Or, in matrix form,

$$\begin{pmatrix} p_D \\ p_T \end{pmatrix} = \begin{pmatrix} x(1) & x(2) \\ y(1) & y(2) \end{pmatrix} \begin{pmatrix} \frac{\pi^*(1)}{R^f} \\ \frac{\pi^*(2)}{R^f} \end{pmatrix}$$

Solve,

$$\begin{pmatrix} \frac{\pi^*(1)}{R^f} \\ \frac{\pi^*(2)}{R^f} \end{pmatrix} = \begin{pmatrix} x(1) & x(2) \\ y(1) & y(2) \end{pmatrix}^{-1} \begin{pmatrix} p_D \\ p_T \end{pmatrix}$$

Put some numbers to it.

$$R^f = \frac{100}{90.625} = 1.1034$$

Suppose in bankrupcy, $x(2) = 0$

$$\begin{pmatrix} \frac{\pi^*(1)}{R^f} \\ \frac{\pi^*(2)}{R^f} \end{pmatrix} = \begin{pmatrix} 130 & 0 \\ 100 & 100 \end{pmatrix}^{-1} \begin{pmatrix} 71.5 \\ 90.625 \end{pmatrix}$$

$$= \begin{pmatrix} 0.55 \\ 0.35625 \end{pmatrix}$$

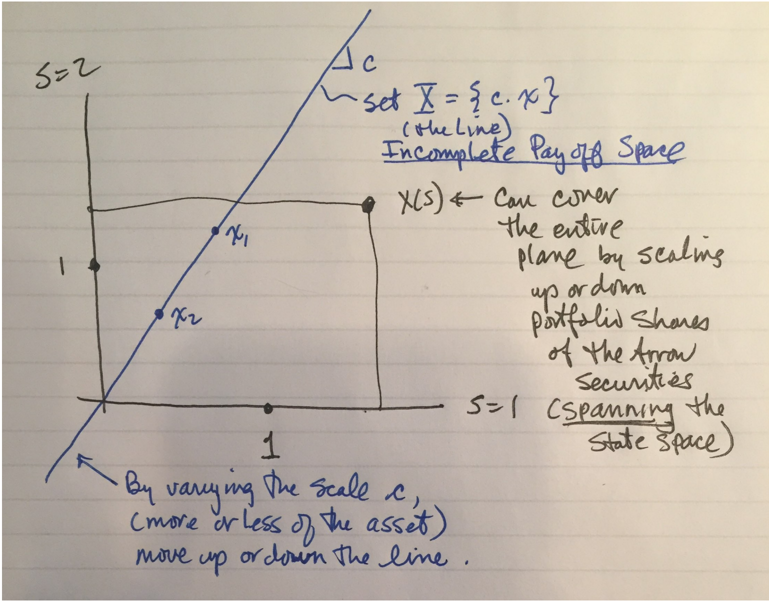
Then

$$\begin{pmatrix} \pi^*(1) \\ \pi^*(2) \end{pmatrix} = \left(1.1034 \begin{array}{c} 0.55 \\ 0.35625 \end{array} \right) = \begin{pmatrix} 0.60687 \\ 0.39309 \end{pmatrix}$$

Less conservative calculation: Moody's says senior unsecured debt is worth about half the face value in the event of bankruptcy. So let's say $x(2) = 65$.

$$\begin{pmatrix} \pi^*(1) \\ \pi^*(2) \end{pmatrix} = 1.1034 \begin{pmatrix} 130 & 65 \\ 100 & 100 \end{pmatrix}^{-1} \begin{pmatrix} 71.5 \\ 90.625 \end{pmatrix}$$
$$= \begin{pmatrix} 0.21378 \\ 0.78617 \end{pmatrix}$$

Incomplete Markets



The LOOP again

- The LOOP (no arbitrage) in this environment, is again, Cochrane's happy meal theorem.
- Let $x_1, x_2 \in X$. These are two payoffs in X. Then we can scale x_1 by a and x_2 by b such that $ax_1 + bx_2 \in X$.
- Let $p(x_1)$ be the price of the payoff x_1 and $p(x_2)$ be the price of the payoff x_2. The LOOP is

$$p(ax_1 + bx_2) = ap(x_1) + bp(x_2)$$

Incomplete Markets

Result: If the LOOP holds, there exists a positive SDF, x^* such that

$$p(x) = E(x^* x).$$

But it may not be unique

Proof. Define $x^* = p(x)\frac{x}{E(x^2)}$. Then by construction,

$$E(x^* x) = E\left(p(x)\frac{x^2}{E(x^2)}\right) = p(x)$$

Another SDF? Let ϵ be a random variable such that $E(\epsilon x) = 0$. Then $x^* + \epsilon$ is another SDF, because

$$p(x) = E[(x^* + \epsilon)x] = E(x^* x)$$