

# Asset Pricing Implications of Demographic Change\*

Thomas A. Maurer<sup>†</sup>

Washington University in St. Louis

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## Abstract

An overlapping generations model featuring stochastic birth and death rates is solved in general equilibrium. I provide sufficient conditions for the interest rate to be decreasing in the birth rate and increasing in the death rate. If preferences are recursive, demographic uncertainty is priced in financial markets, and the equity premium is higher during periods characterized by a high birth rate and low mortality than in times of a low birth and high death rate. Demographic changes explain substantial parts of the time variation in the real interest rate, equity premium and conditional stock price volatility.

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**Keywords:** baby boom, demographic shocks, long run risk, asset pricing, overlapping generations, intertemporal consumption choice

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<sup>†</sup>Olin Business School, Washington University in St. Louis, Email: thomas.maurer@wustl.edu.

# 1 Introduction

In the developed world there is a substantial demographic transition in progress caused by the post World War II baby boom and declining mortality rates. The demographic change is likely to have a significant impact on the global economy, including GDP growth prospects, wealth redistributions between generations, the solvency of social security systems, and financial markets. An often discussed hypothesis is that the past baby boom caused a stock market run-up in the 1980s and 1990s and may cause an asset market meltdown as baby boomers retire over the next two decades.

Most related literature asks how retirement of the baby boomers affects asset prices. In the long-run there is a time variation in asset prices as baby boomers proceed through the life-cycle and the "average savings behavior" across the population slowly and predictably changes. Empirical studies suggest that in the long-run asset prices, price-dividend ratios, the interest rate and equity premia are linked to various demographic quantities (Mankiw and Weil (1989), Yoo (1994), Bakshi and Chen (1994), Erb, Harvey and Viskanta (1997), Bergantino (1998), Poterba (2001), Geanakoplos et al. (2004), Goyal (2004), Ang and Madaloni (2005), Huynh et al. (2006), Favero et al. (2007), Acemoglu and Johnson (2007), Hanewald (2010) and Takats (2010)). A major problem of empirical studies is that available data samples are limited and estimates are noisy (Poterba (2001)).

Given the limitations of empirical tests, it is useful to study the implications of demographic changes in a stylized theoretical model. Calibrations by Brooks (2000, 2004) and Geanakoplos et al. (2004) suggest that predictable baby booms and busts cause the interest rate and equity prices to vary over time as the baby boomers live through the life-cycle. Most of the results are driven by the assumption that consumption-to-wealth ratios differ across cohorts because agents face a fixed lifetime horizon. Abel (2001, 2003) shows in an analytically tractable model that the price of a unit of capital is increasing in the birth rate and follows a mean reverting process. Unfortunately, the risk-free rate and the equity premium cannot be disentangled in his model. Auerbach and Kotlikoff (1987), Kotlikoff et al. (2001), Fehr et al. (2003), and Fehr et al. (2004a, 2004b) use dynamic general equilibrium simulation models to explore the possible impact of deterministic trends in birth and death

rates on long-run economic and fiscal conditions.

I contribute in several ways to the literature. First, I explore how asset prices are affected by demographic transitions and by uncertainty about the timing of future demographic changes. We did not know how long the post-war baby boom was going to last and neither do we know the timing of another large demographic transition. Investors do not know whether and when there will be a war, an epidemic outbreak or a major advancement in (medical) research and technology which could substantially affect life expectancy.

Second, the main economic mechanism in my model is the redistribution risk of wealth and consumption within the population induced by shocks to birth or death rates. Redistributions of consumption within the population are important because pricing depends solely on consumption growth of existing agents in an OLG model.<sup>1</sup> In contrast, in Brooks (2000, 2004) and Geanakoplos et al. (2004) it is not the consumption growth of existing agents which drives the results, but baby booms and busts cause changes in the average life expectancy in the population and hence, changes in the representative agent's marginal propensity to consume or consumption-to-wealth ratio.<sup>2</sup>

Third, I solve in general equilibrium an analytically tractable overlapping generations (OLG) model with stochastic birth and mortality rates. A key contribution is the tractability which allows me to derive novel qualitative results. I am able to disentangle and study various offsetting economic channels through which birth and death rates affect the level of and the time variation in the interest rate and the equity premium. In contrast, most of the literature heavily relies on numerical solutions, which makes it difficult to understand the underlying economics. In addition, my model explains some of the long-run time variations in the interest rate and stock market excess returns which are not explained in previous work.

Fourth, I model births and deaths as Poisson events and stochastic changes in birth and death rates have no effect on the instantaneous variation in the population size, labor supply and aggregate production output. Ignoring total factor productivity (TFP) shocks

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<sup>1</sup>Existing agents are agents which were already alive at time  $t$  and survive over the next  $dt$  time period.

<sup>2</sup>The major difference is the channel of consumption growth of existing agents (my model) versus changes in the marginal propensity to consume (Brooks (2000, 2004) and Geanakoplos et al. (2004), where an agent's life expectancy is decreasing in age). In my model I assume latter mechanism away by assuming age-independent mortality. This is essential to keep the model analytically tractable. In reality both channels are important but for tractability reasons and in order to better isolate my channel from previous work I focus on only the first mechanism. My results should be viewed as a complement to previous findings.

for now, population and production output growth are locally deterministic processes. Since changes in the population size are perfectly predictable over a small instant in time, shifts in the consumption distribution are also perfectly forecast over a short horizon. Demographic shocks do not introduce any instantaneous risk to the economy but only long-run risk, that is, shocks to the expected consumption growth rate of existing agents. A smooth growth in the population size as assumed in the model is close to what is observed in the data. In reality, birth and death rates are subject to unpredictable changes but in the short run the population grows gradually and growth is highly predictable. Finally, I model birth and death rate changes simultaneously, while previous studies usually focus on either birth or death rates.

In my theoretical model the interest rate is decreasing in the birth rate and increasing in the death rate, given a moderate level of relative risk aversion (*RRA*) and a small enough elasticity of intertemporal substitution (*EIS*). A linear regression analysis confirms this pattern in the data. The key driving forces for the result are the following. A high birth rate implies that large new born cohorts are expected to enter the economy in the future. A large new born cohort claims a big share of aggregate consumption and growth in consumption of existing agents is expected to be moderate. In equilibrium, a drop in the expected consumption growth rate of existing agents corresponds to a decline in the interest rate. It is important to understand that the driving force is not a change in expected aggregate consumption growth but the shift in the distribution of aggregate consumption within the population from existing agents to new born cohorts.

In contrast, a high death rate implies a short life expectancy, a high discount of future utility and few savings. In addition, aggregate consumption has to be split among only few survivors if the death rate is high, and consumption growth of existing agents is large. In equilibrium, both channels imply a high interest rate.

Because birth and death rates affect the interest rate through not identical channels, they have to be modelled separately and not as one general state variable that determines total population growth or the average age of the population. This insight is important but mostly neglected in empirical studies.

The stock price volatility exceeds the variation in aggregate consumption growth because

of demographic uncertainty. Stock prices respond to demographic changes through two channels. First, expected growth in labor supply, production output and dividends are sensitive to birth and death rate changes (Barsky and De Long (1993)). Second, demographic changes have a similar effect on the discount rate of stocks as on the real interest rate. Because stock prices instantly incorporate information about changes in future dividend growth and the discount rate, there is instantaneous volatility in stock prices due to demographic shocks.

If agents maximize utility functions of the power utility family, then an immediate implication of a locally deterministic consumption process is that the stochastic discount factor (SDF) has no quadratic variation. As demographic uncertainty only adds long-run risk to the economy, it has an impact on the interest rate and the stock price volatility but the equity premium is not affected.

In the case of recursive utilities pricing depends on the covariation of asset returns and instantaneous and future consumption growth (Bansal and Yaron (2004)). The variation in the current consumption-to-wealth ratio is a sufficient statistic for the variation in future consumption growth. As the consumption-to-wealth ratio is a function of time discounting of future utility and the interest rate (or expected consumption growth of existing agents), it instantly responds to changes in birth and death rates. Accordingly, demographic shocks induce a covariation between stock returns and the consumption-to-wealth ratio, and are priced in financial markets. The equity premium is time varying and I provide sufficient conditions for it to be positive and increasing in the birth rate and decreasing in the death rate. Consistent with the model, I show that US stock market excess returns are positively related to the birth rate and negatively related to the mortality rate.

Shocks to the expected consumption growth rate of existing agents are not only triggered by shocks to expected production or aggregate consumption growth but mainly by shocks to the aggregate consumption share of the new born generation. In other words, rather than long-run risk in labor supply and production output, it is the redistribution risk of aggregate endowment and consumption between new born and old agents which is the main channel for pricing. This is in stark contrast to the long-run risk literature initiated by Bansal and Yaron (2004) where shocks to expected production or aggregate consumption growth generate their results. Given the insight that the economic importance of demographic changes does not

stem from changes in future labor supply and production output, I argue that ignoring endogenous capital accumulation in the model is not a severe problem.

My model suggests that demographic transitions explain substantial parts of the time variation in the interest rate, market price of risk, equity premium and conditional stock price volatility. A large body of empirical literature explores stock return predictability. Returns are found to be more predictable at low frequencies than over the short run, and most of the predictable variation is due to variation in discount rates rather than changes in expected dividend growth (e.g. Keim and Stambaugh (1986), Fama and French (1988a, 1988b), Ammer and Campbell (1993), Goetzman and Jorion (1993), and Cochrane (2011) as an overview). Moreover, Ferson and Harvey (1991) suggest that the time variation in the market price of risk rather than the time variation in the exposure of stocks to systemic risk is the driving force causing a time variation in discount rates. According to my qualitative and quantitative results, these facts may be (partly) explained by demographic changes.

My results require preferences with a low  $EIS$ , which is consistent with a large body of empirical studies. Hall (1988), Campbell and Mankiw (1989), Yogo (2004), and Pakos (2007) use aggregate consumption and financial data to estimate the  $EIS$  from the Euler equation in a representative agent model. They get estimates close to zero. Vissing-Jorgensen (2002) disentangles asset holders from non-asset holders and estimates an  $EIS$  coefficient of 0.3 for stockholders. Hasanov (2007) and Bonaparte (2008) use household-specific consumption and portfolio choice data to take account for heterogeneity. They get an  $EIS$  of about 0.25.

A challenge of my model is that investors have to care about demographic changes when making decisions. DellaVigna and Pollet (2007) use demographic changes to forecast future earnings of firms and show that an investment strategy based on demographic variables can generate abnormal returns. Given the lack of a rational model to explain their findings, they conclude that it is likely that predictable demographic changes are not well forecast by the market.<sup>3</sup> In contrast, Hanewald and Post (2010) provide empirical evidence in favor of my model: (i) investors in the real world are aware of stochastic changes in mortality rates and (ii) changes in mortality rates affect investors' investment and consumption behaviors (see

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<sup>3</sup>I am currently working on a rational model that appears to explain the empirical results of DellaVigna and Pollet (2007), and therefore, I disagree with the conclusion that the market does not incorporate demographic information when pricing assets.

also empirical results by Hugonnier et al. (2012)).

Finally, my model adds in three ways to the long-run risk literature. First, the pricing channel in my model does not work through shocks to expected aggregate consumption growth but demographic uncertainty triggers shocks to the wealth and consumption distribution within the population, which effectively implies shocks to expected consumption growth of existing agents.

Second, I show that a long-run risk source can have positive effects on the equity premium even if the  $EIS$  is small ( $EIS < 1$ ). This result is important because the long-run risk literature is often criticized because estimates of the  $EIS$  appear to be less than 1 in the data but the literature assumes  $EIS > 1$ . In Bansal and Yaron (2004) and my model a risk source must cause a negative relation between stock prices and the consumption-to-wealth ratio to be compensated with a positive risk premium.<sup>4</sup>  $EIS < (>) 1$  implies that the interest rate and consumption-to-wealth ratio are positively (negatively) related to each other because the income (substitution) effect dominates. Clearly, shocks to expected aggregate consumption growth necessarily cause stock prices and the interest rate to move in the same direction.  $EIS > 1$  is required for a negative relation between stock prices and the consumption-to-wealth ratio, and shocks to aggregate consumption growth to be compensated with a positive premium. In contrast, demographic shocks cause stock prices (or expected aggregate consumption growth) and the interest rate (or expected consumption growth of existing agents) to move in opposite directions. Accordingly, I need  $EIS < 1$  for the consumption-to-wealth ratio and stock prices to be negatively related and demographic shocks to carry a positive risk premium.

Third, the long-run risk literature is heavily criticized because shocks to expected aggregate consumption growth are unobservable and the models are not testable. In contrast, birth and death rates are observable in the data, which makes my approach more appealing.

In the following I present my results in three steps. First, I discuss the simplest version of the model with constant birth and death rates and use comparative statics analysis to gain a first intuition about the economic mechanisms. Second, I prove that the intuition

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<sup>4</sup>In Bansal and Yaron (2004) and my model the consumption-to-wealth ratio is positively related to marginal utility.

from the constant case carries over to a dynamic two state Markov switching model and I derive further qualitative results. Third, I generalize the model to include TFP shocks and Brownian uncertainty and show numerically the quantitative importance of my analytical results. Finally, I conclude. In the appendix are details about US birth and death rates, a discussion on limitations and extensions of my model, an additional calibration, and all the proofs.

## 2 The Economy

### 2.1 Demographics and Uncertainty

I consider a continuous time OLG model that generalizes the Blanchard (1985) model. I disentangle birth and death rates and let them change stochastically over time.

The economy is populated with a continuum of agents of measure  $N_t$ . The birth rate is denoted by  $n_t$  and the new born cohort at time  $t$  is of the size  $n_t N_t dt$ .<sup>5</sup> Each agent faces an instantaneous probability of death  $\lambda_t dt$ . Conditional on being alive at time  $t_1$ , an agent's survival probability until time  $t_2 > t_1$  is  $e^{-\int_{t_1}^{t_2} \lambda_v dv}$ .

To keep the model tractable, I do not allow for heterogeneity in the arrival rate of death. Imposing mortality rates to be age-independent is restrictive and counterfactual, but it is a small price to pay when one is interested in the common time variation in death rates. According to the much celebrated Lee and Carter (1992) approach, time variation in age specific death rates is mostly due to one across cohorts common stochastic time component.<sup>6</sup> The main general equilibrium implication of age-independent mortality is that the marginal propensity to consume is independent of age. Arguably the most interesting life-cycle effects on age-dependent consumption and savings behavior do not come from the time variation in the marginal propensity to consume but from the hump-shaped pattern of earnings over the life-cycle, and my model accounts for this feature.

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<sup>5</sup>The instantaneous probability of an existing agent to give birth to a new agent at time  $t$  is  $n_t dt$ .

<sup>6</sup>The Lee and Carter (1992) model is widely used in demographic research and has also gained much attention in other fields of research. In asset pricing and household finance literature many papers employ it (Cox et al., 2006; Chen and Cox, 2009; Cocco and Gomes, 2009; DeNardi et al., 2009; Maurer, 2011; Hanewald and Post, 2010).



The necessity of age-independent mortality for tractability becomes clear when looking at the dynamics in the population size. Suppose the death rate was age-dependent and agents of cohort  $s$  face at time  $t$  mortality  $\lambda_{s,t}dt$ . Population growth is characterized by

$$dN_t = n_t N_t dt - dt \int_{-\infty}^t \lambda_{s,t} n_s N_s e^{-\int_s^t \lambda_{s,u} du} ds$$

The last term depends on the entire history of birth and death rates and  $dN_t$  and the entire model solution are not Markov in my demographic variables. If  $\lambda_{s,t} = \lambda_t$  (age-independent mortality rates), then the above integral adds up to  $dN_t = (n_t - \lambda_t) N_t dt$  and the model will be Markov. Similar problems arise with age-dependent mortality when aggregating labor supply and once I impose market clearing and aggregate consumption (equations (4), (23)).

Timing of death is uncertain to the individual, but on the aggregate the size of a cohort declines non-stochastically over the next instant in time because the economy is populated by a continuum rather than a finite number of agents. The size of cohort  $s$  (agents born at time  $s$ ) shrinks to  $n_s N_s e^{-\int_s^t \lambda_v dv}$  until time  $t > s$ . The population size is  $N_t = \int_{-\infty}^t n_u N_u e^{-\int_u^t \lambda_v dv} du = N_s e^{\int_s^t n_v - \lambda_v dv}$ .<sup>7</sup>  $\frac{dN_t}{N_t} = (n_t - \lambda_t) dt$  is a term in only  $dt$ , and population size  $N_t$  follows a locally deterministic process (zero quadratic variation). A smooth growth in the population size as assumed in the model is close to what is observed in the data. In reality, birth and death rates are subject to unpredictable changes but in the short run the population grows gradually and growth is highly predictable.

In contrast, it is common in literature to model a baby boom and bust in discrete time (with one time period being equivalent to about 25 to 35 years) as a sudden increase respectively decrease in the size of the new born generation. A shortcoming of this approach is that an unpredictable, sudden shock to the population size has an equivalent impact on production as a TFP shock, and assets are compensated with a risk premium. If trading takes place more frequent than every 25 to 35 years, then we mistake long run risk for instantaneous uncertainty in a discrete time model and we price demographic shocks falsely. A discrete time model with sudden shocks to the population size may approximate the unconditional long run variation of production output, the interest rate and asset prices, but

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<sup>7</sup>I need the technical assumption that the economy already exists for an infinite amount of time.

the implications for the equity premium seem spurious.

In the USA the birth plus immigration rate, denoted by  $n_t$  in the model, declined from about 3.5% in 1910 to 1.8% in 2006.<sup>8</sup> Annual changes are subject to an unconditional standard deviation of 3.58%. Shocks are persistent and  $n_t$  appears to follow a process integrated of order 1. In addition to "short-term" uncertainty (annual volatility), there are major "long-term" transitions. Statistics from other developed countries reveal similar patterns. Mankiw and Weil (1988) illustrate the randomness in the birth rate process by pointing out the problems the USA census bureau has to forecast the future evolution of the birth rate. Most of the bureau's projections turned out to be far from actual future realizations (figure 2 in their paper).<sup>9</sup>

The age-independent death rate, denoted by  $\lambda_t$  in the model, can be approximated by the stochastic time component estimated in the Lee and Carter (1992) model.<sup>10</sup> It is mostly decreasing and changes are subject to a yearly unconditional standard deviation of 5.1%. Shocks to the death rate are persistent and  $\lambda_t$  is well described by a geometric Brownian motion (Lee and Carter (1992)). US population statistics are representative for the developed world.

## 2.2 Production

The supply side in the consumption goods market is constituted by a representative firm which is endowed with capital stock  $K_t$  and has access to a technology described by a Cobb-Douglas production function  $Y_t = A_t (G_t)^a (K_t)^{1-a}$ .  $A_t$  denotes TFP,  $G_t$  the employed amount of labor and  $Y_t$  determines the quantity of consumption goods produced by the firm. Except for the last section, I assume  $A_t$  to grow at an exogenously given deterministic rate  $\frac{dA_t}{A_t} = \mu^{(A)} dt$ . The firm is assumed to do not face any economic decision, and I presume for

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<sup>8</sup>Birth rate statistics are provided by the Department of Health and Human Services, National Center for Health Statistics, USA, and The Human Mortality Database, University of California, Berkeley and Max Planck Institute. The birth rate is adjusted for infant mortality (see appendix for details).

<sup>9</sup>Similar difficulties to predict the fertility rate in Japan are reported in the Opening Remark by Masaaki Shirakawa at 2012 BOJ-IMES Conference hosted by the Institute for Monetary and Economic Studies, the Bank of Japan (Chart 6).

<sup>10</sup>Data on the central death rate are provided by the National Center for Health Statistics, USA, and The Human Mortality Database, University of California, Berkeley and Max Planck Institute.

the capital stock  $K_t$  a deterministic growth path according to  $\frac{dK_t}{K_t} = \mu^{(K)} dt$ .<sup>11,12</sup>

### Labor Efficiency Units over the Life-cycle, $G(s, t)$

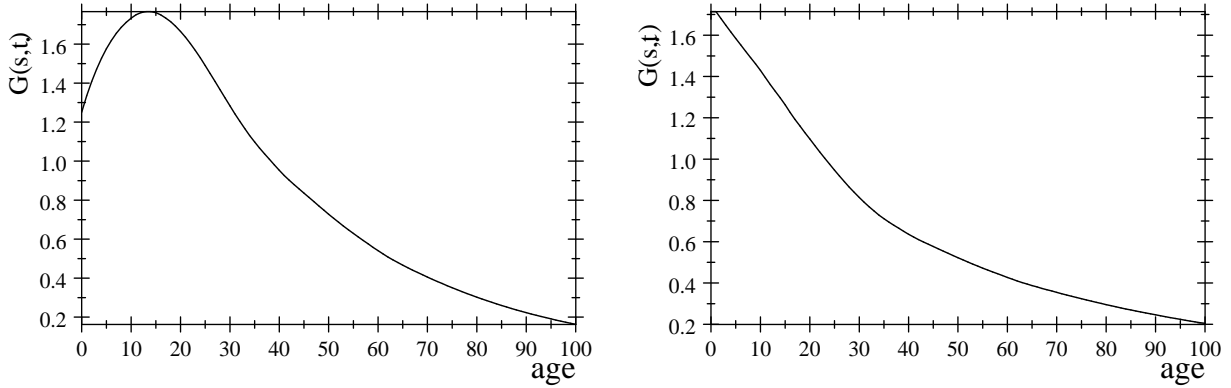


Figure 1: Double exponential function  $G(s, t) = \sum_{i=1}^2 B_i e^{-\delta_i \int_s^t n_u du}$  with the parameterisation  $(B_1, B_2, \delta_1, \delta_2) = (31.25, -30, 2.65, 2.95)$  (left panel), and  $(B_1, B_2, \delta_1, \delta_2) = (1.75, 0, 1.3, 0)$  (right panel).

I suppose full employment in the economy. An agent born at time  $s$  supplies  $G(s, t)$  labor efficiency units at time  $t$ . To match the hump-shaped profile of life-cycle earnings in Hubbard et al. (1993), I let  $G(s, t) = \sum_{i=1}^2 B_i e^{-\delta_i \int_s^t n_u du}$  with the technical assumption of  $\delta_i > -1$ .<sup>13</sup>  $G(s, t)$  generates the desired hump-shape pattern if  $B_1 > |B_2| > B_2$  and  $\delta_1 < \delta_2$ . For some derivations I use a simpler specification with  $B_2 = 0$  (or  $\delta_1 = \delta_2$ ). Aggregation yields the total amount of labor efficiency units employed by the firm,  $G_t = \int_{-\infty}^t G(s, t) n_s N_s e^{-\int_s^t \lambda_u du} ds = N_t \sum_{i=1}^2 \frac{B_i}{1+\delta_i}$ .  $G_t$  is locally deterministic,  $\frac{dG_t}{G_t} = (n_t - \lambda_t) dt$ , but has long-run risk inherent. If there are no TFP shocks, the supply of consumption goods follows a locally deterministic process with the growth rate  $\frac{dY_t}{Y_t} = \mu_t^{(Y)} dt = [\mu^{(A)} + (1-a)\mu^{(K)} + a(n_t - \lambda_t)] dt$ .

Labor is paid according to its marginal productivity  $y_t = a \frac{Y_t}{G_t}$ . An agent of cohort  $s$  earns

<sup>11</sup>No economic decision in the sense that the firm does not invest, employs all supplied labor at a competitive wage equal to the marginal productivity of labor, and pays out all remaining earnings as dividends. Capital growth is understood as a byproduct of production (for free) as is technological progress. I may set  $\mu^{(K)} = 0$  without altering any of my results.

<sup>12</sup>The results do not essentially change if I assume an exogenous capital accumulation process and technological progress which are dependent on population growth, i.e.  $\frac{dK_t}{K_t} = [\mu^{(K)} + \xi^{(K)}(n_t - \lambda_t)] dt$  and  $\frac{dA_t}{A_t} = [\mu^{(A)} + \xi^{(A)}(n_t - \lambda_t)] dt + \sigma^{(A)} d\widetilde{W}_t$ , as long as  $\xi^{(K)}$  and  $\xi^{(A)}$  are not too large.

<sup>13</sup> $\delta_i > -1$  must hold in order to ensure aggregate supply of labor to stay finite,  $|G_t| = \left| \int_{-\infty}^t G(s, t) n_s N_s e^{-\int_s^t \lambda_u du} ds \right| < \infty$ . See also Garleanu and Panageas (2010) for a similar specification.

in exchange for his labor  $y_t^s = aY_t \frac{G(s,t)}{G_t}$ . The firm does not invest and pays the remaining fraction of output  $(1 - a)Y_t$  as dividends  $D_t$  to the shareholders of the firm.

In the model newborn cohorts immediately supply labor but in reality there is roughly a 15 to 20 year lag between birth rate and labor supply changes. Accordingly, I assume a too weak response in capital accumulation to population growth but on the other hand, a too strong response in labor supply growth. In sum the two stylistic and counterfactual assumptions approximately wash out.

### 2.3 Financial Markets: Equities, Bonds, and Insurance Contracts

Financial markets are assumed to be dynamically complete.  $\pi$  denotes the (unique) SDF in the economy and is determined in equilibrium. Agents are born without any financial wealth but are endowed with labor. Financial wealth at time  $t$  of an agent of cohort  $s$  is denoted by  $W_t^s$ , and  $\widehat{W}_t^s$  describes total wealth (financial and human wealth). An agent of cohort  $s$  consume  $c_t^s$  and allocates the remaining part of his financial wealth to equities and bonds. Equities are claims on the stream of aggregate dividends  $D_t$  paid out by the representative firm. Given there are  $d$  independent sources of uncertainty driving demographic and technological changes, dynamic completeness requires  $d$  distinct types of equity contracts. Contract  $j$  is a claim to the dividend stream  $D_t^{(j)}$  and each process  $D_t^{(j)}$  is non-redundant and  $\sum_{j=1}^d D_t^{(j)} = D_t$ . The price of equity  $j$  is denoted by  $P_t^{(j)}$  and the value of the entire stock market is  $P_t = \sum_{j=1}^d P_t^{(j)}$ . The supply of equities is normalized to one.  $X_{j,t}^s$  denotes the number of equities  $j$  purchased by an agent of cohort  $s$ .<sup>14</sup> Bonds are instantaneously risk-free and pay interest  $r_t$ . Bonds are in zero supply. The part of an agent's financial wealth that is not used to buy stocks,  $W_t^s - \sum_{j=1}^d X_{j,t}^s P_t^{(j)}$  is invested in bonds.

Agents have access to annuity contracts supplied by a large, competitive insurance company as in Blanchard (1985). A claim (long position) on an insurance contract pays off as follows: if the agent survives the next time period  $dt$  he receives the premium  $\lambda_t dt$  from the insurer, and if he dies he pays 1. Agents have an incentive to fully annuitize because their

<sup>14</sup>To do not permit arbitrage opportunities I restrict trading activities according to the standard technical assumption  $\frac{X_{j,t}^s P_t^{(j)}}{W_t^s} \in \mathcal{L}^2$ , where  $\mathcal{L}^2 \equiv \left\{ x \in \mathcal{L} : \int_0^T x_t^2 dt < \infty \quad a.s. \right\}$  and  $\mathcal{L}$  is the set of processes adapted to the filtration  $\mathcal{F}^P$  generated by asset prices,  $\mathcal{F}_t^P \equiv \sigma \{P_s : s \leq t\}$ .

objective functions are strictly increasing in consumption and they do not draw utility from bequest. The insurer breaks even almost surely (earnings and liabilities coincide).

## 2.4 Agents' Objective Functions and Budget Constraints

An agent's financial wealth  $W_t^s$  evolves according to the dynamics

$$dW_t^s = \underbrace{W_t^s \lambda_t dt}_{\text{insurance premium}} + \underbrace{W_t^s r_t dt}_{\text{risk-free return}} + \underbrace{\sum_{j=1}^d X_{j,t}^s \left( dP_t^{(j)} + D_t^{(j)} dt - P_t^{(j)} r_t dt \right)}_{\text{stock market excess return}} + \underbrace{y_t^s dt}_{\text{labor income}} - \underbrace{c_t^s dt}_{\text{consumption}} \quad (1)$$

with the initial condition  $W_s^s = 0$ . As in Blanchard (1985) I impose the transversality condition (given the agent is still alive at time  $u$ )  $\lim_{u \rightarrow \infty} e^{-\int_s^u \lambda_v dv} \frac{\pi_u}{\pi_s} W_u^s = 0$ . This ensures that agents do not borrow without limit, accumulate an infinite amount of debt, and protect themselves by buying annuity contracts.

The set of feasible cash flows is  $(M + y^s + W_s^s) \equiv \left\{ x^s : F_s^{(\lambda)}(x^s) - F_s^{(\lambda)}(y^s) - W_s^s \in M \right\}$ .  $F_s^{(\lambda)}$  is a discount function such that  $F_s^{(\lambda)}(x_t) = e^{-\int_s^t \lambda_u du} x_t$ , and  $M$  denotes the set of all marketable cash flows.<sup>15</sup> The set of admissible cash flows is  $\mathfrak{S} \equiv (M + y^s + W_s^s) \cap \mathcal{L}^+$ .  $\mathcal{L}^+$  includes all non-negative processes adapted to  $\mathcal{F}^P$  (filtration generated by asset prices). An agent's consumption process  $c^s$  has to be an element of the set of admissible cash flows  $\mathfrak{S}$ .

Agents are assumed to feature homogeneous preferences and the only heterogeneity in the model is timing of birth and death and wealth between agents across cohorts (but not within the same cohort). Preferences are described by a stochastic differential utility function of the Kreps and Porteus (1978) type introduced by Duffie and Epstein (1992a, 1992b). Following Duffie and Epstein (1992a) and adding the feature of lifetime uncertainty (for the formal derivation see appendix), the utility specification is

$$V_t^s = E_t \left[ \int_t^\infty f(c_u^s, V_u^s) du \right] \quad (2)$$

<sup>15</sup>A cash flow is marketable if it is financed by a trading strategy  $X^s \in \mathcal{L}^2$ .

with the aggregator function  $f(\cdot)$  given by

$$f(c_u^s, V_u^s) = \frac{\beta (c_u^s)^\rho - \left(\beta + \frac{\rho}{1-\gamma} \lambda_u\right) [(1-\gamma) V_u^s]^{1-\gamma}}{\rho [(1-\gamma) V_u^s]^{1-\gamma} - 1} \quad (3)$$

The term  $\frac{1}{1-\rho}$  equals the *EIS*,  $\gamma$  controls risk aversion, and  $\beta$  specifies time discounting.

The term  $\frac{\rho}{1-\gamma} \lambda_u$  discounts future utility due to risk aversion towards uncertainty about the timing of death. Intuitively, the probability of dying early creates an incentive to save less than an infinitely-lived agent (or an agent with a fixed lifetime) because there is no bequest motive. In contrast, the possibility of surviving longer than life expectancy (state of high marginal utility) creates an incentive for precautionary savings. The former intuition corresponds to a positive discount of utility from future consumption, while the latter one implies that the agent cares relatively more about future consumption. It depends on the preference parameters whether the first or the second effect dominates and the discount is positive or negative. Under time additive utility only the first intuition matters while the second intuition is irrelevant. In the case of power utility agents are risk neutral towards uncertainty about the timing of death (Bommier (2003), Hugonnier et al. (2012)).

An agent's objective is to maximize the value function subject to the dynamic or equivalently the static budget constraint,

$$\sup_{\{c^s, X^s\} \in (\mathfrak{S} \times \mathcal{L}^2)} \left\{ V_s^s(c^s) = E_s \left[ \int_s^\infty f(c_u^s, V_u^s) du \right] \right\}, \quad s.t. \quad d\lambda_s, \quad dn_s \quad (\mathbf{P1})$$

## 3 The Equilibrium

### 3.1 Definition of Equilibrium

An equilibrium is defined by a set of adapted processes  $\{c, X, \pi\}$  such that (i) for every agent utility is maximized subject to the dynamic budget constraint, problem **(P1)** is solved  $\forall s$ , (ii) consumption markets clear,  $Y_t = C_t = \int_{-\infty}^t c_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds$ , and (iii) financial markets clear,  $1 = \int_{-\infty}^t \sum_{j=1}^d X_{j,t}^s n_s N_s e^{-\int_s^t \lambda_u du} ds$  and  $0 = \int_{-\infty}^t \left( W_t^s - \sum_{j=1}^d X_{j,t}^s P_t^{(j)} \right) n_s N_s e^{-\int_s^t \lambda_u du} ds$ .

### 3.2 General Remarks about the Equilibrium Analysis

The optimal consumption-to-wealth ratio is given by the function  $\psi_t(\lambda, n) = \frac{c_t^s}{W_t^s}$ , which is constant across cohorts. The functions  $F^{y,t,(i)}(\lambda, n, t)$ ,  $\forall i \in \{1, 2\}$  define the present value of labor income of a new born agent,  $\widehat{W}_t^t = \frac{Y_t}{N_t} \sum_{i=1}^2 F^{y,t,(i)}(\lambda, n, t)$ . These quantities are essential to determine the aggregate consumption share of the new born cohort,  $n_t \sum_{i=1}^2 F^{y,t,(i)} \psi_t = \frac{c_t^t n_t N_t}{C_t}$ . In equilibrium, the interest rate depends crucially on  $\frac{c_t^t n_t N_t}{C_t}$ . Moreover, the variation in  $\psi_t(\lambda, n)$  is a sufficient statistic for the variation in future consumption growth, which is needed for pricing risky assets.

To understand what affects the SDF  $\pi$ , I provide a short sketch of its derivation. I employ the market clearing condition in the consumption goods market, which must hold almost surely at all times and implies  $dY_t = dC_t$ . Growth in aggregate consumption depends on three terms: aggregation of optimal consumption growth of individuals, dying agents who abruptly stop their stream of consumption, and consumption of the new born cohort,

$$dC_t = \int_{-\infty}^t \frac{dc_t^s}{c_t^s} c_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds - \lambda_t C_t dt + c_t^t n_t N_t dt \quad (4)$$

Applying Ito's lemma to the first order condition of the optimal consumption choice problem (P1) implies that the dynamics of an individual's optimal consumption are independent of his cohort but dependent on the current birth and death rates and the dynamics of the SDF,  $\frac{dc_t^s}{c_t^s} = \Xi \left( \frac{d\pi_t}{\pi_t}, n_t, \lambda_t, dn_t, d\lambda_t \right)$ .<sup>16</sup> Plugging into (4), I can solve for the dynamics of the SDF,

$$\frac{d\pi_t}{\pi_t} = \Xi^{-1} \left( \underbrace{\frac{dY_t}{Y_t} + \lambda_t dt - \frac{c_t^t n_t N_t}{Y_t} dt}_{\text{consumption growth of existing agents}}, n_t, \lambda_t, dn_t, d\lambda_t \right) \quad (5)$$

According to (5), it is the Euler equation of existing agents that matters for pricing. Because shocks to birth and death rates cause a redistribution of aggregate consumption within the population (which affects the consumption growth rate of existing agents), they are crucial for pricing. It is important to understand that shocks to the distribution of consumption within the population is a pricing channel which differs from shocks to expected labor supply

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<sup>16</sup>Function  $\Xi \left( \frac{d\pi_t}{\pi_t}, n_t, \lambda_t, dn_t, d\lambda_t \right)$  represents the left hand side of equation (24) combined with (20).

and aggregate production/ consumption growth  $dY_t$ , that is, the usual shocks in the long-run risk literature. I show below in more detail that my results are driven by this consumption redistribution mechanism which implies changes to the expected consumption growth rate of existing agents. This is also in contrast to the mechanism in Brooks (2000, 2004) and Geanakoplos et al. (2004), where baby booms and busts imply changes in the average life expectancy and the representative agent's marginal propensity to consume.

**Lemma 1** *In equilibrium the stock price  $P_t$  is given by*

$$P_t = \underbrace{\frac{Y_t}{\psi_t(\lambda, n)}}_{\text{aggregate total wealth}} - \underbrace{Y_t \sum_{i=1}^2 \frac{F^{y,t,(i)}(\lambda, n, t)}{1 + \delta_i}}_{\text{aggregated present value of labor income}}$$

**Proof.** See appendix. ■

The stock price is determined in financial market clearing and is equal to aggregate financial wealth (total wealth minus present value of labor income).

**Lemma 2** *The expected excess return of an asset paying the stream of dividends  $D_t$  is*

$$E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt = - \frac{dP_t}{P_t} \frac{d\pi_t}{\pi_t}$$

**Proof.** See appendix. ■

An asset is compensated with a positive (negative) risk premium if its instantaneous returns are negatively (positively) correlated with the marginal utility process.

### 3.3 Constant Birth and Mortality Rates

As a benchmark, I consider the case of no demographic changes and no TFP shocks. I explore the differences in the dependence of the interest rate on birth and death rates. Comparative statics analyses help to get an intuition of how demographic changes affect the economy.

**Proposition 1** *Consider an economy as described. Suppose that the birth rate and the mortality rate are constant over time and the two assumptions (i)  $\mu^{(Y)} - (1 + \delta_i) n < r$*



$\forall i \in \{1, 2\}$  and (ii)  $\frac{\rho\gamma}{1-\gamma}\lambda + \beta > \rho r$  hold. There exists an equilibrium with a constant interest rate  $r$  which is a root to the equation

$$r = \underbrace{\beta}_{\text{time discounting}} + \underbrace{(1-\rho)}_{=\frac{1}{EIS}} \underbrace{\left[ \mu^{(Y)} + \lambda - \frac{nN_t c_t^t(r)}{C_t} \right]}_{\text{consumption growth of existing agents}} - \underbrace{\lambda}_{\text{annuity payoff}} + \underbrace{\frac{\rho}{1-\gamma}\lambda}_{\text{time discounting due to uncertain lifetime}} \quad (6)$$

with  $c_t^t(r)$  specified in the appendix. The SDF is non-stochastic and the return on equities is constant and equal to the risk-free interest rate  $r$ .

**Proof.** See appendix. ■

Conditions (i) and (ii) are required to ensure total wealth and the consumption-to-wealth ratio to be finite. The equity premium and the volatility of asset prices are zero because there is no uncertainty on the aggregate. The focus lies on the interest rate.

The interest rate in an equivalent economy populated by an infinitely-lived representative agent is  $r_* = \beta + (1-\rho)\mu_*^{(Y)}$  with  $\mu_*^{(Y)} = \mu^{(A)} + (1-a)\mu^{(K)}$ . It differs from the rate in the OLG economy by the term

$$r - r_* = \underbrace{(1-\rho) \left[ \lambda - \frac{nN_t c_t^t(r)}{C_t} \right]}_{(I)} + \underbrace{(1-\rho)a(n-\lambda)}_{(II)} - \underbrace{\lambda}_{(III)} + \underbrace{\frac{\rho}{1-\gamma}\lambda}_{(IV)} \quad (7)$$

(I) Following equation (5),  $r \neq r_*$  holds because aggregate consumption growth (consumption of the infinitely-lived agent) differs from consumption growth of existing agents in the OLG economy. Deaths of existing agents have a positive effect on consumption growth of surviving agents because survivors have to share total production output with less peers. Births of new agents mean a decline in the older cohorts' share of aggregate consumption and their consumption growth because new agents claim a fraction of aggregate consumption. The death rate increases and the birth rate decreases the interest rate compared to the rate found in the infinitely-lived agent economy. This channel captures the consumption distribution within the population and its effects on consumption growth of existing agents, which is the essence of this paper.

(II) In an OLG economy the growth rate of total output depends on population and labor supply growth  $(n_t - \lambda_t) dt$ . A high birth rate causes total output to grow fast which positively affects the interest rate. A high death rate results the opposite. In reality this labor supply channel is expected to be weaker than in my stylized model because newborns do not enter the workforce immediately but only many years after birth. Since (II) works into the opposite direction of (I), many of my results would be even stronger if the labor supply channel was assumed to be negligible.

(III) The insurance premium has the same impact on an agent's wealth dynamics (equation (1)) and optimal consumption path as the risk-free interest rate. As the insurance premium works as a substitute to the interest rate, in equilibrium the interest rate is not required to be as high as in a world without insurance payments. Accordingly, the interest rate in an OLG economy is lower than the rate in an infinitely-lived agent economy due to annuity contracts.

(IV) In an OLG economy an agent faces risk aversion towards uncertainty about the length of his life. There is a trade-off between how much savings an agent requires for consumption until death and how much he is willing to risk when facing the probability of an early death. The first reason tells that an agent saves more under lifetime uncertainty than if he knew the exact time of death because there is a chance that he will live an unexpectedly long life and his marginal utility is high in future (precautionary savings). The latter reason says that an agent consumes bigger parts of his wealth early in time under lifetime uncertainty because he faces a probability that he will not be alive to consume his savings in future and draw utility from it. In an OLG economy agents save more (less) and the interest rate is smaller (larger) than in an infinitely-lived agent economy, if the discount of future utility due to agents' risk aversion towards uncertainty about the timing of death is negative (positive) and  $\frac{\rho}{1-\gamma}\lambda < (>) 0$  holds (see section 2.4).

**Lemma 3** *Suppose (i)  $\gamma > 1$ , (ii)  $\frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} > 1 - \frac{1+\frac{1-a}{a}\gamma}{1-\gamma} \frac{\lambda}{n}$ , and the technical conditions in the appendix hold. There exists a cut-off value  $\overline{EIS}^{(r)}$  such that the condition  $EIS < \overline{EIS}^{(r)}$  suffices for the interest rate in an OLG economy to be smaller than the rate in an equivalent economy populated by an infinitely-lived representative agent ( $r < r_*$ ).*

**Proof.** See appendix.<sup>17</sup> ■

Condition (ii) requires life-cycle earnings to be sufficiently decreasing in age. For  $B_2 = 0$ , the condition is  $\delta_1 > -\frac{1+\frac{1-a}{\gamma}\lambda}{1-\gamma} \frac{\lambda}{n}$ . Agents have to save for retirement if life-cycle earnings are decreasing in age, and a big supply in savings implies a low interest rate (Blanchard, 1985).

Under a strong motive for consumption smoothing (small  $\rho$ , *EIS*), an agent seeks to flatten his consumption path over the life-cycle, which corresponds to a large consumption-to-wealth ratio and few savings (few financial wealth).<sup>18</sup> Given a large consumption-to-wealth ratio, the new born cohort claims a big fraction of aggregate consumption and  $\frac{nN_t c_t^t}{C_t}$  is large enough to ensure expression (7) to be negative and  $r < r_*$  to hold.

To get an intuition how a change in the birth rate affects the interest rate I take the first derivative of  $r$  with respect to  $n$ ,

$$\frac{\partial r}{\partial n} = (1 - \rho) \frac{\underbrace{a}_{\text{faster growth in total output}} - \underbrace{\frac{N_t c_t^t}{C_t}}_{\text{increase in new born generation}} + \underbrace{nN_t \frac{\partial}{\partial n} \left( -\frac{c_t^t}{C_t} \right)}_{\text{decrease in consumption share of individual new born agent}}}{1 + (1 - \rho) nN_t \frac{\partial}{\partial r} \left( \frac{c_t^t(r)}{C_t} \right)} \quad (8)$$

with  $\frac{\partial}{\partial n} \left( -\frac{c_t^t}{C_t} \right) = \frac{1}{N_t} \sum_{i=1}^2 \frac{-a+1+\delta_i}{r-\mu^{(Y)}+(1+\delta_i)n} F^{y,(i)} \psi$  ( $F^{y,(i)}$  and  $\psi$  are specified in the appendix).

To ensure that the denominator in equation (8) is positive, I let  $f^{(r)}(x) = -x + \beta + (1 - \rho) \left[ \mu^{(Y)} + \lambda - \frac{nN_t c_t^t(x)}{C_t} \right] - \frac{1-\gamma-\rho}{1-\gamma} \lambda$  and suppose  $f^{(r)}(x)$  to be decreasing at  $x = r$ . The requirement on the slope of  $f^{(r)}(\cdot)$  is not a strong assumption. For instance, under the conditions in Lemma 3 there exists  $r \leq r_*$  that satisfies the requirement.

There are three offsetting effects of the birth rate on the interest rate. First, the workforce and production output grow faster as the birth rate increases, which has a positive impact on the consumption growth of existing agents ( $r \nearrow$ ). As mentioned earlier, this labor supply channel is expected to be weaker in reality than in my stylized model.

Second, holding  $c_t^t$  constant, an increase in the size of the new born generation causes the new born cohort's claim on aggregate consumption to rise. The interest rate is negatively

<sup>17</sup>The conditions are sufficient but not necessary. The technical conditions in the appendix are easy to satisfy and I do not worry about them. The same is true for all Lemmas that follow.

<sup>18</sup>I suppose that an agent's consumption grows with age ( $\frac{c_t^s}{c_s^s} > 1$ , for  $\forall s < t$ ). This is a natural assumption and is true for a large enough growth in GDP.  $\frac{c_t^s}{c_s^s} > 1$  implies  $\frac{\partial}{\partial(-\rho)} \left( \frac{c_t^s}{c_s^s} \right) < 0$ ,  $\forall s < t$ .

affected by an increase in the aggregate consumption share of the new born cohort as it slows down consumption growth of existing agents ( $r \searrow$ ).

Third, labor income is declining in the birth rate.<sup>19</sup> A boost in the workforce causes the marginal productivity of labor and wages to drop. A new born agent's total wealth is equal to the present value of his life-cycle earnings, which is sensitive to changes in labor income. In contrast, total wealth of an old agent is less prone to labor income shocks because a large fraction of his endowment consists of financial wealth. A negative shock to labor income implies a relatively stronger decline in total wealth of a new born agent than in total wealth of an old agent. Since the consumption-to-wealth ratio remains unchanged, the aggregate consumption share of a new born agent declines, and the consumption growth of existing agents increases as the birth rate rises ( $r \nearrow$ ).

**Lemma 4** *Suppose  $\gamma > 1$  and the technical conditions in the appendix hold. There exists a cut-off value  $\overline{EIS}^{(n)}$  such that the condition  $EIS < \overline{EIS}^{(n)}$  suffices for  $\frac{\partial r}{\partial n} < 0$ .*

**Proof.** See appendix. ■

Intuitively, if I decrease the  $EIS$ , agents save less financial wealth. An old agent's total wealth becomes more sensitive to labor income shocks, and the relative difference in a drop of total wealth of old versus young agents due to an increase in the birth rate and a decline in labor income gets smaller. Accordingly the magnitude of  $\frac{\partial}{\partial n} \left( -\frac{c_t^t}{C_t} \right)$  is small if the  $EIS$  is small.

A strong motive for consumption smoothing implies a large consumption-to-wealth ratio and  $c_t^t$  (much consumption at birth). A large  $c_t^t$  ensures that (on the margin) the additional new born agent consumes more than what he "produces" ( $a \frac{Y_t}{N_t} - c_t^t = \left[ a - \frac{N_t c_t^t}{C_t} \right] \frac{Y_t}{N_t} < 0$ ).

As a result, if the  $EIS$  is small enough, there is one key channel through which a change in the birth rate affects the interest rate. A rise in the birth rate causes more new born agents to enter the economy and to claim a bigger fraction of aggregate consumption. Accordingly, consumption growth of existing agents slows down and the interest rate declines.

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<sup>19</sup>I assume that  $1 + \delta_1 > a$ , so that the positive effect of an increase in output and aggregate labor income (due to an increase in  $n$ ) is dominated by the negative effect of a decrease in marginal productivity of labor and productivity of agents. This is satisfied for a decreasing life-cycle earnings profile. For now I ignore feedback effects through the interest rate.

Taking the first derivative of  $r$  with respect to  $\lambda$  yields,

$$\frac{\partial r}{\partial \lambda} = \frac{(1-\rho) \left[ \underbrace{-a}_{\text{slower growth in output}} + \underbrace{1}_{\text{surviving agents share aggregate consumption with less peers}} - \underbrace{nN_t \frac{\partial}{\partial \lambda} \left( \frac{c_t^t}{C_t} \right)}_{\text{change in consumption share of new born cohort}} \right] - \underbrace{1}_{\text{higher payments from annuities}} + \underbrace{\frac{\rho}{1-\gamma}}_{\text{time discounting due to lifetime uncertainty}}}{1 + (1-\rho) nN_t \frac{\partial}{\partial r} \left( \frac{c_t^t(r)}{C_t} \right)} \quad (9)$$

with  $\frac{\partial}{\partial \lambda} \left( \frac{c_t^t}{C_t} \right) = -\frac{1}{N_t} \sum_{i=1}^2 \frac{a}{r-\mu^{(Y)}+(1+\delta_i)n} F^{y,(i)} \psi + \frac{1}{N_t} \frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \sum_{i=1}^2 F^{y,(i)}$ .

The expression  $(1-\rho)(-a+1) - 1 + \frac{\rho}{1-\gamma}$  summarizes the following four effects. An increase in mortality (i) decreases the workforce and production output ( $r \searrow$ ), (ii) increases the growth in consumption of survivors ( $r \nearrow$ ), (iii) implies a high insurance premium ( $r \searrow$ ), and (iv) increases the magnitude of time discounting of future utility due to risk aversion towards lifetime uncertainty (if  $\frac{\rho}{1-\gamma} > (<) 0$ , then  $r \nearrow (\searrow)$ ).

Keeping the interest rate constant, the consumption share of the new born cohort changes with fluctuations in the death rate for two reasons. First, an increase in the death rate causes production output and the present value of labor income to decline. Following the argument in the discussion of a change in the birth rate, the new born cohort's aggregate consumption share decreases because of the negative labor income shock (term  $-\sum_{i=1}^2 \frac{an}{r-\mu^{(Y)}+(1+\delta_i)n} F^{y,(i)} \psi$ ,  $r \nearrow$ ). Second, as mortality increases agents discount future utility more positively (negatively) and increase (decrease) their consumption-to-wealth ratio, if  $\frac{\rho}{1-\gamma} > (<) 0$ . Accordingly, the consumption level at birth and the aggregate consumption share of the new born cohort increase (decline) (term  $n \frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \sum_{i=1}^2 F^{y,(i)}$ ,  $r \searrow (\nearrow)$ ).

**Lemma 5** *Suppose  $\gamma > 1$  and the technical conditions in the appendix hold. There exists a cut-off value  $\overline{EIS}^{(\lambda)}$  such that the condition  $EIS < \overline{EIS}^{(\lambda)}$  suffices for  $\frac{\partial r}{\partial \lambda} > 0$ .*

**Proof.** See appendix. ■

Intuitively, the term  $(1-\rho)(-a+1) - 1$  is positive if the  $EIS$  is small enough.

As I shrink the  $EIS$ , the aggregate consumption share of the new born cohort becomes less sensitive to changes in mortality. The intuition is similar to the discussion on changes in the birth rate.

For  $\gamma > 1$  and  $EIS < 1$ , agents (positively) discount future utility because of risk aversion towards lifetime uncertainty. The discount and the (positive) effect on the interest rate are large if the  $EIS$  is small.

As a result, for a small enough  $EIS$ , I end up with the following two key driving forces that causes the interest rate to be increasing in mortality. As the death rate increases, agents face a higher probability of dying early and discount future utility stronger. As a consequence they consume more of their wealth early in life and save less financial wealth, which causes the interest rate to increase in equilibrium. In addition, aggregate consumption has to be split among less survivors and the consumption growth of existing agents and the interest rate increase.

For the remaining discussion I use the results:  $\frac{\partial r}{\partial n} < 0$  and  $\frac{\partial r}{\partial \lambda} > 0$ .

From Lemma 1 and 2 it is straightforward to derive the Gordon growth model

$$P_t = \frac{D_t}{r - \mu^{(Y)}} \quad (10)$$

The stock price is increasing in the birth rate. More agents enter the workforce and growth in total output and future stock dividends increase. In addition, the discount rate declines.<sup>20</sup>

$$\frac{\partial P}{\partial n} = \frac{\overbrace{a}^{\text{increase in growth of future dividends}} - \overbrace{\frac{\partial r}{\partial n}}^{\text{decrease in discount rate}}}{r - \mu^{(Y)}} P_t \quad (11)$$

The stock price is decreasing in the death rate. An increase in mortality causes growth in output and future dividends to decline and the discount rate to increase.

$$\frac{\partial P}{\partial \lambda} = \frac{\overbrace{-a}^{\text{decrease in growth of future dividends}} - \overbrace{\frac{\partial r}{\partial \lambda}}^{\text{increase in discount rate}}}{r - \mu^{(Y)}} P_t$$

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<sup>20</sup>Equivalently, the stock price increases because the demand for stocks hikes driven (i) by an increase in aggregate savings (boost in present value of aggregate labor income) and (ii) by a drop in interest paid by the risk-free asset, which makes stocks more attractive as an alternative investment to the riskless bond market.

The consumption-to-wealth ratio  $\psi$  depends crucially on agents' preferences. Time discounting of future utility has a positive impact on  $\psi$ . Depending on the dominance of either income or substitution effect ( $EIS < 1$  or  $EIS > 1$ ), the rate of return on wealth (from bonds and annuities)<sup>21</sup> is positively or negatively related to  $\psi$ .

$$\psi = \frac{\overbrace{1}^{=EIS}}{1-\rho} \left[ \underbrace{\beta}_{\text{time discounting}} + \underbrace{\frac{\rho}{1-\gamma}\lambda}_{\text{discounting due to uncertain lifetime}} - \underbrace{\rho(\underbrace{r}_{\text{interest rate}} + \underbrace{\lambda}_{\text{annuity payoff}})}_{\text{trade off between income and substitution effect}} \right] \quad (12)$$

The consumption-to-wealth ratio is decreasing in the birth rate if  $EIS < 1$ . As the interest rate declines the agent experiences a negative income shock, savings grow slower and the agent can afford less consumption in future. If the agent cares enough about consumption smoothing, he will save more and consume less today to compensate for the negative shock to future endowment/ consumption (income effect dominates substitution effect).

$$\frac{\partial \psi}{\partial n} = -\frac{\rho}{1-\rho} \frac{\partial r}{\partial n}$$

The consumption-to-wealth ratio is increasing in the death rate if  $EIS < 1$  and  $\gamma > 1$ . Agents discount utility from future consumption stronger due to an increase in mortality ( $\frac{\rho}{1-\gamma} > 0$ ) and prefer to consume a larger part of their wealth early in life. In addition, future consumption becomes cheaper as the interest rate and the insurance premium increase and agents instantly consume part of the "newly gained income" (income effect).

$$\frac{\partial \psi}{\partial \lambda} = \frac{1}{1-\rho} \left[ \frac{\rho}{1-\gamma} - \rho \left( \frac{\partial r}{\partial \lambda} + 1 \right) \right]$$

### 3.4 Regime Shifts in the Birth Rate: Two State Markov Switching Model

I keep mortality constant and let the birth rate randomly jump between two levels. For illustrative purposes I still assume no TFP shocks. Random switches capture the long-run

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<sup>21</sup>Or equivalently the inverse of the price of future consumption.

profile of baby boom and bust transitions found in US birth rate data. Once the birth rate process is stochastic, long-run risk is introduced in the economy and I can explore the impact of demographic uncertainty on pricing stocks.

I let the birth rate process be  $dn_t = s^{(n)}dS_t^{(n)}$ , with  $s^{(n)} = n_H - n_L$ .  $S_t^{(n)} \in \{1, 0\}$  follows a two state, continuous time Markov switching process with transition probability matrix between time  $t$  and  $t + \Delta$  given by  $\Theta^{(S,n)}(\Delta) = \begin{pmatrix} 1 - \bar{\theta}_H^{(n)}\Delta & \bar{\theta}_H^{(n)}\Delta \\ \bar{\theta}_L^{(n)}\Delta & 1 - \bar{\theta}_L^{(n)}\Delta \end{pmatrix}$ . The birth rate switches between the two values  $n_t \in \{n_L, n_H\}$ . Because the model has only two states, key variables, which depend on the birth rate, switch between two distinct values.

There are minor changes to the utility specification as described in the appendix. Agents' objectives stay the same.

**Proposition 2** *Suppose an economy as described. In general, there exists an equilibrium with a SDF  $\pi$  that follows a stochastic process driven by the same two state Markov switching process  $S^{(n)}$  as the birth rate. The equilibrium interest rate  $r_t$  switches between two distinct levels,  $r_t \in \{r_L^{(n)}, r_H^{(n)}\}$  defined by*

$$r_j^{(n)} = \beta + (1 - \rho) \left[ \mu_j^{(Y,n)} + \lambda - n_j \sum_{i=1}^2 F_j^{y,(i),(n)} \psi_j^{(n)} \right] - \frac{1 - \gamma - \rho}{1 - \gamma} \lambda \\ - \bar{\theta}_j^{(n)} \left( \left( \frac{\psi_k^{(n)}}{\psi_j^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) + \frac{1 - \gamma - \rho}{1 - \gamma} \bar{\theta}_j^{(n)} \left( \left( \frac{\psi_k^{(n)}}{\psi_j^{(n)}} \right)^{-\frac{1-\gamma}{\rho}} - 1 \right)$$

$\forall (j, k) \in \{(L, H), (H, L)\}$ , with  $r_t | [n_t = n_L] = r_L^{(n)}$  and  $r_t | [n_t = n_H] = r_H^{(n)}$ . The market price of risk jumps between two distinct values,  $\kappa_t \in \{\kappa_L^{(n)}, \kappa_H^{(n)}\}$  given by

$$\kappa_j = - \left( \left( \frac{\psi_k^{(n)}}{\psi_j^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right)$$

$\forall (j, k) \in \{(L, H), (H, L)\}$ , with  $\kappa_t | [n_t = n_L] = \kappa_L^{(n)}$  and  $\kappa_t | [n_t = n_H] = \kappa_H^{(n)}$ . Demographic uncertainty is priced in equilibrium and the equity premium is non-zero. In the special case of power utility, the SDF follows a locally deterministic process and the equity premium disappears. The functions  $F_L^{y,(1),(n)}$ ,  $F_L^{y,(2),(n)}$ ,  $F_H^{y,(1),(n)}$ ,  $F_H^{y,(2),(n)}$ ,  $\psi_L^{(n)}$ , and  $\psi_H^{(n)}$  are determined



in a system of 6 non-linear equations provided in the appendix.

**Proof.** See appendix. ■

To understand why the market price of risk is non-zero in the general case of recursive utility and zero in the special case of CRRA preferences, it is best to look at the optimal consumption path for an individual agent

$$\underbrace{\frac{C_t^s}{C_s^s}}_{\text{variation in current consumption growth}} = e^{\frac{1}{1-\rho} \int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} \underbrace{\left( \frac{V_t^s(\psi_t, \pi_t)}{V_s^s(\psi_s, \pi_s)} \right)^{\frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}}}_{\text{variation in future consumption growth}} \underbrace{\left( \frac{\pi_t}{\pi_s} \right)^{-\frac{1}{1-\rho}}}_{\text{variation in marginal utility process}} \quad (13)$$

Suppose that the SDF had zero quadratic variation. Because the value function  $V^s$  features a discontinuity at the time of a regime shift, optimal consumption must jump as a regime shift occurs. As each agent is affected the same (dynamics of the value function are independent of the cohort), the aggregate consumption process features jumps. But, the aggregate supply of consumption goods has no discontinuities and markets could not possibly clear ( $dY_t \neq dC_t$ ). To resolve the problem it must be that the SDF is driven by a jump process such that all discontinuities in  $V^s$  are exactly offset and optimal consumption of the individual follows a locally deterministic process (compare equation (27)).

In other words, the SDF is defined as a deterministic multiple of the marginal utility process (Gateau derivative of the utility function), which depends on current and future consumption. As current consumption follows a locally deterministic process it does not introduce any stochastics in the marginal utility process and its dynamics are irrelevant for the derivation of the equity premium. The variation in the consumption-to-wealth ratio is a sufficient statistic of the variation in future consumption growth. As a result the market price of risk is a non-linear function of the ratio  $\frac{\psi_H}{\psi_L}$ .

In the case of CRRA preferences optimal consumption does not depend on the agent's value function,  $c_t^s = c_s^s e^{-\frac{1}{\gamma} \beta(t-s)} \left( \frac{\pi_t}{\pi_s} \right)^{-\frac{1}{\gamma}}$ . The consumption path of an individual agent, who survives over the next instant in time, is locally deterministic, and the SDF must not be stochastic to ensure market clearing. The market price of risk is zero and pricing of risky

assets is not affected by stochastic changes in the birth rate.<sup>22</sup>

**Lemma 6** *Suppose  $\gamma \in (1, 1 - \rho)$  ( $\rho < 0$ ) and the technical conditions in the appendix hold. There exists a cut-off value  $\overline{EIS}_1^{(n)}$  such that the condition  $EIS < \overline{EIS}_1^{(n)}$  suffices for the interest rate during a period characterized by a high birth rate (baby boom) to be lower than the rate during times of a low birth rate (baby bust),  $r_L^{(n)} > r_H^{(n)}$ . The consumption-to-wealth ratio is decreasing and the magnitude of the market price of risk is increasing in the birth rate,  $\psi_L^{(n)} > \psi_H^{(n)}$  and  $|\kappa_L^{(n)}| < |\kappa_H^{(n)}|$ .*

**Proof.** See appendix. ■

The intuition for  $r_L^{(n)} > r_H^{(n)}$  and  $\psi_L^{(n)} > \psi_H^{(n)}$  is equivalent to the argument provided in the static case.

The stock price is a state dependent multiple of GDP (Lemma 1),

$$P_t^{(n_j)} = P_t | [n_t = n_j] = Y_t \left[ \frac{1}{\psi_j^{(n)}} - \sum_{i=1}^2 \frac{F_j^{y,(i),(n)}}{1 + \delta_i} \right]$$

$\forall j \in \{L, H\}$ . The growth rate is stochastic and conditional on the state of the world

$$\frac{dP_t}{P_t} \Big| [n_t = n_j] = \mu_j^{(Y,n)} dt + \frac{Y_t}{P_t^{(n_j)}} \left[ \frac{1}{\psi_k^{(n)}} - \frac{1}{\psi_j^{(n)}} - \sum_{i=1}^2 \frac{F_k^{y,(i),(n)} - F_j^{y,(i),(n)}}{1 + \delta_i} \right] \Big| dS_t^{(n)}$$

$\forall (j, k) \in \{(L, H), (H, L)\}$ . GDP follows a locally deterministic process because demographic uncertainty introduces only long-run risk in the economy. In contrast, the stock price has non-zero quadratic variation since stocks are forward looking and incorporate changes in growth prospects of the economy (information about future growth in dividends and future changes in the discount rate). Demographic uncertainty introduces in a natural way excess volatility in asset returns over the variation in aggregate consumption growth.

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<sup>22</sup>Another way to understand that the SDF is locally deterministic is by noticing that in case of time additive utilities, marginal utility depends solely on current consumption but not future consumption.

Following Lemma 2, the equity premium is

$$\frac{1}{dt} E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t \Big| [n_t = n_j] = \bar{\theta}_j^{(n)} \frac{Y_t}{P_t^{(n_j)}} \left( \left( \frac{\psi_k^{(n)}}{\psi_j^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \cdot \left[ \frac{1}{\psi_j^{(n)}} - \frac{1}{\psi_k^{(n)}} + \sum_{i=1}^2 \frac{F_k^{y,(i),(n)} - F_j^{y,(i),(n)}}{1 + \delta_i} \right]$$

$\forall (j, k) \in \{(L, H), (H, L)\}$ . Demographic uncertainty is priced in equilibrium and the equity premium switches between two distinct values. In the special case of power utility with  $1 - \gamma - \rho = 0$  the equity premium is zero.

**Lemma 7** *Suppose  $\frac{1-\gamma-\rho}{\rho} < 0$  and the technical conditions in the appendix hold. There exists a cut-off value  $\overline{EIS}_2^{(n)}$  such that the condition  $EIS < \overline{EIS}_2^{(n)}$  suffices for the equity premium to be positive in both states of the world.*

**Proof.** See appendix. ■

A key result is the inequality  $\left| \frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}} \right| - \left| \sum_{i=1}^2 \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1 + \delta_i} \right| > 0$ . The stock price moves into the opposite (same) direction as the consumption-to-wealth ratio (total wealth). This is consistent with the developed intuition from the comparative statics analyses in the previous section.

By equation (18) and (27),  $(1 - \gamma) V_t^s$  is decreasing (increasing) in the consumption-to-wealth ratio if  $\frac{1-\gamma}{\rho} > (<) 0$ . Combining equations (3) and (14), marginal utility is decreasing (increasing) in  $(1 - \gamma) V_t^s$  if  $\frac{1-\gamma-\rho}{1-\gamma} < (>) 0$ . Accordingly, condition  $\frac{1-\gamma-\rho}{\rho} < 0$  is necessary for changes in the stock price and the SDF to be negatively correlated. The payoff of stocks is low (high) in states of the world when marginal utility is high (low) and more (less) wealth is desired, and investors require a positive compensation for holding stocks.

**Lemma 8** *Suppose  $\bar{\theta}_H^{(n)} > (<) \bar{\theta}_L^{(n)}$ , and the conditions in Lemma 6 and 7 hold. There exists a cut-off value  $\overline{EIS}_3^{(n)}$  such that the condition  $EIS < \overline{EIS}_3^{(n)}$  ensures that the equity premium is larger (lower) during a baby boom than the premium during times of a low birth rate.*

**Proof.** See appendix. ■

The parameters  $\bar{\theta}_L^{(n)}$  and  $\bar{\theta}_H^{(n)}$  determine the probability of a regime switch conditional on being in a low and high birth rate state. The ratio  $\frac{\bar{\theta}_H^{(n)}}{\bar{\theta}_L^{(n)}}$  describes the ratio between the instantaneous risk in stock returns during a high and a low birth rate state. The equity premium is higher during a high birth rate state, if a baby boom lasts on average shorter (the risk for a regime switch is higher) compared to a low birth rate state. US population data over the last 100 years reveal that this seems true.

Consistent with the result in the previous section, Lemma 6 and 7 imply that the stock price is increasing in the birth rate,

$$P_t^{(n_H)} - P_t^{(n_L)} = Y_t \left[ \frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}} - \sum_{i=1}^2 \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1 + \delta_i} \right] > 0$$

A baby boom causes the stock market to boom and the growth rates of the stock price and dividends are high. There is an immediate stock market bust (negative jump) as soon as the baby boom stops (at the time of a regime shift from a high to a low birth rate). The model implies a slow growth in asset prices and in dividends when the birth rate is low, but it does not imply a major stock market bust as the baby boom generation "retires".<sup>23</sup> This follows because all key quantities are Markov processes and immediately adjust at the time of a regime shift, when agents are surprised by a change in the economic environment.

The result that the retirement of the baby boom generation does not have an impact on asset prices can be challenged on different grounds. First, capital accumulation (with convex adjustment costs) is likely to alter the result because a slow-down in population growth causes disinvestment. Because of convex adjustment costs there is not one immediate cut in the capital stock as the birth rate drops, but disinvestment continues over a long horizon and the desired capital stock is approached slowly (Abel, 2003).

Second, the specification of the life-cycle earnings profile enforces by construction the Markov property of aggregate supply of labor efficiency units, which implies the consumption-to-wealth ratio and total wealth to be Markov processes. A choice of a more general path for life-cycle earnings (e.g. newborns do not work until age 20 or some discontinuity in

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<sup>23</sup>Retirement can be defined as the age when an agent is endowed with less labor efficiency units than some level  $x$ , for instance the data in Hubbard et al. (1993) suggest that people at age 65 earn about 35-40% of the maximal labor income over the life-cycle.

labor supply at time of retirement) causes the consumption-to-wealth ratio and total wealth to be history-dependent functions (in particular I have to keep track which cohort enters the workforce or retires at which point in time). The introduction of age-dependent death rates also causes the variables to depend on the past. If the consumption-to-wealth ratio and total wealth are not Markov processes, then asset prices depend on past observations of the birth rate, and baby booms and busts have implications on asset prices for a long time after a regime shift occurs. As a result the model's answer to the question whether the retirement of the baby boomers causes a stock market meltdown has to be treated with caution. Brooks (2000, 2004) and Geanakoplos et al. (2004) complement my model with respect to these issues and deliver an answer to this question. My model is setup to explore how demographic uncertainty affects asset pricing in addition to the effects documented by Brooks (2000, 2004) and Geanakoplos et al. (2004).

In the appendix I derive another Markov switching model where I fix the birth rate while letting the mortality rate switch between a high and a low level. The analysis is equivalent to the above discussion.

### 3.5 General Model with Brownian Uncertainty: Calibration

I illustrate the quantitative magnitude of my results in a calibration exercise. I model the birth rate and the death rate as Brownian diffusion processes,

$$\begin{aligned} dn_t &= \mu_t^{(n)} dt + \sigma_t^{(n)} d\widetilde{W}_t = n_t \mu^{(n)} dt + n_t \sigma^{(n)} d\widetilde{W}_t \\ d\lambda_t &= \mu_t^{(\lambda)} dt + \sigma_t^{(\lambda)} d\widetilde{W}_t = \lambda_t \mu^{(\lambda)} dt + \lambda_t \sigma^{(\lambda)} d\widetilde{W}_t \end{aligned}$$

$\mu^{(i)}$  and  $\sigma^{(i)}$  denote constant drift and diffusion terms of process  $i \in \{n, \lambda\}$ , and  $\widetilde{W}_t$  is a  $d$  dimensional Brownian motion. Demographic literature suggests that a geometric Brownian motion describes in particular death rate data well (Lee and Carter, 1992). I introduce TFP shocks to the economy and let

$$\frac{dA_t}{A_t} = \mu^{(A)} dt + \sigma^{(A)} d\widetilde{W}_t$$

**Proposition 3** *Suppose an economy as described. In general, there exists an equilibrium with a SDF  $\pi$  with the dynamics*

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \kappa_t d\widetilde{W}_t$$

The interest rate  $r_t$  is

$$r_t = \beta + (1 - \rho) \left[ \mu_t^{(Y)} + \lambda_t - n_t \sum_{i=1}^2 F^{y,t,(i)} \psi_t \right] - \frac{1 - \gamma - \rho}{1 - \gamma} \lambda_t$$

$$+ \underbrace{\frac{1 - \gamma - \rho}{2\rho} \frac{\sigma_t^{(\psi)}}{\psi_t} \left( \frac{\sigma_t^{(\psi)}}{\psi_t} \right)^T - \frac{\gamma(2 - \rho)}{2} \sigma^{(A)} (\sigma^{(A)})^T - \frac{1 - \gamma - \rho}{\rho} \sigma^{(A)} \left( \frac{\sigma_t^{(\psi)}}{\psi_t} \right)^T}_{\text{precautionary savings}}$$

and the market price of risk  $\kappa_t$  takes the form

$$\kappa_t = \underbrace{\frac{1 - \gamma - \rho}{\rho} \frac{\sigma_t^{(\psi)}}{\psi_t}}_{\text{pricing of long-run risk/ demographic uncertainty}} + \underbrace{\gamma \sigma^{(A)}}_{\text{pricing of instantaneous risk}}$$

Demographic uncertainty is priced in equilibrium except in the special case of power utility and  $\sigma^{(A)} (\sigma^{(n)})^T = \sigma^{(A)} (\sigma^{(\lambda)})^T = 0$ . The functions  $F^{y,t,(1)}(\lambda, n, t)$ ,  $F^{y,t,(2)}(\lambda, n, t)$ , and  $\psi_t(\lambda, n)$  are determined in a system of 3 differential equations provided in the appendix.

**Proof.** See appendix. ■

Precautionary savings induced by TFP shocks and demographic uncertainty have a negative impact on the interest rate for the parameterization considered in the calibration exercise below.

Following Lemma 1, the stock price volatility is

$$\frac{\sigma_{P,t}^{(P)}}{P_t} = \sigma^{(A)} + \frac{Y_t}{P_t} \left[ -\frac{\sigma_t^{(\psi)}}{\psi_t^2} - \sum_{i=1}^2 \frac{\sigma_t^{(F^{y,t,(i)})}}{1 + \delta_i} \right]$$

The volatility is stochastically changing over time, and there is instantaneous excess volatility of financial assets over consumption growth.

The equity premium follows from Lemma 2,

$$\begin{aligned}
E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt &= \gamma \sigma^{(A)} (\sigma^{(A)})^T - \frac{1 - \gamma - \rho}{\rho} \frac{Y_t \sigma_t^{(\psi)}}{P_t \psi_t} \left( \frac{\sigma_t^{(\psi)}}{\psi_t^2} + \sum_{i=1}^2 \frac{\sigma_t^{(F^{y,(i)})}}{1 + \delta_i} \right)^T \\
&\quad + \frac{1 - \gamma - \rho}{\rho} \frac{\sigma_t^{(\psi)} (\sigma^{(A)})^T}{\psi_t} - \frac{Y_t}{P_t} \gamma \sigma^{(A)} \left( \frac{\sigma_t^{(\psi)}}{\psi_t^2} + \sum_{i=1}^2 \frac{\sigma_t^{(F^{y,(i)})}}{1 + \delta_i} \right)^T
\end{aligned}$$

If TFP shocks are independent of the demographic variables, then the argument of the previous discussion carries over to explain why demographic uncertainty is priced under recursive preferences but not in case of power utility. If  $\sigma^{(A)} (\sigma^{(n)})^T \neq 0$  or  $\sigma^{(A)} (\sigma^{(\lambda)})^T \neq 0$ , then demographic shocks are priced even in the case of power utility. In contrast to the conditions in the previous sections the equity premium may be positive (negative) in the case of  $\frac{1-\gamma-\rho}{\rho} > (<) 0$ , depending on the size and sign of the correlation between TFP and demographic shocks. The data suggests that the correlation is indeed non-zero (see calibration below).

**Lemma 9** *Suppose  $\sigma^{(A)} (\sigma^{(n)})^T = \sigma^{(A)} (\sigma^{(\lambda)})^T = 0$  and  $\gamma \in (1, 1 - \rho)$  ( $\rho < 0$ ). There exists  $\overline{EIS}(n_t, \lambda_t)$  such that  $EIS < \overline{EIS}(n_t, \lambda_t)$  suffices for the interest rate to be decreasing in the birth rate and increasing in the mortality rate and the equity premium to be positive.*

**Proof.** See appendix. ■

The result is similar to the findings in the earlier discussion, but weaker.  $\overline{EIS}(n_t, \lambda_t)$  depends on the current level of the birth rate and the death rate.

For the calibration below I also want to compute 1-year, 5-year and 30-year bond yields. The  $q$ -year bond yield at time  $t$  is given by no arbitrage,

$$y_{t,q} = -\frac{1}{q} \ln \left( E_t \left[ \frac{\pi_{t+q}}{\pi_t} \right] \right)$$

Yields are hard to compute analytically in my model but easy to approximate numerically using Monte Carlo simulations.

I calibrate the model to illustrate the quantitative importance of demographic changes. To match the first two unconditional moments of US population statistics I set  $\mu^{(n)} =$

$-0.0055$ ,  $\sigma^{(n)} = 0.0358$ ,  $\mu^{(\lambda)} = -0.0071$ ,  $\sigma^{(\lambda)} = 0.051$ , and  $\text{corr}\left(\frac{dn_t}{n_t}, \frac{d\lambda_t}{\lambda_t}\right) = -0.225$ . I set  $\text{corr}\left(\frac{dn_t}{n_t}, \frac{dA_t}{A_t}\right) = -0.2$  and  $\text{corr}\left(\frac{d\lambda_t}{\lambda_t}, \frac{dA_t}{A_t}\right) = 0.35$ , which roughly matches my estimates of the unconditional correlations between real per capita GDP growth and changes in birth respectively death rates (see also for instance Hanewald (2010), Jones and Tertilt (2007)). However, some parts of the literature suggest that lagged fertility is pro-cyclical (see for instance Sobotka, Skirbekk and Philipov, 2010). As a robustness check I repeat the calibrations with  $\text{corr}\left(\frac{dn_t}{n_t}, \frac{dA_t}{A_t}\right) = 0.2$ ; the results are essentially unchanged (see appendix). I choose  $\beta = 0.005$ ,  $\mu^{(A)} + (1 - a)\mu^{(K)} = 2.1\%$ ,  $\sigma^{(A)} = 3.5\%$ ,  $a = 0.9$  (see Garleanu and Panageas (2010) as a justification of the magnitude of  $a = 0.9$ ).  $\mu^{(A)} + (1 - a)\mu^{(K)}$  and  $\sigma^{(A)}$  are chosen to match the unconditional moments of US GDP growth data (see table 1). I calibrate the model for  $\gamma = 7.5$  and two different values of the  $EIS \in \{0.1, 0.125\}$ .

The calibration shows that the interest rate is increasing in the death rate and decreasing in the birth rate, provided the current birth rate is moderate. The interest rate is increasing in the birth rate if the current birth rate is large. Intuitively, a high birth rate implies a low consumption-to-wealth ratio and  $c_t^t$  is small. An additional incremental increase in the birth rate leads only to a slight increase in the aggregate consumption share of the new born cohort (as  $c_t^t$  is small), and puts moderate downward pressure on the interest rate. On the other side, an increase in the birth rate also affects the interest rate positively due to the acceleration in production output growth. The latter positive effect is independent of the current level in the birth rate and becomes dominant if the current birth rate is large. Accordingly, the interest rate becomes increasing in the birth rate, given  $n_t$  is large enough. Lemma 12 is difficult to be satisfied (a lower  $EIS$  is required) if the current birth rate is large.

The equity premium is decreasing in the death rate and increasing in the birth rate, if the current birth rate is moderate. Similar to the equity premium, the market price of risk compensating uncertainty in the birth rate is increasing in the birth rate. The market price of risk compensating uncertainty in the death rate is slightly decreasing in mortality. The exposure of the risky asset to uncertainty in the birth rate is increasing in the birth rate as long as the current level in the birth rate is not too large, while the exposure to risk in the death rate is almost independent of the level in the death rate.



Changes in the birth rate cause a variation of considerable magnitude in the market price of birth rate risk, the exposure of the risky asset to uncertainty in the birth rate, and the equity premium. In contrast, changes in the death rate cause less variation in the market price of mortality risk, the exposure of the stock to uncertainty in the death rate, and the equity premium.

Given the numerical solution of the model I can calculate model implied economic quantities corresponding to any pair of birth and death rates. I compute for each data point of historical US birth and death rates (figure 4) the model implied annualized GDP growth rate ( $\ln \left( \frac{Y_{t+1}}{Y_t} \right)$ ), real short rate ( $r_t$ ), bond yields ( $y_{t,1}, y_{t,5}, y_{t,30}$ ), and stock returns ( $\ln \left( \frac{P_{t+1}+D_t}{P_t} \right)$ ). As a calibration input for the birth rate I use the adjusted birth rate plus the historical immigration rate. I approximate the age-independent death rate by the stochastic time component estimated in a Lee and Carter (1992) model. Because my model does not account for financial leverage, I multiply the model implied annualized stock excess returns  $\ln \left( \frac{P_{t+1}+D_t}{P_t} \right) - r_t$  by a factor of 1.5 when comparing the calibration results to the data (Barro (2006), Frank and Goyal (2008)).

Table 1 compares unconditional moments in the data to unconditional moments of model implied quantities.  $E[.]$  denotes the unconditional average and  $Std[.]$  denotes the unconditional standard deviation. I further compare the calibration results to the quantities in an OLG economy without demographic changes ( $dn_t = d\lambda_t = 0$ ) and constant birth and death rates  $n_t = 2\%$  and  $\lambda_t = 0.9\%$ , and to an equivalent economy populated with an infinitely-lived representative agent ( $n_t = \lambda_t = 0$ ). For the case of  $n_t = \lambda_t = 0$ , I match expected consumption growth and its volatility with the unconditional moments estimated from the calibration with demographic uncertainty. The quantities of the economy without demographic changes are denoted with a tilde and the ones of the infinitely-lived agent economy with a star superscript in the bottom panel of table 1.

The data moments in table 1 are estimated from the following data sources. I have downloaded US data on real Gross Domestic Product (GDP), Treasury Inflation Protected Securities (TIPS), Consumer Price index (CPI) and government bond yields from the St. Louis Fed, and 1-month Treasury Bills and stock returns from CRSP. Real GDP data is available from 1929, CPI and 1-month Treasury Bills from 1920, TIPS from 2003, 30-year

bond yields from 1925, 1- and 5-year bond yields from 1953, and stock returns from 1926. I approximate the risk-free real interest rate (short rate) as follows: for the period 1920 – 1970 I use the difference between the return on 1-month Treasury Bills and the realized inflation, for 1971 – 2002 the real interest rate estimates provided by Chernov and Mueller (2012), and for 2003 – 2006 TIPS data. For the estimation of the average and variance of  $y_{t,30} - y_{t,1}$  (yield spread between 30- and 1-year bonds) in the pre 1950 period I use 1-month Treasury Bills as an approximation of the 1-year yield. All returns and growth rates are annualized and continuously compounded.<sup>24</sup>

The chosen parameterization produces reasonable unconditional moments in real GDP growth. The expected GDP growth rate in table 1 implies an expected GDP per capita growth rate of roughly 2%, which is also consistent with the literature (for instance Campbell and Cochrane, 1999).

The model implied risk-free real interest rate (short rate)  $r_t$  is on average close to what I observe in the data. This means a big improvement over the large rate  $r^*$  in an equivalent infinitely-lived representative agent economy. As pointed out earlier by Garleanu and Panageas (2010) an OLG structure reduces the risk-free rate substantially compared to an infinitely-lived agent economy (drop from over 20% to roughly 4%). Demographic shocks lead to an additional reduction in the risk-free rate by about 2.5%, which is more than 50% of the 4% interest rate in an OLG model without demographic changes. An OLG structure in combination with stochastic birth and death rates appears to resolve Weil’s (1989) risk-free rate puzzle (given we assume a reasonably low  $EIS$ ).

In an economy without demographic changes (including an infinitely-lived agent economy) the short rate  $\tilde{r}$  or  $r^*$  is constant and there is neither a term premium nor volatility in bond markets. In contrast, demographic shocks give rise to a positive term premium. The model implied premium is slightly smaller but not far from the data. The volatilities in the short rate and in bond yields are a little lower in the model than in the data. The difference between model and data may be explained by inflation risks. I do not model inflation or volatility in inflation. Accordingly, it is reasonable to expect that the model generates a

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<sup>24</sup>The presented results do not consider features like financial constraints and limited asset market participation as discussed in Brooks (2004) and Geanakoplos et al. (2004) which are likely to improve the results.

**Table 1: Unconditional Moments - Calibration vs Data**

	Data		Model	
	1926-2006	1950-2006		
<i>EIS</i>			0.1	0.125
$E \left[ \ln \left( \frac{Y_{t+1}}{Y_t} \right) \right]$	3.2%	3.1%	3.1%	3.1%
$Std \left[ \ln \left( \frac{Y_{t+1}}{Y_t} \right) \right]$	4.9%	2.4%	3.5%	3.5%
$E [r_t]$	1.1%	1.7%	2.08%	1.58%
$Std [r_t]$	3.8%	1.8%	1.67%	0.74%
$E [y_{t,30} - y_{t,1}]$	1.3%	1.1%	0.75%	0.43%
$Std [y_{t,30} - y_{t,1}]$	1.4%	1.5%	0.37%	0.26%
$E [y_{t,30} - y_{t,5}]$	N/A	0.5%	0.62%	0.37%
$Std [y_{t,30} - y_{t,5}]$	N/A	0.9%	0.30%	0.19%
$E \left[ \ln \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) \right]$	8.5%	8.3%	5.10%	5.25%
$E \left[ \ln \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) - r_t \right]$	7.4%	6.6%	3.02%	3.66%
$Std \left[ \ln \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) - r_t \right]$	19.7%	17.2%	23.2%	17.0%
$Corr \left[ \ln \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) - r_t, \ln \left( \frac{Y_{t+1}}{Y_t} \right) \right]$	0.31	0.60	0.22	0.29
Economy without demographic shocks ( $n_t = 2\%$ , $\lambda_t = 0.9\%$ , $dn_t = d\lambda_t = 0$ ):				
$\tilde{r}$	1.1%	1.7%	4.47%	4.03%
$E \left[ \ln \left( \frac{\tilde{P}_{t+1} + \tilde{D}_t}{\tilde{P}_t} \right) - \tilde{r} \right]$	8.1%	7.6%	1.38%	1.38%
$Std \left[ \ln \left( \frac{\tilde{P}_{t+1} + \tilde{D}_{t+1}}{\tilde{P}_t} \right) - \tilde{r} \right]$	21.0%	18.5%	5.25%	5.25%
$Corr \left[ \ln \left( \frac{\tilde{P}_{t+1} + \tilde{D}_{t+1}}{\tilde{P}_t} \right) - \tilde{r}, \ln \left( \frac{\tilde{Y}_{t+1}}{\tilde{Y}_t} \right) \right]$	0.31	0.60	1.00	1.00
Infinitely-lived representative agent ( $n_t = \lambda_t = 0$ ):				
$r^*$	1.1%	1.7%	26.4%	21.2%

Table 1: Estimation of unconditional moments:  $E[.]$  denotes the unconditional average, and  $Std[.]$  denotes the unconditional standard deviation. Quantities with a tilde or a star superscripts are obtained in an equivalent OLG economy without demographic changes respectively an economy populated with a single infinitely-lived representative agent. I have downloaded real GDP, TIPS, CPI and government bond yield data from the St. Louis Fed, and 1-month Treasury Bills and stock returns from CRSP. Real GDP data is available from 1929, CPI and 1-month Treasury Bills from 1920, TIPS from 2003, 30-year bond yields from 1925, 1- and 5-year bond yields from 1953, and stock returns from 1926. The model implied moments are calculated by plugging historical US birth and death rates from 1910 to 2006 into the model. As the historical birth rate I use the adjusted birth rate plus the immigration rate as described in the appendix and as the age-independent death rate I use the stochastic time component estimated in a Lee-Carter model.

lower volatility in bond markets and a smaller term premium compared to the data.

Demographic shocks appear to matter substantially for the pricing of stocks. Stochastic birth and death rates introduce a sizeable stock price volatility in excess of the volatility in aggregate consumption growth (and bond market returns). This is in stark contrast to a standard Lucas economy where the volatility in stock returns (not adjusted for leverage) is equal to the volatility in aggregate consumption growth. The equity premium is also higher due to demographic shocks.<sup>25</sup> Given the chosen parameter values, the model does not explain the entire equity premium puzzle but it provides a natural explanation for a part of it. Moreover, historical estimates of the equity premium are noisy and the calibration results lie within a 95% confidence interval of the estimation.

Table 1 shows that the equity premium is higher in the case of  $EIS = 0.125$  than in the case of  $EIS = 0.1$ . Notice that this does not contradict the result in Lemma 9. First, in the calibration (and the data) TFP and demographic shocks are correlated, while Lemma 9 assumes that they are uncorrelated. Second, although the equity premium is continuous in the  $EIS$ , we do not know whether it is monotonic. Third, I show that the equity premium is positive if the  $EIS$  is small enough, but it is analytically difficult to determine whether it is positive or negative if the  $EIS$  is large. Lemma 9 only provides us with a sufficient condition.

Finally, uncertainty in birth and death rates helps to resolve the low correlation puzzle between stock market returns and GDP (or aggregate consumption) growth. This is because prices are forward looking and adjust instantly to sudden changes in discount rates and future expected growth rates while GDP growth does not. Demographic shocks introduce positive quadratic variation in the asset price process, but they only cause shocks to the expected GDP growth rate. The model implied correlation is slightly lower than the estimated correlation between real US GDP growth and US stock market returns. It is, however, close to correlation estimates between real US consumption growth of nondurables and services and US stock returns provided by Campbell (2003). His estimates are between 0.23 and 0.34 for the sample period 1947 – 1998.

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<sup>25</sup>The larger equity premium is attributed to the higher unconditional stock price volatility, rather than an increase in the unconditional Sharpe ratio.

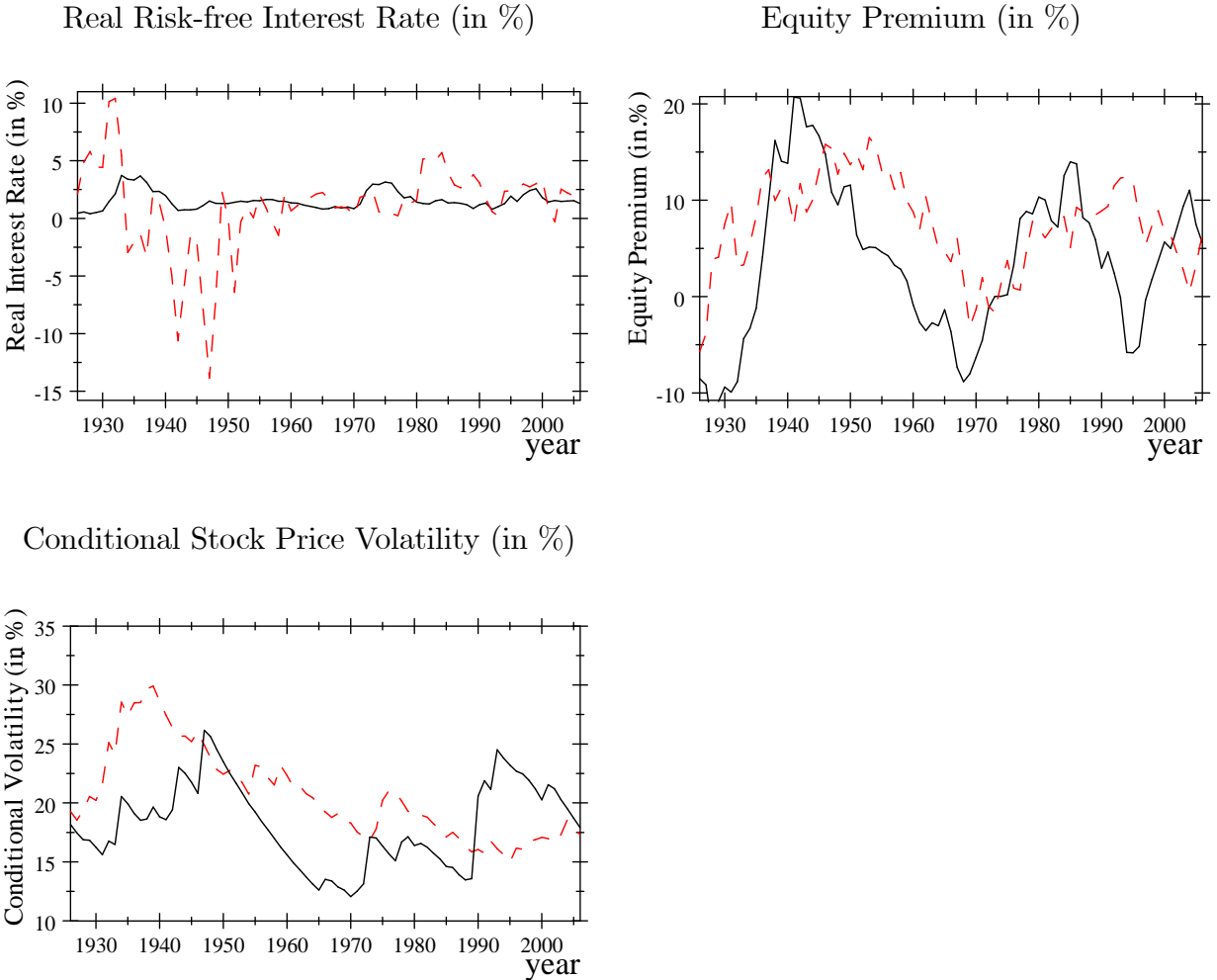


Figure 2: Top left panel: Real interest rate in USA (red dashed line) versus model implied interest rate using historical US birth and mortality data (black solid line). Top right panel: 10 year moving average of US stock market excess returns (red dashed line) and 10 year moving average of model implied stock excess returns (black solid line). Bottom left panel: Conditional stock price volatilities approximated by an exponentially weighted moving average (EWMA) model with an exponential weighting or decay factor of 0.915. Estimated conditional volatilities from US stock excess returns are presented by the red dashed line and volatilities of model implied stock excess returns by the black solid line.

In figure 2, I explore the time variations in the real interest rate and the first two conditional moments in stock returns. The graphs present the calibration results with  $EIS = 0.125$ .

The top left panel compares the real interest rate in the USA (red dashed line) with the model implied interest rate (black solid line). The model implied interest rate is too smooth. In particular, it deviates substantially from the data within the first 25 years of the sample. This time interval includes the Great Depression with large unexpected deflation (1930 – 1933) and subsequent periods of large unexpected inflation and large volatility in inflation (1940 – 1952). The large differences between model and data during the two periods may be explained by the noisy measure of the real interest rate, that is, nominal rate minus realized inflation. Since inflation was extremely volatile in the two periods, it is likely that realized inflation is a poor approximation for expected inflation and my estimates are subject to large measurement errors. The model performs fairly well in the period after 1952, for which I have more reliable data on the real interest rate.

The other two graphs attempt to capture the time variation in the equity premium and the stock price volatility. The top right panel contrasts a 10-year moving average of US stock market excess returns (red dashed line) with a 10-year moving average of model implied stock excess returns (black solid line).<sup>26</sup> The estimated correlation between the moving average excess returns in the data versus the model is 0.44.

The bottom left panel measures conditional stock price volatilities approximated by an exponentially weighted moving average (EWMA) model with an exponential weighting or decay factor of 0.915. Estimated conditional volatilities from US stock excess returns are presented by the red dashed line and volatilities of model implied stock excess returns by the black solid line. The correlation between the conditional volatility estimates is only 0.19.

Despite many simplifying assumptions, the model does a remarkable job capturing the long-term time variation in the first two moments of stock market returns. Large cycles in expected stock market excess returns and a major decline in stock price volatility over the past century appear to be explained by demographic transitions.

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<sup>26</sup>Moving averages are centered around each point in time, that is, for year  $t$  I compute an average return over the annual excess returns between year  $t - 4$  and year  $t + 5$ .

I further explore the relation between the levels in financial quantities and demographic variables. The model suggests that the interest rate is decreasing in the birth rate and increasing in the death rate while the equity premium is increasing in the birth rate and decreasing in mortality. I regress real interest rates, stock market excess returns and conditional volatilities on demographic variables for the sample 1926 – 2006. I estimate the linear regressions for both financial quantities in the data and my calibration outputs (with  $EIS = 0.125$ ) as dependent variables. I use historical US birth and death rates (figure 4) as the demographic regressors.

My model suggests that the financial quantities and demographic variables are non-stationary processes but they are cointegrated. The linear regressions should be understood as Engle-Granger (1987) cointegration regressions. Indeed, all regressions (for quantities in the data and the model) suggest that the regression errors are stationary.

Table 2 suggests that in both the data and the model the interest rate is indeed decreasing in the birth rate and increasing in the death rate. For the interest rate in the data the regression coefficient estimate on the birth rate is significantly negative. The estimated slope coefficient on the death rate is only slightly positive and insignificantly different from zero. Unfortunately, my data sample is short and estimates are noisy which makes it difficult to detect relatively weak relationships at all (see also Poterba, 2001). In the case of the model implied interest rate both regression coefficient estimates on birth and death rate are significantly different from zero.

**Table 2:**  $r_t = \alpha^{(r)} + \beta_n^{(r)}n_t + \beta_\lambda^{(r)}\lambda_t + \beta_{\Delta n}^{(r)}\Delta n_t + \beta_{\Delta\lambda}^{(r)}\Delta\lambda_t + \epsilon_t^{(r)}$

	$\alpha^{(r)}$	$\beta_n^{(r)}$	$\beta_\lambda^{(r)}$	$\beta_{\Delta n}^{(r)}$	$\beta_{\Delta\lambda}^{(r)}$
<b>Data</b>	0.073	-3.879	0.546	-0.460	-0.044
(t-stats)	(4.17)	(3.76)	(0.60)	(4.93)	(0.33)
<b>Model</b>	0.028	-0.927	0.415	-0.001	-0.032
(t-stats)	(6.92)	(3.83)	(1.96)	(0.06)	(1.00)

Table 2: Linear regression of real interest rate on demographic variables for the sample 1926-2006.  $r_t$  represents the real interest rate (either from the data or the calibration with  $EIS = 0.125$ ),  $a^{(r)}$  is a constant term,  $\beta_n^{(r)}$ ,  $\beta_\lambda^{(r)}$ ,  $\beta_{\delta n}^{(r)}$  and  $\beta_{\delta\lambda}^{(r)}$  are regression coefficients,  $n_t$  and  $\lambda_t$  are the birth rate and the death rate in the data,  $\delta$  is a lag operator, and  $\epsilon_t^{(r)}$  is an error term.

Table 3 suggests that stock market excess returns are increasing in the birth rate and decreasing in mortality. The regression estimates in the data are noisy and not significantly different from zero with any reasonable confidence. The estimates using model implied returns as a dependent variable are significantly different from zero with a 90% confidence. Although the estimates are noisy, the direction of the regression coefficient are consistent with my analytical results and support the identified mechanics of the model.

**Table 3:**  $r_t^{(x)} = \alpha^{(r(x))} + \beta_n^{(r(x))} n_t + \beta_\lambda^{(r(x))} \lambda_t + \epsilon_t^{(r(x))}$

	$\alpha^{(r(x))}$	$\beta_n^{(r(x))}$	$\beta_\lambda^{(r(x))}$
<b>Data</b>	-0.035	7.383	-2.497
(t-stats)	(0.32)	(1.13)	(0.43)
<b>Model</b>	-0.059	8.593	-7.231
(t-stats)	(0.57)	(1.40)	(1.34)

Table 3: Linear regression of stock market excess returns on demographic variables for the sample 1926-2006.  $r_t^{(x)}$  represents the stock excess return (either from the data or the calibration with  $EIS = 0.125$ ),  $a^{(r(x))}$  is a constant term,  $\beta_n^{(r(x))}$ ,  $\beta_\lambda^{(r(x))}$  are regression coefficients,  $n_t$  and  $\lambda_t$  are the birth rate and the death rate in the data, and  $\epsilon_t^{(r(x))}$  is an error term.

Finally, table 4 suggest that the conditional stock market volatility is significantly, positively related to both birth and death rates in the data. Model implied conditional volatility appears positively related to the birth rate but negatively related to mortality, though the latter relation is insignificant.

**Table 4:**  $\sigma_t = \alpha^{(\sigma)} + \beta_n^{(\sigma)} n_t + \beta_\lambda^{(\sigma)} \lambda_t + \epsilon_t^{(\sigma)}$

	$\alpha^{(\sigma)}$	$\beta_n^{(\sigma)}$	$\beta_\lambda^{(\sigma)}$
<b>Data</b>	0.109	2.898	5.362
(t-stats)	(6.95)	(3.13)	(6.60)
<b>Model</b>	0.152	1.826	-0.453
(t-stats)	(7.60)	(1.54)	(0.43)

Table 4: Linear regression of conditional stock market volatilities (EWMA with decay factor 0.925) on demographic variables for the sample 1926-2006.  $\sigma_t$  represents the stock excess return (either from the data or the calibration with  $EIS = 0.125$ ),  $a^{(\sigma)}$  is a constant term,  $\beta_n^{(\sigma)}$ ,  $\beta_\lambda^{(\sigma)}$  are regression coefficients,  $n_t$  and  $\lambda_t$  are the birth rate and the death rate in the data, and  $\epsilon_t^{(\sigma)}$  is an error term.



## 4 Conclusion

I answer the question how demographic transitions affect the value of financial assets and whether demographic uncertainty is priced in financial markets. I solve an analytically tractable general equilibrium model with stochastic birth and death rates.

The interest rate is time varying due to demographic changes. For a small enough  $EIS$  and a moderate  $RRA$  the interest rate is decreasing in the birth rate and increasing in the death rate. The equity premium is stochastically changing over time and I provide conditions that suffice for the equity premium to be increasing in the birth rate and decreasing in the death rate.

An important result for future empirical research is that the identified asset pricing implications of changes in death and birth rates work through not identical channels and it is essential to model birth and death rates separately and not as one general state variable that determines total population growth or the average age of the population.

Numerical calibrations suggest that stochastic changes in the birth rate have stronger implications on asset pricing than changes in the death rate. Demographic uncertainty explains part of the equity premium puzzle and the excess volatility of asset returns over volatility in aggregate consumption growth. Demographic transitions appear to explain much of the long-term time variation in the interest rate and stock market returns in the USA in the 20th century.

An interesting result for the long-run risk literature is that a risk source can be compensated by a positive premium even if  $EIS < 1$ . The requirement on the risk source is that a shock causes interest rate and stock prices to move in opposite directions (see also Maurer (2013)).

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## 5 Appendix

### 5.1 Historical Birth and Death Rates

Birth rate statistics are provided by the Department of Health and Human Services, National Center for Health Statistics, USA, and The Human Mortality Database, University of California, Berkeley and Max Planck Institute. Data on the central death rate are provided by the National Center for Health Statistics, USA, and The Human Mortality Database, University of California, Berkeley and Max Planck Institute. I adjust the birth rate for infant mortality to get a better estimate of the "birth rate of economic agents". The adjustment is done by multiplying the observed birth rate with the expected survival probability of a newborn child to reach age 15. The survival probability of a newborn to reach age 15 is shown in the left panel in figure 3. The observed birth rate from the aforementioned data providers (red dotted line) and the adjusted birth rate (black solid line) are shown in figure 3 in the right panel. The green dashed line shows the sum of the adjusted birth rate and the immigration rate which arguably is the best measure of the birth/entry of economic agents.

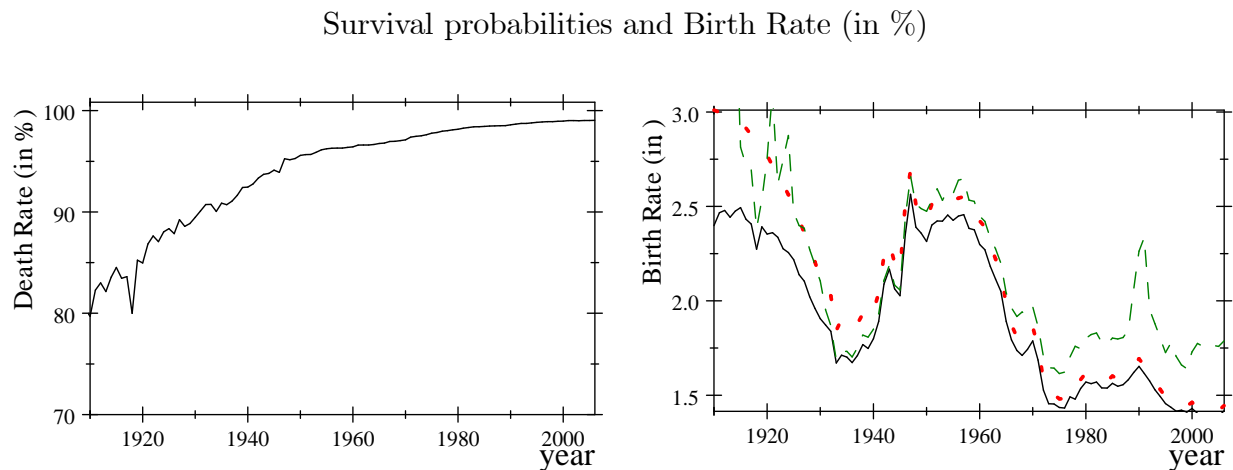


Figure 3: Left panel: Survival probability of newborn to reach age 15 (in %). Source: The Human Mortality Database, University of California, Berkeley and Max Planck Institute for Demographic Research. Right panel: Crude birth rate (in %; red dashed line), adjusted birth rate (in %; black solid line) and adjusted birth rate + immigration rate (green dotted line) in the USA from 1910 until 2006. Source: Department of Health and Human Services, National Center for Health Statistics, USA; and The Human Mortality Database, University of California, Berkeley and Max Planck Institute for Demographic Research. The adjusted birth rate is calculated by multiplying the crude birth rate with the survival probability of a newborn to reach age 15.

### 5.2 Additional Calibration

Some parts of the literature suggest that lagged fertility is slightly pro-cyclical (see Sobotka, Skirbekk and Philipov (2010) for an overview). As a robustness check I repeat the calibrations above but set

Birth Rate and Lee and Carter (1992) Stochastic Time Component in Death Rates (in %)

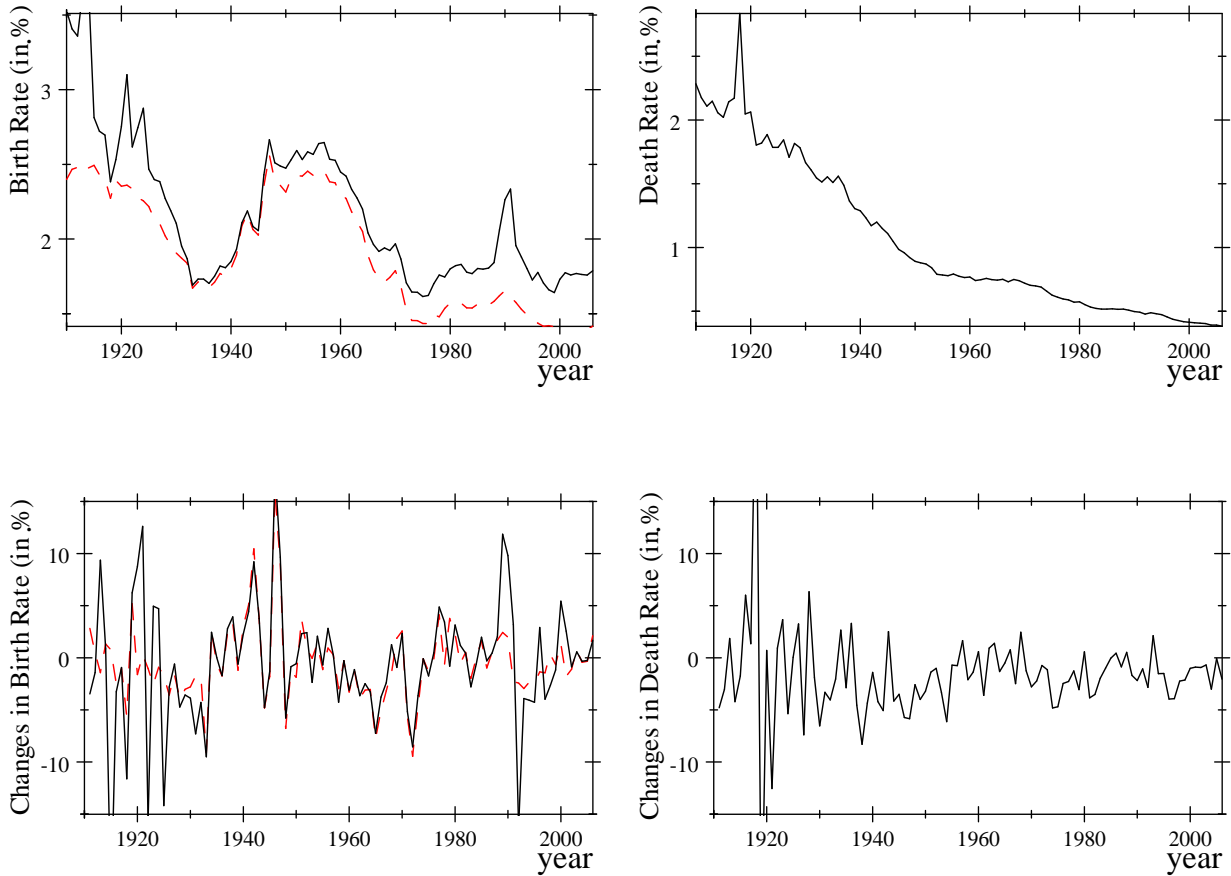


Figure 4: Top-left panel: Adjusted birth rate (in %; red dashed line; multiplication of crude birth rate and survival probability of a newborn to reach age 15) and adjusted birth rate + immigration rate (black solid line) in the USA from 1910 until 2006 (see appendix for details). Source: Department of Health and Human Services, National Center for Health Statistics, USA; and The Human Mortality Database, University of California, Berkeley and Max Planck Institute for Demographic Research. Top-right panel: Lee and Carter (1992) model output (in %) for US mortality data from 1900 to 2006 provided by National Center for Health Statistics. Estimation of common stochastic time component across generations. Bottom-left panel: Percentage changes in adjusted birth rate (red dashed line) and adjusted birth rate + immigration rate (black solid line) . Bottom-right panel: Percentage changes in common stochastic time component of Lee and Carter (1992) model estimation.

**Table 5: Unconditional Moments - Calibration vs Data**

	Data		Model	
	1926-2006	1950-2006		
<i>EIS</i>			0.1	0.125
$E \left[ \ln \left( \frac{Y_{t+1}}{Y_t} \right) \right]$	3.2%	3.1%	3.1%	3.1%
$Std \left[ \ln \left( \frac{Y_{t+1}}{Y_t} \right) \right]$	4.9%	2.4%	3.5%	3.5%
$E [r_t]$	1.1%	1.7%	2.08%	1.31%
$Std [r_t]$	3.6%	1.8%	1.67%	0.77%
$E [y_{t,30} - y_{t,1}]$	1.3%	1.1%	0.75%	0.85%
$Std [y_{t,30} - y_{t,1}]$	1.4%	1.5%	0.37%	0.55%
$E [y_{t,30} - y_{t,5}]$	N/A	0.5%	0.62%	0.73%
$Std [y_{t,30} - y_{t,5}]$	N/A	0.9%	0.30%	0.44%
$E \left[ \ln \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) \right]$	8.5%	8.3%	5.10%	5.60%
$E \left[ \ln \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) - r_t \right]$	7.4%	6.6%	3.02%	4.29%
$Std \left[ \ln \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) - r_t \right]$	19.7%	17.2%	23.2%	17.0%
$Corr \left[ \ln \left( \frac{P_{t+1} + D_{t+1}}{P_t} \right) - r_t, \ln \left( \frac{Y_{t+1}}{Y_t} \right) \right]$	0.31	0.60	0.22	0.25
<hr/>				
Economy without demographic shocks				
$(n_t = 2\%, \lambda_t = 0.9\%, dn_t = d\lambda_t = 0):$				
$\tilde{r}$	1.1%	1.7%	4.47%	4.03%
$E \left[ \ln \left( \frac{\tilde{P}_{t+1} + \tilde{D}_t}{\tilde{P}_t} \right) - \tilde{r} \right]$	8.1%	7.6%	1.38%	1.38%
$Std \left[ \ln \left( \frac{\tilde{P}_{t+1} + \tilde{D}_{t+1}}{\tilde{P}_t} \right) - \tilde{r} \right]$	21.0%	18.5%	5.25%	5.25%
$Corr \left[ \ln \left( \frac{\tilde{P}_{t+1} + \tilde{D}_{t+1}}{\tilde{P}_t} \right) - \tilde{r}, \ln \left( \frac{\tilde{Y}_{t+1}}{\tilde{Y}_t} \right) \right]$	0.31	0.60	1.00	1.00
Infinitely-lived representative agent				
$(n_t = \lambda_t = 0):$				
$r^*$	1.1%	1.7%	26.4%	21.2%

Table 5: Estimation of unconditional moments:  $E[.]$  denotes the unconditional average, and  $Std[.]$  denotes the unconditional standard deviation. Quantities with a tilde or a star superscripts are obtained in an equivalent OLG economy without demographic changes respectively an economy populated with a single infinitely-lived representative agent. I have downloaded real GDP, TIPS, CPI and government bond yield data from the St. Louis Fed, and 1-month Treasury Bills and stock returns from CRSP. Real GDP data is available from 1929, CPI and 1-month Treasury Bills from 1920, TIPS from 2003, 30-year bond yields from 1925, 1- and 5-year bond yields from 1953, and stock returns from 1926. The model implied moments are calculated by plugging historical US birth and death rates from 1910 to 2006 into the model. As the historical birth rate I use the adjusted birth rate plus the immigration rate as described in the appendix and as the age-independent death rate I use the stochastic time component estimated in a Lee-Carter model.



$\text{corr}\left(\frac{dn_t}{n_t}, \frac{dA_t}{A_t}\right) = 0.2$  instead of  $\text{corr}\left(\frac{dn_t}{n_t}, \frac{dA_t}{A_t}\right) = -0.2$ . The results are summarized in table 5 (the graphs show the case of  $EIS = 0.125$ ).

### 5.3 Regime Shifts in the Mortality Rate: Two State Markov Switching Model

To analyze the impact of random changes in the death rate on financial markets, I fix the birth rate while letting the mortality rate switch between a high and a low level. The results and the discussion are similar to the previous section.

I let the death rate process be  $d\lambda_t = s^{(\lambda)}dS_t^{(\lambda)}$ , with  $s^{(\lambda)} = \lambda_H - \lambda_L$ .  $S_t^{(\lambda)} \in \{1, 0\}$  follows a two state, continuous time Markov switching process with transition probability matrix between time  $t$  and  $t + \Delta$  given by  $\Theta^{(S, \lambda)}(\Delta) = \begin{pmatrix} 1 - \bar{\theta}_H^{(\lambda)}\Delta & \bar{\theta}_H^{(\lambda)}\Delta \\ \bar{\theta}_L^{(\lambda)}\Delta & 1 - \bar{\theta}_L^{(\lambda)}\Delta \end{pmatrix}$ . The death rate switches between the two values  $\lambda_t \in \{\lambda_L, \lambda_H\}$ .

There are minor changes to the utility specification as described below. Agents' objectives stay the same.

**Proposition 4** *Suppose an economy as described. In general, there exists an equilibrium with a SDF  $\pi$  that follows a stochastic process driven by the same two state Markov switching process  $S^{(\lambda)}$  as the death rate. The equilibrium interest rate  $r_t$  switches between two distinct levels,  $r_t \in \{r_L^{(\lambda)}, r_H^{(\lambda)}\}$  defined by*

$$\begin{aligned} r_j^{(\lambda)} &= \beta + (1 - \rho) \left[ \mu_j^{(Y, \lambda)} + \lambda_j - n \sum_{i=1}^2 F_j^{y, (i), (\lambda)} \psi_j^{(\lambda)} \right] - \lambda_j + \frac{\rho}{1 - \gamma} \lambda_j \\ &\quad - \bar{\theta}_j^{(\lambda)} \left( \left( \frac{\psi_k^{(\lambda)}}{\psi_j^{(\lambda)}} \right)^{-\frac{1 - \gamma - \rho}{\rho}} - 1 \right) + \frac{1 - \gamma - \rho}{1 - \gamma} \bar{\theta}_j^{(\lambda)} \left( \left( \frac{\psi_k^{(\lambda)}}{\psi_j^{(\lambda)}} \right)^{-\frac{1 - \gamma}{\rho}} - 1 \right) \end{aligned}$$

$\forall (j, k) \in \{(L, H), (H, L)\}$ , with  $r_t | [\lambda_t = \lambda_L] = r_L^{(\lambda)}$  and  $r_t | [\lambda_t = \lambda_H] = r_H^{(\lambda)}$ . The market price of risk also jumps between two distinct values,  $\kappa_t \in \{\kappa_L^{(\lambda)}, \kappa_H^{(\lambda)}\}$  given by

$$\kappa_j^{(\lambda)} = - \left( \left( \frac{\psi_k^{(\lambda)}}{\psi_j^{(\lambda)}} \right)^{-\frac{1 - \gamma - \rho}{\rho}} - 1 \right)$$

$\forall (j, k) \in \{(L, H), (H, L)\}$ , with  $\kappa_t | [\lambda_t = \lambda_L] = \kappa_L^{(\lambda)}$  and  $\kappa_t | [\lambda_t = \lambda_H] = \kappa_H^{(\lambda)}$ . Demographic uncertainty is priced in equilibrium and the equity premium is non-zero, except for the special case of power utility. The functions  $F_L^{y, (1), (\lambda)}$ ,  $F_L^{y, (2), (\lambda)}$ ,  $F_H^{y, (1), (\lambda)}$ ,  $F_H^{y, (2), (\lambda)}$ ,  $\psi_L^{(\lambda)}$ , and  $\psi_H^{(\lambda)}$  are determined in a system of 6 non-linear equations provided in the appendix.

**Proof.** See below. ■

**Lemma 10** *Suppose  $\gamma \in (1, 1 - \rho)$  ( $\rho < 0$ ) and the technical conditions in the appendix hold. There exists a cut-off value  $\overline{EIS}_1^{(\lambda)}$  such that the condition  $EIS < \overline{EIS}_1^{(\lambda)}$  suffices for the interest rate during a period characterized by a high death rate to be higher than the rate during times of low mortality,  $r_L^{(\lambda)} < r_H^{(\lambda)}$ . The consumption-to-wealth ratio is increasing and the magnitude of the market price of risk is decreasing in the death rate,  $\psi_L^{(\lambda)} < \psi_H^{(\lambda)}$  and  $|\kappa_L^{(\lambda)}| > |\kappa_H^{(\lambda)}|$ .*

**Proof.** See below. ■

The result is equivalent to the finding in the static case.  
The stock price is a multiple of GDP (Lemma 1),

$$P_t^{(\lambda_j)} = P_t | [\lambda_t = \lambda_j] = Y_t \left[ \frac{1}{\psi_j^{(\lambda)}} - \sum_{i=1}^2 \frac{F_j^{y,(i),(\lambda)}}{1 + \delta_i} \right]$$

$\forall j \in \{L, H\}$  and the growth rate is

$$\frac{dP_t}{P_t} \Big|_{[\lambda_t = \lambda_j]} = \mu_j^{(Y,\lambda)} dt + \frac{Y_t}{P_t^{(\lambda_j)}} \left[ \frac{1}{\psi_k^{(\lambda)}} - \frac{1}{\psi_j^{(\lambda)}} - \sum_{i=1}^2 \frac{F_k^{y,(i),(\lambda)} - F_j^{y,(i),(\lambda)}}{1 + \delta_i} \right] \Big|_{dS_t^{(\lambda)}}$$

$\forall (j, k) \in \{(L, H), (H, L)\}$ . GDP is locally deterministic, while stock returns are subject to instantaneous volatility due to the forward looking property of the stock price.

According to Lemma 2, the equity premium is

$$\begin{aligned} \frac{1}{dt} E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t \Big|_{[\lambda_t = \lambda_j]} &= \bar{\theta}_j^{(\lambda)} \frac{Y_t}{P_t^{(\lambda_j)}} \left( \left( \frac{\psi_k^{(\lambda)}}{\psi_j^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \\ &\quad \cdot \left[ \left( \frac{1}{\psi_j^{(\lambda)}} - \frac{1}{\psi_k^{(\lambda)}} \right) + \sum_{i=1}^2 \frac{F_k^{y,(i),(\lambda)} - F_j^{y,(i),(\lambda)}}{1 + \delta_i} \right] \end{aligned}$$

$\forall (j, k) \in \{(L, H), (H, L)\}$ . In the special case of CRRA utility with  $1 - \gamma - \rho = 0$ , there is no equity premium.

**Lemma 11** *Suppose  $\frac{1-\gamma-\rho}{\rho} < 0$  and the technical conditions in the appendix hold. There exists a cut-off value  $\overline{EIS}_2^{(\lambda)}$  such that the condition  $EIS < \overline{EIS}_2^{(\lambda)}$  suffices for the equity premium to be positive in both states of the world.*

**Proof.** See below. ■

Lemma 10 is equivalent to Lemma 7.

Consistent with the comparative statics analysis in section 3.3, the stock price is decreasing in the death rate (under the conditions in Lemma 9 and 10),

$$P_t^{(\lambda_L)} - P_t^{(\lambda_H)} = Y_t \left[ \frac{1}{\psi_L^{(\lambda)}} - \frac{1}{\psi_H^{(\lambda)}} - \sum_{i=1}^2 \frac{F_L^{y,(i),(\lambda)} - F_H^{y,(i),(\lambda)}}{1 + \delta_i} \right] > 0$$

**Lemma 12** *Suppose  $\bar{\theta}_L^{(\lambda)} > (<) \bar{\theta}_H^{(\lambda)}$  and the conditions in Lemma 9 and 10 hold. There exists a cut-off value  $\overline{EIS}_3^{(\lambda)}$  such that the condition  $EIS < \overline{EIS}_3^{(\lambda)}$  ensures that the equity premium is larger (lower) during a period characterized by a low death rate than the premium in times of high mortality.*

**Proof.** See below. ■

## 5.4 Extensions and Comments

I heuristically discuss three shortcomings/ extensions of the model.

1) *Generalization of Birth and Death Rate Processes*

For simplicity I have modelled birth and death rates as geometric Brownian motions with constant drifts and diffusions. Instead I may consider for instance autoregressive processes. The US birth rate seems to be slightly positively autocorrelated. However, it is not clear from demographic literature whether an auto-regressive process is preferred to a geometric Brownian motion. Changes in the death rate are not autocorrelated.

If changes in the birth rate are described by an autoregressive process rather than white noise, I expect the consumption-to-wealth ratio and the interest rate to depend (in addition to the current level and volatility) on recent changes in the birth rate.

The static model in section 3.1 provides a good intuition. Assume that changes in the birth rate are positively autocorrelated. Consider a large past increase (decrease) in the birth rate. Accordingly, another increase (decrease) in the birth rate is expected in the near future. Because the consumption-to-wealth ratio is negatively related to the birth rate (for a small enough *EIS*), a large expected increase (decrease) in the birth rate creates an incentive for a forward-looking agent to choose a low (high) current consumption-to-wealth ratio. The property  $\frac{\partial \psi}{\partial n} < 0$  and the positive relation between  $\psi_t$  and  $\mu_t^{(\psi)} = \frac{1}{dt} E_t(d\psi_t)$  in equation (20) formalizes the intuition. In equilibrium a low (high) consumption-to-wealth ratio corresponds to a low (high) interest rate (if *EIS* < 1). As a result the consumption-to-wealth ratio and the interest rate are negatively related to recent changes in the birth rate. I show in the first section in the appendix that there is empirical evidence for a negative relationship between the level of the current interest rate and past changes in the birth rate. Geanakoplos et al. (2004) document a similar relation between changes in demographic quantities and the level in the interest rate.

## 2) Social Security and other Intergenerational Transfers

The simplest way to model a social security system is by letting agents pay a (possibly age-dependent) labor income tax which is redistributed to the entire population.<sup>27</sup> In the limit when all labor income is collected and agents receive/ consume per capita GDP, the consumption goods allocation is identical to the first best allocation in an Arrow economy with (intergenerational) market completeness.<sup>28</sup>

Other intergenerational transfers are modelled by assuming that agents care about other agents' utilities. For instance, a parent may care about how much utility his children obtain and vice versa. In the extreme case when agents care about other agents' utilities the same as about their own utility, the economy achieves the first best allocation.

I look at the extreme case when the intergenerational wealth redistribution leads to the first best allocation. Noticing that under first best  $c_t^t = \frac{C_t}{N_t}$ , equation (8) and equation (9) become  $\frac{\partial r}{\partial n} = (1 - \rho)(a - 1)$  and  $\frac{\partial r}{\partial \lambda} = (1 - \rho)(1 - a) + \frac{\rho}{1 - \gamma} - 1$ . The comparative statics analysis suggests that for  $\min \left\{ -\frac{a(1 - \gamma)}{(1 - a)(1 - \gamma) - 1}, 0 \right\} > \rho$  the interest rate and the consumption-to-wealth ratio are decreasing in the birth rate and increasing in the death rate. The result is stronger than Lemma 4 and 5. The interpretation and the key driving forces for the result remain the same: an increase in the birth rate causes a redistribution of consumption and wealth from existing agents to the new born cohort. I expect the intuition to continue to hold in a dynamic model with stochastic changes in birth and death rates. Because the consumption-to-wealth ratio is sensitive to

<sup>27</sup>Alternatively, I may consider a set-up as in Gertler (1997) where agents randomly switch from a working state to retirement and social security is a transfer between workers and retirees. In that case, to keep my model tractable I must introduce a new set of contracts to let agents hedge the new retirement risk and to keep markets dynamically complete. However, in Gertler (1997) the results are driven by the market incompleteness due to retirement risk.

<sup>28</sup>See also Abel (2003) for a discussion on how a social security system can be employed to approach the Golden Rule in the economy

demographic changes, I expect demographic uncertainty to be priced and the equity premium to be time varying.

Since my results are not affected even if I impose the first best allocation, I do not expect that the introduction of a (reasonable) social security system or other intergenerational transfers alter the fundamental qualitative results of my model. Though the quantitative magnitude of the effects might change. For instance, Brooks (2004) argues that the introduction of a social security system may have important quantitative asset pricing implications.

### 3) *Endogenous Capital Accumulation*

For simplicity I have assumed that firms cannot invest. The assumption is to some extent justified because the main driving force of my results is the redistribution of wealth and consumption between young and old generations in response to demographic shifts rather than changes in future labor supply and GDP growth. Moreover, in reality there is a 15 – 20 year lag between changes in the birth rate and changes in the size of the workforce which further complicates the relationship between the birth rate and the optimal capital accumulation policy.

Nevertheless, demographic changes have a long-term impact on the labor supply, and it is reasonable that a firm optimally adjusts its capital stock in response to highly predictable long-run changes in the labor market. One way to tackle the problem is to approximate endogenous capital accumulation by a reasonable exogenous process like  $\frac{dK_t}{K_t} = [\mu^{(K)} + \xi^{(K)}(n_t - \lambda_t)] dt$  as mentioned in section 2.2. As long as  $\xi^{(K)}$  is not too large the qualitative results in the paper remain unchanged.

Another approach is to consider endogenous capital accumulation with convex adjustment costs as in Abel (2003). Suppose the birth rate increases (decreases) or the death rate decreases (increases). The stock price increases (declines). In response the firm starts to invest (disinvest) and less (more) units of production output will be available to consumers. Under time additive utilities, a drop (increase) in current aggregate consumption implies a high (low) marginal utility state. As a result I expect a positive correlation between the marginal utility process and stock returns which implies a negative equity premium.

In contrast, under recursive utilities it is not clear whether capital accumulation has a negative or a positive impact on the equity premium. It is still true that current aggregate consumption drops (increases) due to investment (disinvestment) by the representative firm, which has a positive (negative) effect on marginal utility. But future aggregate consumption will grow faster (slower) due to the initial investment (disinvestment) and under certain restrictions on the parameterization of the recursive preferences, this has a negative (positive) impact on marginal utility. The two effects are offsetting and it is not clear whether there is a positive or negative correlation between the marginal utility process and stock returns.

I expect capital accumulation to reduce the sensitivity of the interest rate and the consumption-to-wealth ratio towards changes in birth and death rates. Capital accumulation causes growth in production output to react stronger in response to changes in birth and death rates because an increase (decrease) in the birth rate or a decrease (increase) in the death rate comes with additional investment (disinvestment). Given equation (8) and (9) I conjecture that  $|\frac{\partial r}{\partial n}|$ ,  $|\frac{\partial \psi}{\partial n}|$ ,  $|\frac{\partial r}{\partial \lambda}|$  and  $|\frac{\partial \psi}{\partial \lambda}|$  are smaller in a model with endogenous capital accumulation compared to the model above. Accordingly, a decrease in the sensitivity of the consumption-to-wealth ratio to changes in birth and death rates causes the market price of risk and asset price volatility to decline.

## 5.5 Derivation of Kreps and Porteus (1978) Type Stochastic Differential Utilities given Uncertain Lifetimes

Stochastic differential utilities are a continuous time counterpart to the recursive utilities discussed by Epstein and Zin (1989, 1991). Duffie and Epstein (1992a) restrict their derivation to the case of Brownian information. In a model with uncertain lifetimes the dynamics of the value function include a Poisson jump term that sets the value function to zero when the agent passes away. If information is generated by a Brownian motion and a Poisson jump process (due to lifetime uncertainty), then I have to make some adaptations to the utility specification introduced in Duffie and Epstein (1992a). Following the notation in Duffie and Epstein (1992a), the dynamics of the utility (given the agent is still alive at time  $t$  and will die at time  $\tau$ ) have to be rewritten as (given  $t \leq \tau$ )

$$dV_t = \mu_t dt + \sigma_t dB_t - V_t dQ_t$$

$B$  is a Brownian motion,  $Q$  is a compensated Poisson jump process with hazard rate  $\lambda_t$ . The agent dies if  $Q$  jumps the first time since the agent is born and I denote the time of the first jump by  $\tau$ . The arrival rate of death is time varying and stochastic, i.e.  $Q$  is a doubly stochastic process (Cox process). Following the lines in Duffie and Epstein (1992a) this implies

$$\begin{aligned} \mu_t &= -f(c_t, V_t) - \frac{1}{2} A(V_t) \sigma_t^2 - \lambda_t [M(V_t, V_{t-}) - M(V_{t-}, V_{t-})] \\ &= -f(c_t, V_t) - \frac{1}{2} A(V_t) \sigma_t^2 - \lambda_t [M(V_{t-} - V_{t-}, V_{t-}) - M(V_{t-}, V_{t-})] \\ &= -f(c_t, V_t) - \frac{1}{2} A(V_t) \sigma_t^2 + \lambda_t M(V_{t-}, V_{t-}) \end{aligned}$$

$M(y, x) = \frac{h(y)}{h'(x)}$  is the local gradient representation of the certainty equivalent  $m$ , i.e.  $\nabla m(\delta_x; p) = \int M(y, x) dp(y)$ , and  $h(\cdot)$  is defined as  $h(m(V)) = E(h(V))$ .

Since

$$V_t = E_t[V_T | \tau > T] + E_t \left[ \int_t^T -\mu_s ds \right]$$

it follows that as  $T$  goes to infinity

$$V_t = E_t \left[ \int_t^T f(c_s, V_s) + \frac{1}{2} A(V_s) \sigma_s^2 - \lambda_s M(V_s, V_s) ds \right]$$

I can show as in Duffie and Epstein (1992a) that the following transformation leads to an equivalent utility function  $\bar{V}_t = \phi(V_t)$  with

$$\begin{aligned} f(c_t, z) &= \frac{\bar{f}(c_t, \phi(z))}{\phi'(z)} \\ m(z) &= \phi^{-1}(\bar{m}[\phi(z)]) \\ \phi'(z) M(y, z) &= \bar{M}(\phi(y), \phi(z)) \\ A(x) &= \phi'(x) \bar{A}(\phi(x)) + \frac{\phi''(x)}{\phi'(x)} \end{aligned}$$

This follows from

$$\begin{aligned}
d\bar{V}_t &= \left( \phi'(V_t) \left[ -f(c_t, V_t) - \frac{1}{2}A(V_t)\sigma_t^2 + \lambda_t M(V_t, V_t) \right] + \frac{1}{2}\phi''(V_t)\sigma_t^2 \right) dt \\
&\quad + \phi'(V_t)\sigma_t dB_t - \phi(V_t)dQ_t \\
&= \left[ -\bar{f}(c_t, \bar{V}_t) - \frac{1}{2}\bar{A}(\bar{V}_t)\bar{\sigma}_t^2 + \lambda_t \bar{M}(\bar{V}_t, \bar{V}_t) \right] dt + \bar{\sigma}_t dB_t - \bar{V}_t dQ_t
\end{aligned}$$

with

$$\bar{\sigma}_t = \phi'(V_t)\sigma_t$$

and

$$\begin{aligned}
&-f(c_t, V_t) - \frac{1}{2}A(V_t)\sigma_t^2 + \lambda_t M(V_t, V_t) \\
&= -\frac{\bar{f}(c_t, \bar{V}_t)}{\phi'(V_t)} - \frac{1}{2}\frac{\bar{A}(\bar{V}_t)}{\phi'(V_t)}\bar{\sigma}_t^2 - \frac{1}{2}\frac{\phi''(V_t)}{\phi'(V_t)}\sigma_t^2 + \lambda_t \frac{\bar{M}(\bar{V}_t, \bar{V}_t)}{\phi'(V_t)} \\
&= -\frac{\bar{f}(c_t, \bar{V}_t)}{\phi'(V_t)} - \frac{1}{2}\left[ \bar{A}(\bar{V}_t)\phi'(V_t) + \frac{\phi''(V_t)}{\phi'(V_t)} \right]\sigma_t^2 + \lambda_t \frac{\bar{M}(\bar{V}_t, \bar{V}_t)}{\phi'(V_t)}
\end{aligned}$$

Choosing  $\phi''(x) = \phi'(x)A(x)$  implies  $\bar{A}(x) = 0$ , and thus,  $\bar{m}[x] = E[x]$ .

For the specification introduced in Duffie and Epstein (1992a), featuring the Kreps and Porteus (1978) property of preferences over the timing of risk resolution,

$$\begin{aligned}
f(c_s, V_s) &= \frac{\beta c_s^\rho - V_s^\rho}{\rho V_s^{\rho-1}} \\
m(x) &= (E[x^{1-\gamma}])^{\frac{1}{1-\gamma}} \\
V_t &= E_t \left[ \int_t^\infty f(c_s, V_s) + \frac{1}{2}A(V_s)\sigma_s^2 - \lambda_s M(V_s, V_s) ds \right] \\
&= E_t \left[ \int_t^\infty \frac{\beta c_s^\rho - V_s^\rho}{\rho V_s^{\rho-1}} - \frac{1}{2}\frac{\gamma}{V_s}\sigma_s^2 - \lambda_s \frac{V_s}{1-\gamma} ds \right]
\end{aligned}$$

Letting  $\phi(x) = \frac{1}{1-\gamma}x^{1-\gamma}$  to get an equivalent utility function  $\bar{V}_t = \phi(V_t)$ , I end up with

$$\begin{aligned}
\bar{f}(c_s, \bar{V}_s) &= \phi'(\phi^{-1}(\bar{V}_s))f(c_s, \phi^{-1}(\bar{V}_s)) \\
&= \frac{\beta c_s^\rho - [(1-\gamma)\bar{V}_s]^{\frac{\rho}{1-\gamma}}}{\rho [(1-\gamma)\bar{V}_s]^{\frac{\rho}{1-\gamma}-1}} \\
\bar{m}(x) &= E[x]
\end{aligned}$$

I get the utility specification

$$\begin{aligned}
\bar{V}_t &= E_t \left[ \int_t^\infty \bar{f}(c_s, \bar{V}_s) + \frac{1}{2} \bar{A}(\bar{V}_s) \sigma_s^2 - \lambda_s \bar{M}(\bar{V}_s, \bar{V}_s) ds \right] \\
&= E_t \left[ \int_t^\infty \frac{\beta c_s^\rho - [(1-\gamma)\bar{V}_s]^{\frac{\rho}{1-\gamma}}}{\rho [(1-\gamma)\bar{V}_s]^{\frac{\rho}{1-\gamma}-1}} - \lambda_s \bar{V}_s ds \right] \\
&= E_t \left[ \int_t^\infty \frac{\beta c_s^\rho - \left( \beta + \frac{\rho}{(1-\gamma)} \lambda_s \right) [(1-\gamma)\bar{V}_s]^{\frac{\rho}{1-\gamma}}}{\rho [(1-\gamma)\bar{V}_s]^{\frac{\rho}{1-\gamma}-1}} ds \right] \\
&= E_t \left[ \int_t^\infty \hat{f}(c_s, \bar{V}_s) ds \right]
\end{aligned}$$

As shown in the online appendix of Garleanu and Panageas (2010) the same specification can be obtained as a continuous time limit of the discrete time recursive utility function

$$\begin{aligned}
V_t &= \left\{ c_t^\rho + (1-\beta) E_t \left[ (1_s V_{t+1})^{1-\gamma} \right]^{\frac{\rho}{1-\gamma}} \right\}^{\frac{1}{\rho}} \\
&= \left\{ c_t^\rho + (1-\beta) E_t \left[ (1-\lambda_t) V_{t+1}^{1-\gamma} \right]^{\frac{\rho}{1-\gamma}} \right\}^{\frac{1}{\rho}} \\
&= \left\{ c_t^\rho + (1-\beta) (1-\lambda_t)^{\frac{\rho}{1-\gamma}} E_t \left[ V_{t+1}^{1-\gamma} \right]^{\frac{\rho}{1-\gamma}} \right\}^{\frac{1}{\rho}}
\end{aligned}$$

$1_s$  is an indicator function determining whether the agent survives ( $1_s = 1$ ) or passes away ( $1_s = 0$ ). The non-linear "discounting" term  $(1-\lambda_t)^{\frac{\rho}{1-\gamma}}$  captures risk aversion towards the timing of death. This relates to the discussion by Bommier (2003). Depending on the preference parameters,  $\frac{\rho}{1-\gamma} > (<) 0$ , an agent is less (more) concerned about future consumption (utility) and wants to save less (more) than under a certain length of life. The utility specification in the paper of Garleanu and Panageas (2010) differs from my specification because (opposed to my approach) they exclude risk aversion towards the timing of death.

Because the utility function is a continuous time version of the recursive utility function introduced by Epstein and Zin (1989, 1991), in order for the agent to have a preference for early (late) resolution of risk (in the sense of Kreps and Porteus (1978)), I need

$$1 - \gamma < (>) \rho$$

This insight becomes clear when considering the discrete time recursive utility function

$$V_t = \left[ c_t^\rho + (1-\beta) E_t \left[ 1_s V_{t+1}^{1-\gamma} \right]^{\frac{\rho}{1-\gamma}} \right]^{\frac{1}{\rho}}$$

I define  $V_t = \bar{V}_t^{\frac{1}{\rho}}$  and rewrite the discrete time utility specification as

$$\bar{V}_t = \left[ c_t^\rho + (1-\beta) E_t \left[ 1_s \bar{V}_{t+1}^{\frac{1-\gamma}{\rho}} \right]^{\frac{\rho}{1-\gamma}} \right]$$

For  $\rho > 0$ ,  $\arg \sup_{\{c^s, X^s\} \in (\mathbb{S} \times \mathcal{L}^2)} \{V_t\} = \arg \sup_{\{c^s, X^s\} \in (\mathbb{S} \times \mathcal{L}^2)} \{\bar{V}_t\}$  and by Jensen's inequality early (late) resolution of risk is preferred if  $\frac{1-\gamma}{\rho} < (>) 1$  or  $1 - \gamma < (>) \rho$ . For  $\rho < 0$ ,  $\arg \sup_{\{c^s, X^s\} \in (\mathbb{S} \times \mathcal{L}^2)} \{V_t\} = \arg \sup_{\{c^s, X^s\} \in (\mathbb{S} \times \mathcal{L}^2)} \{-\bar{V}_t\}$  and by Jensen's inequality early (late) resolution is preferred if  $\frac{1-\gamma}{\rho} > (<) 1$  or  $1 - \gamma < (>) \rho$ .

The specification nests the special case of a time additive expected utility function featuring a CRRA profile with  $\gamma = \frac{1}{E\bar{S}}$ . Indeed setting  $\gamma = 1 - \rho$  reduces to the familiar specification for power utilities

$$\begin{aligned}\bar{V}_t &= E_t \left[ \int_t^\infty \frac{\beta c_s^{1-\gamma}}{1-\gamma} - (\beta + \lambda_s) \bar{V}_s ds \right] \\ &= E_t \left[ \int_t^\infty \frac{\beta c_s^{1-\gamma}}{1-\gamma} e^{-\int_t^s \beta + \lambda_u du} ds \right]\end{aligned}$$

The condition  $\gamma = 1 - \rho$  implies indifference with respect to timing of risk resolution; neither early nor late resolution of risk is preferred. In the case of time additive utility agents also become risk neutral towards uncertainty about the timing of death (cf. also Bommier (2003)).

There are a few comments on the specification. As the utility function may be defined on the negative space, it might seem that being dead is desirable. I can rule out this problem by not giving the agent the option to commit suicide. One may also circumvent the problem of suicidal agents by adding a large enough constant term to the aggregator function  $f(\cdot)$ , so that the agent draws utility from simply being alive. Such a constant term does not matter in the utility maximization problem. Further, the specification here excludes bequest motives. This is restrictive, but in turn a too large bequest motive may give rise to suicidal behavior of an agent and counter-intuitive results (for more details see Maurer, 2011).

The derivation of the utility specification in the economy with regime shifts (Markov switching model) follows the same steps. Let the state of the world be indicated by the state variable  $S_t \in \{0, 1\}$ , which jumps when a regime shift occurs. Adjustments have to be done with respect to the dynamics in the value function,

$$dV_t = \mu_t dt + 1_{\{S_t=1\}} s_1^{(V)} |d\widehat{S}_t| + 1_{\{S_t=0\}} s_0^{(V)} |d\widehat{S}_t| - V_t dQ_t$$

$\widehat{S}_t$  is a compensated Poisson jump process corresponding to the Markov switching process  $S_t$ , and  $s_i^{(V)}$  denotes the change in the value function due to a jump from state  $i \in \{1, 0\}$  to the other state. The drift term is given by

$$\begin{aligned}\mu_t &= -f(c_t, V_t) + 1_{\{S_t=1\}} \bar{\theta}_H \left[ M(V_{t-} + s_1^{(V)} V_{t-}) - M(V_{t-}, V_{t-}) \right] \\ &\quad + 1_{\{S_t=0\}} \bar{\theta}_L \left[ M(V_{t-} + s_0^{(V)} V_{t-}) - M(V_{t-}, V_{t-}) \right] + \lambda_t M(V_{t-}, V_{t-})\end{aligned}$$

The remaining of the derivation follows by applying the same lines of argument as above. The specification of the SDU in case of regime shifts in the birth rate becomes

$$V_s^s = E_s \left[ \int_s^\infty f(c_u^s, V_u^s) du \right]$$

with

$$f(c_u^s, V_u^s) = \frac{\beta (c_u^s)^\rho - \left( \beta + \frac{\rho}{1-\gamma} \lambda \right) [(1-\gamma) V_u^s]^{\frac{\rho}{1-\gamma}}}{\rho [(1-\gamma) V_u^s]^{\frac{\rho}{1-\gamma} - 1}} - \left[ \begin{aligned} &1_{\{S_u^{(n)}=1\}} \bar{\theta}_H^{(n)} s_1^{(V^s, n)} \\ &+ 1_{\{S_u^{(n)}=0\}} \bar{\theta}_L^{(n)} s_0^{(V^s, n)} \end{aligned} \right]$$

The specification in case of regime shifts in the death rate is written as

$$V_s^s = E_s \left[ \int_s^\infty f(c_u^s, V_u^s) du \right]$$



with

$$f(c_u^s, V_u^s) = \frac{\beta (c_u^s)^\rho - \left(\beta + \frac{\rho}{1-\gamma} \lambda_u\right) [(1-\gamma) V_u^s]^{1-\frac{\rho}{1-\gamma}}}{\rho [(1-\gamma) V_u^s]^{1-\frac{\rho}{1-\gamma}-1}} - \left[ \begin{array}{l} 1_{\{S_u^{(\lambda)}=1\}} \bar{\theta}_H^{(\lambda)} s_1^{(V^s, \lambda)} \\ + 1_{\{S_u^{(\lambda)}=0\}} \bar{\theta}_L^{(\lambda)} s_0^{(V^s, \lambda)} \end{array} \right]$$

## 5.6 Proofs of Propositions

Propositions 1 is a special cases of Proposition 4. Proposition 2 and 3 are closely related to Proposition 4. I provide a proof for the general case and show afterwards how to get from there the other Propositions.

**Proof of Proposition 4.** Following Duffie and Skiadas (1994, Theorem 2), the Gateau derivative (directional derivative) of the utility function in equation (2) at  $\bar{c}^s$  in the direction  $x$  is

$$\begin{aligned} \nabla V_s^s(\bar{c}^s; x) &\equiv \lim_{\alpha \rightarrow 0} \frac{V_s^s(\bar{c}^s + \alpha x) - V_s^s(\bar{c}^s)}{\alpha} \\ &= E_s \left[ \int_s^\infty e^{\int_s^t \frac{\partial}{\partial V_u^s} f(\bar{c}_u^s, V_u^s) du} \frac{\partial}{\partial c_t} f(\bar{c}_t^s, V_t^s) x dt \right] \\ &= E_s \left[ \int_s^\infty R_t x dt \right] \end{aligned} \tag{14}$$

The Riesz representation process  $R_t$  is defined as

$$R_t = e^{\int_s^t \frac{\partial}{\partial V_u^s} f(\bar{c}_u^s, V_u^s) du} \frac{\partial}{\partial c_t} f(\bar{c}_t^s, V_t^s)$$

Optimality implies for any agent born at time  $s$  (assuming that the optimal consumption plan  $c^{s*}$  is in the interior)

$$\nabla V_s^s(c^{s*}; [c^s - c^{s*}]) = 0$$

for all admissible consumption plans  $c^s \in \mathfrak{F}$ . Since  $F_s^{(\lambda)}(c^s - c^{s*})$  spans the set of all marketable cash flows  $M$ ,

$$\nabla V_s^s(c^{s*}; F_s^{(\lambda)-1}(x)) = 0$$

holds for all marketable cash flows  $x \in M$ . This implies that the Riesz representation process is a multiple of a SDF  $\pi$ ,

$$R_t e^{\int_s^t \lambda_u du} = \eta^s \pi_t$$

for some constant  $\eta^s$ . Since markets are dynamically complete, the found pricing kernel is unique. I can solve for the optimal consumption plan for any agent born at time  $s$  by plugging in the expression for the Riesz representation process (from now I drop the notation indicating the optimum by a star)

$$\begin{aligned} \eta^s \pi_t &= e^{\int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} \frac{\partial}{\partial c_t} f(c_t^s, V_t^s) \\ c_t^s &= (\eta^s)^{-\frac{1}{1-\rho}} \beta^{\frac{1}{1-\rho}} e^{\frac{1}{1-\rho} \int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} [(1-\gamma) V_t^s]^{\frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}} (\pi_t)^{-\frac{1}{1-\rho}} \\ c_t^s &= c_s^s e^{\frac{1}{1-\rho} \int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} \left( \frac{V_t^s}{V_s^s} \right)^{\frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}} \left( \frac{\pi_t}{\pi_s} \right)^{-\frac{1}{1-\rho}} \end{aligned} \tag{15}$$

Using dynamic programming to solve the utility maximization problem of an agent born at time  $s$ , I can

state the Hamilton-Jacobi-Bellman equation as follows

$$0 = \sup_{\{c_t^s, X_t^s\}} \left\{ f \left( c_t^s, V^s \left( \widehat{W}^s, \lambda, n, t \right) \right) dt + E_t \left[ dV^s \left( \widehat{W}^s, \lambda, n, t \right) \right] \right\}$$

with  $\widehat{W}_t^s = W_t^s + E_t \left[ \int_t^\infty e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} y_u^s du \right]$  representing the agent's total wealth while  $W^s$  indicates his financial wealth. The first order condition with respect to optimal consumption is given by

$$\frac{\partial}{\partial c_t^s} f \left( c_t^s, V^s \left( \widehat{W}^s, \lambda, n, t \right) \right) = \frac{\partial}{\partial \widehat{W}^s} V^s \left( \widehat{W}^s, \lambda, n, t \right)$$

This holds conditional on survival. In the following I also condition on survival and although it is not written explicitly, I keep in mind that the variables  $c^s$ ,  $W^s$ , and  $V^s$  jump to zero when the agent dies. I make the following conjecture for the value function

$$V^s \left( \widehat{W}^s, \lambda, n, t \right) = \frac{\left( \widehat{W}_t^s \right)^{1-\gamma}}{1-\gamma} \beta^{\frac{1-\gamma}{\rho}} \psi_t \left( \lambda, n \right)^{-\frac{(1-\gamma)(1-\rho)}{\rho}} \quad (16)$$

Plugging the conjectured value function into the FOC yields

$$c_t^s = \widehat{W}_t^s \psi_t \quad (17)$$

Plugging back into the conjectured value function and solving for  $c_t^s$ , allows us to rewrite the expression for optimal consumption as

$$c_t^s = \left[ (1-\gamma) V_t^s \right]^{\frac{1}{1-\gamma}} \psi_t^{\frac{1}{\rho}} \beta^{-\frac{1}{\rho}}$$

Combining this with the expression obtained from the martingale approach (equation (15)) and solving for the value function leaves us with

$$V_t^s = \frac{1}{1-\gamma} \left( \eta^s \right)^{-\frac{1-\gamma}{\gamma}} e^{\frac{1-\gamma}{\gamma} \int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} \beta^{\frac{1-\gamma}{\gamma\rho}} \psi_t^{-\frac{(1-\gamma)(1-\rho)}{\gamma\rho}} \pi_t^{-\frac{1-\gamma}{\gamma}} \quad (18)$$

$\frac{\partial}{\partial V_t^s} f(c_t^s, V_t^s)$  is independent of the agent's time of birth, but is a function of only  $\lambda$  and  $n$ , and it is a Markov process,

$$\frac{\partial}{\partial V_t^s} f(c_t^s, V_t^s) = \frac{1-\gamma-\rho}{\rho} \psi_t - \frac{1-\gamma}{\rho} \beta - \lambda_t$$

Solving for optimal consumption yields

$$\begin{aligned} c_t^s &= \left( \eta^s \right)^{-\frac{1}{\gamma}} \beta^{\frac{1-\gamma}{\gamma\rho}} e^{\frac{1}{\gamma} \int_s^t \frac{1-\gamma-\rho}{\rho} \psi_u - \frac{1-\gamma}{\rho} \beta du} \psi_t^{-\frac{1-\gamma-\rho}{\gamma\rho}} \pi_t^{-\frac{1}{\gamma}} \\ c_t^s &= c_s^s e^{\frac{1}{\gamma} \int_s^t \frac{1-\gamma-\rho}{\rho} \psi_u - \frac{1-\gamma}{\rho} \beta du} \left( \frac{\psi_t}{\psi_s} \right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left( \frac{\pi_t}{\pi_s} \right)^{-\frac{1}{\gamma}} \end{aligned} \quad (19)$$

The dynamics of the utility function are given by (assuming the agent survives over the next instant of time)

$$\begin{aligned} \frac{dV_t^s}{V_t^s} &= \frac{1-\gamma}{\gamma} \left( \frac{\partial}{\partial V_t^s} f(c_t^s, V_t^s) + \lambda_t \right) dt - \frac{(1-\gamma)(1-\rho)}{\gamma\rho} \frac{d\psi_t}{\psi_t} - \frac{1-\gamma}{\gamma} \frac{d\pi_t}{\pi_t} \\ &\quad + \frac{(1-\gamma)^2(1-\rho)}{\gamma^2\rho} \frac{d\psi_t}{\psi_t} \frac{d\pi_t}{\pi_t} + \frac{1}{2} \left( \frac{(1-\gamma)^2}{\gamma^2} + \frac{1-\gamma}{\gamma} \right) \left( \frac{d\pi_t}{\pi_t} \right)^2 \\ &\quad + \frac{1}{2} \left( \frac{(1-\gamma)^2(1-\rho)^2}{\gamma^2\rho^2} + \frac{(1-\gamma)(1-\rho)}{\gamma\rho} \right) \left( \frac{d\psi_t}{\psi_t} \right)^2 \end{aligned}$$

According to the definition of the value function, the drift term has to equal  $-f(c_t^s, V_t^s) dt$ , which boils down to a PDE determining the function  $\psi_t(\lambda, n)$  and at the same time verifies my conjecture about the value function (given a solution for the stated PDE exists)

$$0 = \frac{1-\rho}{\rho} \psi_t(\lambda, n) - \frac{\beta}{\rho} - \frac{\gamma}{1-\gamma} \lambda_t - \frac{1-\rho}{\rho} \frac{1}{dt} E_t \left[ \frac{d\psi_t}{\psi_t} \right] + \frac{1}{dt} E_t \left[ -\frac{d\pi_t}{\pi_t} \right] \quad (20)$$

$$+ \frac{1}{2\gamma} \frac{1}{dt} \left( \frac{d\pi_t}{\pi_t} \right)^2 + \frac{(1-\gamma)(1-\rho)}{\gamma\rho} \frac{1}{dt} \frac{d\psi_t}{\psi_t} \frac{d\pi_t}{\pi_t} + \frac{1}{2} \left( \frac{(1-\gamma)(1-\rho)^2}{\gamma\rho^2} + \frac{1-\rho}{\rho} \right) \frac{1}{dt} \left( \frac{d\psi_t}{\psi_t} \right)^2$$

The last step is equivalent to solving the HJB equation.

Next, I make use of the static budget constraint,

$$0 = W_s^s = E_s \left[ \int_s^\infty e^{-\int_s^t \lambda_v dv} \frac{\pi_t}{\pi_s} (c_t^s - y_t^s) dt \right]$$

to solve for the optimal consumption level of new born agents,  $c_s^s$ . Plugging in expression (19) for optimal consumption yields

$$c_s^s = \frac{E_s \left[ \int_s^\infty e^{-\int_s^t \lambda_v dv} \frac{\pi_t}{\pi_s} y_t^s dt \right]}{E_s \left[ \int_s^\infty e^{-\int_s^t \lambda_v dv} e^{\frac{1}{\gamma} \int_s^t \frac{1-\gamma-\rho}{\rho} \psi_u - \frac{1-\gamma}{\rho} \beta du} \left( \frac{\psi_t}{\psi_s} \right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left( \frac{\pi_t}{\pi_s} \right)^{-\frac{1-\gamma}{\gamma}} dt \right]}$$

I define the following functions

$$F^{c,s}(\lambda, n, s) = E_s \left[ \int_s^\infty e^{-\int_s^t \lambda_v dv} e^{\frac{1}{\gamma} \int_s^t \frac{1-\gamma-\rho}{\rho} \psi_u - \frac{1-\gamma}{\rho} \beta du} \left( \frac{\psi_t}{\psi_s} \right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left( \frac{\pi_t}{\pi_s} \right)^{-\frac{1-\gamma}{\gamma}} dt \right]$$

and

$$F^{y,s,(i)}(\lambda, n, s) = E_s \left[ \int_s^\infty \frac{\pi_t}{\pi_s} a \frac{Y_t}{Y_s} \frac{B_i e^{-(1+\delta_i) \int_s^t n_v dv}}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} dt \right]$$

with

$$\frac{Y_s}{N_s} \sum_{i=1}^2 F^{y,s,(i)}(\lambda, n, s) = E_s \left[ \int_s^\infty e^{-\int_s^t \lambda_v dv} \frac{\pi_t}{\pi_s} y_t^s dt \right]$$

Thus, I have

$$c_s^s = \frac{Y_s}{N_s} \frac{\sum_{i=1}^2 F^{y,s,(i)}(\lambda, n, s)}{F^{c,s}(\lambda, n, s)}$$

I define the variables

$$Z_s^c = E_s \left[ \int_{-\infty}^\infty e^{-\int_{-\infty}^t \lambda_v dv} e^{\frac{1}{\gamma} \int_{-\infty}^t \frac{1-\gamma-\rho}{\rho} \psi_u - \beta \frac{1-\gamma}{\rho} du} \left( \frac{\psi_t}{\psi_{-\infty}} \right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left( \frac{\pi_t}{\pi_{-\infty}} \right)^{-\frac{1-\gamma}{\gamma}} dt \right]$$

$$= e^{-\int_{-\infty}^s \lambda_v dv} e^{\frac{1}{\gamma} \int_{-\infty}^s \frac{1-\gamma-\rho}{\rho} \psi_u - \beta \frac{1-\gamma}{\rho} du} \left( \frac{\psi_s}{\psi_{-\infty}} \right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left( \frac{\pi_s}{\pi_{-\infty}} \right)^{-\frac{1-\gamma}{\gamma}} F_s^{c,s}$$

$$+ \int_{-\infty}^s e^{-\int_{-\infty}^t \lambda_v dv} e^{\frac{1}{\gamma} \int_{-\infty}^t \frac{1-\gamma-\rho}{\rho} \psi_u - \beta \frac{1-\gamma}{\rho} du} \left( \frac{\psi_t}{\psi_{-\infty}} \right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left( \frac{\pi_t}{\pi_{-\infty}} \right)^{-\frac{1-\gamma}{\gamma}} dt$$

and

$$\begin{aligned}
Z_s^{y,(i)} &= E_s \left[ \int_{-\infty}^{\infty} \frac{\pi_t}{\pi_{-\infty}} a \frac{Y_t}{Y_{-\infty}} \frac{B_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} e^{-(1+\delta_i) \int_{-\infty}^t n_v dv} dt \right] \\
&= \frac{\pi_s}{\pi_{-\infty}} \frac{Y_s}{Y_{-\infty}} e^{-(1+\delta_i) \int_{-\infty}^s n_v dv} F_s^{y,s} + \int_{-\infty}^s \frac{\pi_t}{\pi_{-\infty}} a \frac{Y_t}{Y_{-\infty}} \frac{B_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} e^{-(1+\delta_i) \int_{-\infty}^t n_v dv} dt
\end{aligned}$$

Noticing that the newly defined quantities,  $Z_s^c$ ,  $Z_s^{y,(1)}$  and  $Z_s^{y,(2)}$  are (local) martingales and (by the tower property of conditional expectations) their drift terms equal zero, I get PDE's that determine the functions  $F^{c,s}(\lambda, n, s)$ ,  $F^{y,s,(1)}(\lambda, n, s)$  and  $F^{y,s,(2)}(\lambda, n, s)$

$$\begin{aligned}
0 &= \left[ -\lambda_s + \frac{1-\gamma-\rho}{\gamma\rho} \psi_s - \frac{1}{\gamma} \beta \frac{1-\gamma}{\rho} \right] ds F_s^{c,s} + ds \\
&+ E_s [dF_s^{c,s}] - \frac{1-\gamma}{\gamma} E_s \left[ \frac{d\pi_s}{\pi_s} \right] F_s^{c,s} - \frac{1-\gamma-\rho}{\gamma\rho} E_s \left[ \frac{d\psi_s}{\psi_s} \right] F_s^{c,s} \\
&+ \frac{1}{2} \frac{(1-\gamma-\rho)(1-\gamma)(1-\rho)}{\gamma^2 \rho^2} \left( \frac{d\psi_s}{\psi_s} \right)^2 F_s^{c,s} + \frac{1}{2} \frac{1-\gamma}{\gamma^2} \left( \frac{d\pi_s}{\pi_s} \right)^2 F_s^{c,s} \\
&+ \frac{(1-\gamma-\rho)(1-\gamma)}{\gamma^2 \rho} \frac{d\psi_s}{\psi_s} \frac{d\pi_s}{\pi_s} F_s^{c,s} - \frac{1-\gamma-\rho}{\gamma\rho} \frac{d\psi_s}{\psi_s} dF_s^{c,s} - \frac{1-\gamma}{\gamma} \frac{d\pi_s}{\pi_s} dF_s^{c,s}
\end{aligned} \tag{21}$$

and  $\forall i \in \{1, 2\}$

$$\begin{aligned}
0 &= \left[ \frac{1}{ds} E_s \left[ \frac{d\pi_s}{\pi_s} \right] + \mu_s^{(Y)} + \frac{1}{ds} \frac{d\pi_s}{\pi_s} \frac{dY_s}{Y_s} - (1+\delta_i) n_s \right] ds F_s^{y,s,(i)} + E_s [dF_s^{y,s,(i)}] \\
&+ dF_s^{y,s,(i)} \frac{d\pi_s}{\pi_s} + dF_s^{y,s,(i)} \frac{dY_s}{Y_s} + a \frac{B_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} ds
\end{aligned} \tag{22}$$

Alternatively, using equation (17) I can derive  $F^{c,s}(\lambda, n, s)$  as follows

$$\begin{aligned}
c_s^s &= \widehat{W}_s^s \psi_s = \left( W_s^s + E_s \left[ \int_s^{\infty} e^{-\int_s^t \lambda_v dv} \frac{\pi_t}{\pi_s} y^s dt \right] \right) \psi_s \\
&= E_s \left[ \int_s^{\infty} e^{-\int_s^t \lambda_v dv} \frac{\pi_t}{\pi_s} c_t^s dt \right] \psi_s = c_s^s F_s^{c,s} \psi_s \\
F_s^{c,s} &= \frac{1}{\psi_s}
\end{aligned}$$

Combining with the PDE determining  $F^{c,s}(\lambda, n, s)$  (equation (21)) I end up with

$$\begin{aligned}
0 &= \frac{1-\rho}{\rho} \psi_s ds - \frac{\beta}{\rho} ds - \frac{\gamma}{1-\gamma} \lambda_s ds - \frac{1-\rho}{\rho} E_s \left[ \frac{d\psi_s}{\psi_s} \right] + E_s \left[ -\frac{d\pi_s}{\pi_s} \right] \\
&+ \frac{1}{2\gamma} \left( \frac{d\pi_s}{\pi_s} \right)^2 + \frac{1}{2} \left( \frac{(1-\gamma)(1-\rho)^2}{\gamma\rho^2} + \frac{1-\rho}{\rho} \right) \left( \frac{d\psi_s}{\psi_s} \right)^2 + \frac{(1-\gamma)(1-\rho)}{\gamma\rho} \frac{d\psi_s}{\psi_s} \frac{d\pi_s}{\pi_s}
\end{aligned}$$

which is the same as the PDE (20) that determines  $\psi_t(\lambda, n)$ . This verifies the conjecture about the value function (16). Equation (17) also tells us that  $\psi_t(\lambda, n)$  describes the consumption to wealth ratio.

Market clearing in the consumption market implies

$$dY_t = dC_t$$

Growth in aggregate output is exogenously given and for aggregate consumption I have

$$\begin{aligned} dC_t &= d\left(\int_{-\infty}^t c_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds\right) \\ &= c_t^s n_t N_t dt + \int_{-\infty}^t \frac{dc_t^s}{c_t^s} c_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds - \lambda_t C_t dt \end{aligned} \quad (23)$$

Notice that if death rates were age-dependent then the last term in equation (23) becomes messy because the integral over individual consumptions will not add up to aggregate consumption anymore. I can use expression (19) to get the dynamics of the optimal consumption process

$$\begin{aligned} \frac{dc_t^s}{c_t^s} &= \frac{1-\gamma-\rho}{\rho\gamma} \psi_t dt - \beta \frac{1-\gamma}{\gamma\rho} dt - \frac{1-\gamma-\rho}{\gamma\rho} \frac{d\psi_t}{\psi_t} - \frac{1}{\gamma} \frac{d\pi_t}{\pi_t} + \frac{1+\gamma}{2\gamma^2} \left(\frac{d\pi_t}{\pi_t}\right)^2 \\ &\quad + \frac{(1-\gamma-\rho)(1-\gamma)(1-\rho)}{2\gamma^2\rho^2} \left(\frac{d\psi_t}{\psi_t}\right)^2 + \frac{1-\gamma-\rho}{\gamma^2\rho} \frac{d\psi_t}{\psi_t} \frac{d\pi_t}{\pi_t} \end{aligned} \quad (24)$$

Plugging back into the market clearing condition and solving for growth in the SDF gives

$$\begin{aligned} \frac{d\pi_t}{\pi_t} &= -\gamma \frac{dY_t}{Y_t} - \gamma \lambda_t dt + \gamma n_t \sum_{i=1}^2 F^{y,t,(i)} \psi_t dt + \frac{1-\gamma-\rho}{\rho} \psi_t dt - \beta \frac{1-\gamma}{\rho} dt + \frac{1+\gamma}{2\gamma} \left(\frac{d\pi_t}{\pi_t}\right)^2 \\ &\quad - \frac{1-\gamma-\rho}{\rho} \frac{d\psi_t}{\psi_t} + \frac{(1-\gamma-\rho)(1-\gamma)(1-\rho)}{2\gamma\rho^2} \left(\frac{d\psi_t}{\psi_t}\right)^2 + \frac{1-\gamma-\rho}{\gamma\rho} \frac{d\psi_t}{\psi_t} \frac{d\pi_t}{\pi_t} \end{aligned}$$

Using (20) and (22), and plugging in the expression for the SDF, I can derive a system of 3 differential equations that determines the quantities  $\psi_t$  and  $F^{y,t,(i)}$ ,  $\forall i \in \{1, 2\}$

$$\begin{aligned} 0 &= -\left[r_t - \mu_t^{(Y)} + \gamma \left(\sigma^{(A)}\right)^2 + \frac{1-\gamma-\rho}{\rho} \frac{\sigma_t^{(\psi)}}{\psi_t} \sigma^{(A)} + (1+\delta_i) n_t\right] F_t^{y,t,(i)} \\ &\quad + \mu_t^{(F^{y,(i)})} - \frac{1-\gamma-\rho}{\rho} \sigma_t^{(F^{y,(i)})} \frac{\sigma_t^{(\psi)}}{\psi_t} + (1-\gamma) \sigma^{(A)} \sigma_t^{(F^{y,(i)})} + \frac{aB_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \\ 0 &= -\psi_t + \frac{1}{1-\rho} \beta + \frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \lambda_t - \frac{\rho}{1-\rho} r_t + \frac{\mu_t^{(\psi)}}{\psi_t} \\ &\quad - \frac{1}{2} \left(\frac{1-\gamma}{\rho} + \frac{\gamma}{1-\rho}\right) \left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^2 - \frac{1}{2} \frac{\gamma\rho}{1-\rho} \left(\sigma^{(A)}\right)^2 + \frac{\gamma\rho}{1-\rho} \sigma^{(A)} \frac{\sigma_t^{(\psi)}}{\psi_t} \end{aligned} \quad (25)$$

with the dynamics of  $F^{y,t,(i)}$  and  $\psi_t$  defined as

$$\begin{aligned} \mu_t^{(F^{y,(i)})} &= F_{\lambda}^{y,t,(i)} \mu_t^{(\lambda)} + F_n^{y,t,(i)} \mu_t^{(n)} \\ &\quad + \frac{1}{2} F_{\lambda\lambda}^{y,t,(i)} \left(\sigma_t^{(\lambda)}\right)^2 + \frac{1}{2} F_{nn}^{y,t,(i)} \left(\sigma_t^{(n)}\right)^2 + F_{\lambda n}^{y,t,(i)} \sigma_t^{(\lambda)} \left(\sigma_t^{(n)}\right)^T \\ \sigma_t^{(F^{y,(i)})} &= F_{\lambda}^{y,t,(i)} \sigma_t^{(\lambda)} + F_n^{y,t,(i)} \sigma_t^{(n)} \\ \mu_t^{(\psi)} &= \psi_{\lambda} \mu_t^{(\lambda)} + \psi_n \mu_t^{(n)} \\ &\quad + \frac{1}{2} \psi_{\lambda\lambda} \left(\sigma_t^{(\lambda)}\right)^2 + \frac{1}{2} \psi_{nn} \left(\sigma_t^{(n)}\right)^2 + \psi_{\lambda n} \sigma_t^{(\lambda)} \left(\sigma_t^{(n)}\right)^T \\ \sigma_t^{(\psi)} &= \psi_{\lambda} \sigma_t^{(\lambda)} + \psi_n \sigma_t^{(n)} \end{aligned}$$

By definition of  $r_t = E_t \left[ -\frac{d\pi_t}{\pi_t} \right]$  and  $\kappa_t = -\frac{d\pi_t}{\pi_t} - E_t \left[ -\frac{d\pi_t}{\pi_t} \right]$ , Proposition 4 follows,

$$\begin{aligned} r_t &= \beta + (1 - \rho) \left[ \mu_t^{(Y)} + \lambda_t - n_t \sum_{i=1}^2 F^{y,t,(i)} \psi_t \right] - \frac{1 - \gamma - \rho}{1 - \gamma} \lambda_t \\ &\quad + \frac{1 - \gamma - \rho}{2\rho} \left( \frac{\sigma_t^{(\psi)}}{\psi_t} \right)^2 - \frac{\gamma(2 - \rho)}{2} \left( \sigma^{(A)} \right)^2 - \frac{1 - \gamma - \rho}{\rho} \sigma^{(A)} \frac{\sigma_t^{(\psi)}}{\psi_t} \\ \kappa_t &= \frac{1 - \gamma - \rho}{\rho} \frac{\sigma_t^{(\psi)}}{\psi_t} + \gamma \sigma^{(A)} \end{aligned}$$

In the following I describe how the model can be solved numerically under the assumption that

$$\begin{aligned} dn_t &= \mu_t^{(n)} dt + \sigma_t^{(n)} d\widetilde{W}_t = n_t \mu^{(n)} dt + n_t \sigma^{(n)} d\widetilde{W}_t \\ d\lambda_t &= \mu_t^{(\lambda)} dt + \sigma_t^{(\lambda)} d\widetilde{W}_t = \lambda_t \mu^{(\lambda)} dt + \lambda_t \sigma^{(\lambda)} d\widetilde{W}_t \end{aligned}$$

I solve the differential equations numerically using the finite difference method. To derive boundary conditions I first set  $\lambda_t = n_t = 0$  and the model collapses to the special case of one infinitely-lived representative agent. The system of differential equations is

$$\begin{aligned} F_t^{y,t,(i)} &= \frac{1}{r_t - \mu^{(A)} - (1 - a) \mu^{(K)} + \gamma \left( \sigma^{(A)} \right)^2} \frac{aB_i}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \\ \psi_t &= \frac{1}{1 - \rho} \beta - \frac{\rho}{1 - \rho} r_t - \frac{1}{2} \frac{\gamma \rho}{1 - \rho} \left( \sigma^{(A)} \right)^2 \end{aligned}$$

and

$$\begin{aligned} r_t &= \beta + (1 - \rho) \left( \mu^{(A)} + (1 - a) \mu^{(K)} \right) - \frac{\gamma(2 - \rho)}{2} \left( \sigma^{(A)} \right)^2 \\ \kappa_t &= \gamma \sigma^{(A)} \end{aligned}$$

Next I assume  $(1 - \rho)(1 - a) - \frac{1 - \gamma - \rho}{1 - \gamma} \neq 0$  and  $\frac{\rho}{1 - \gamma} - \rho(1 - a) \neq 0$ . For  $\lim_{\lambda_t \rightarrow \infty} \frac{n_t}{\lambda_t} = 0$  I conjecture that  $\lim_{\lambda_t \rightarrow \infty} \frac{r_t}{\lambda_t} \neq 0$ ,  $\left| \lim_{\lambda_t \rightarrow \infty} \frac{r_t}{\lambda_t} \right| < \infty$ ,  $\lim_{\lambda_t \rightarrow \infty} \frac{\psi_t}{\lambda_t} \neq 0$ ,  $\left| \lim_{\lambda_t \rightarrow \infty} \frac{\psi_t}{\lambda_t} \right| < \infty$ ,  $\lim_{\lambda_t \rightarrow \infty} F_t^{y,t,(i)} = 0$ . This implies

$$\begin{aligned} \lim_{\lambda_t \rightarrow \infty} \psi_\lambda &= \lim_{\lambda_t \rightarrow \infty} \frac{\psi_t}{\lambda_t} \\ \lim_{\lambda_t \rightarrow \infty} \psi_{\lambda\lambda} &= 0 \\ \lim_{\lambda_t \rightarrow \infty} F_\lambda^{y,t,(i)} &= \lim_{\lambda_t \rightarrow \infty} F_{\lambda\lambda}^{y,t,(i)} = \lim_{\lambda_t \rightarrow \infty} F_{\lambda n}^{y,t,(i)} = 0 \end{aligned}$$

The system of differential equations becomes

$$\begin{aligned} \lim_{\lambda_t \rightarrow \infty} \frac{r_t}{\lambda_t} &= (1 - \rho)(1 - a) - \frac{1 - \gamma - \rho}{1 - \gamma} \\ \lim_{\lambda_t \rightarrow \infty} \frac{\psi_t}{\lambda_t} &= \frac{\rho}{1 - \rho} \frac{\gamma}{1 - \gamma} - \frac{\rho}{1 - \rho} \lim_{\lambda_t \rightarrow \infty} \frac{r_t}{\lambda_t} = \frac{\rho}{1 - \gamma} - \rho(1 - a) \\ \lim_{\lambda_t \rightarrow \infty} F_t^{y,t,(i)} &= 0 \end{aligned}$$

The conjectures are indeed consistent with the solution. I have boundaries at  $(\lambda_t, n_t) = (0, 0)$  and  $(\lambda_t, n_t) = (\lambda_{\max}, 0)$  with  $\lambda_{\max} \rightarrow \infty$  and can numerically solve the above differential equation over the space spanned by  $\lambda_t$  and keeping  $n_t = 0$ . I choose  $n_{\max}$  small enough and solve the differential equations on the space

(grid) spanned by  $\lambda_t \in (0, \lambda_{\max})$  and  $n_t \in (0, n_{\max})$ . From the above discussion I have boundaries at  $n_t = 0$  and  $\lambda_t \in (0, \lambda_{\max})$ , and  $n_t \in (0, n_{\max})$  and  $\lambda_{\max}$ . ■

**Proof of Proposition 1.** Proposition 1 is a special case of Proposition 4 and follows immediately when using  $\sigma^{(A)} = 0$ ,  $d\lambda_t = 0$  and  $dn_t = 0$ . Moreover, I rewrite

$$\begin{aligned} c_t^t(r) &= \frac{C_t}{N_t} \sum_{i=1}^2 F^{y,t,(i)} \psi_t \\ &= \frac{C_t}{N_t} a \frac{\int_t^\infty \sum_{i=1}^2 e^{-(r+n(1+\delta_i)-\mu^{(Y)})(s-t)} \frac{B_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} ds}{\int_t^\infty e^{-\left(\frac{\rho\gamma}{(1-\gamma)(1-\rho)}\lambda + \frac{1}{1-\rho}\beta - \frac{\rho}{1-\rho}r\right)(s-t)} ds} \\ &= \frac{C_t}{N_t} \sum_{i=1}^2 \frac{aB_i \left( \frac{1}{1-\rho}\beta + \frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\lambda - \frac{\rho}{1-\rho}r \right)}{\left( \frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2} \right) (r - \mu^{(Y)} + (1 + \delta_i)n)} \end{aligned}$$

■

**Proof of Proposition 2 and 3.** The proof of Proposition 2 and 3 are basically the same. The main difference to Proposition 4 is that the function  $\psi_t(\lambda, n)$  does not follow a continuous diffusion process, but jumps between two distinct values. The argument follows basically the same lines. The derivation from equation (14) to equation (19) is carried over without any change. The further derivation differs slightly. To derive optimal consumption of new born agents I make use of the static budget constraint and define the functions  $F^{c,s}(\lambda, n, s)$ ,  $F^{y,s,(i)}(\lambda, n, s)$ ,  $Z_s^c$  and  $Z_s^{y,(i)}$  as before. Using the martingale property of the  $Z$  functions and setting the drift zero yields

$$\begin{aligned} 0 &= \left[ -\lambda_{s-} + \frac{1-\gamma-\rho}{\gamma\rho} \psi_{s-} - \frac{1-\gamma}{\gamma\rho} \beta \right] ds F^{c,s-} + ds - \frac{1-\gamma}{\gamma} E_s \left[ \frac{d\pi_{s-}^{(cp)}}{\pi_{s-}} \right] F^{c,s-} \\ &\quad + \frac{1}{2} \frac{1-\gamma}{\gamma^2} \left( \frac{d\pi_{s-}^{(cp)}}{\pi_{s-}} \right)^2 F^{c,s-} + E_s \left[ \left( \left( \frac{\psi_s}{\psi_{s-}} \right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left( \frac{\pi_s}{\pi_{s-}} \right)^{-\frac{1-\gamma}{\gamma}} \frac{F^{c,s}}{F^{c,s-}} - 1 \right) F^{c,s-} |dS_s| \right] \end{aligned}$$

and

$$\begin{aligned} 0 &= \left[ E_s \left[ \frac{d\pi_{s-}^{(cp)}}{\pi_{s-}} \right] + \mu_{s-}^{(Y)} ds + \frac{d\pi_{s-}^{(cp)}}{\pi_{s-}} \frac{dY_{s-}}{Y_{s-}} - (1 + \delta_i) n_{s-} ds \right] F^{y,s-,(i)} \\ &\quad + E_s \left[ \left( \frac{\pi_s}{\pi_{s-}} \frac{F^{y,s,(i)}}{F^{y,s-,(i)}} - 1 \right) F^{y,s-,(i)} |dS_s| \right] + \frac{aB_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} ds \end{aligned}$$

where the superscript  $(cp)$  denotes the continuous part of the process (for notational details compare Shreve, 2004). Using the relation  $F^{c,s} = \frac{1}{\psi_s}$  gives the equation determining  $\psi_t$

$$\begin{aligned} 0 &= \frac{1-\rho}{\rho} \psi_{s-}(\lambda, n) - \frac{\beta}{\rho} - \frac{\gamma}{1-\gamma} \lambda_{s-} - \frac{1}{ds} E_s \left[ \frac{d\pi_{s-}^{(cp)}}{\pi_{s-}} \right] \\ &\quad + \frac{1}{2\gamma} \left( \frac{d\pi_{s-}^{(cp)}}{\pi_{s-}} \right)^2 + \frac{\gamma}{1-\gamma} E_s \left[ \left( \left( \frac{\psi_s}{\psi_{s-}} \right)^{-\frac{(1-\gamma)(1-\rho)}{\gamma\rho}} \left( \frac{\pi_s}{\pi_{s-}} \right)^{-\frac{1-\gamma}{\gamma}} - 1 \right) |dS_s| \right] \end{aligned}$$

Given these functions, it holds

$$c_s^s = \frac{Y_s}{N_s} \sum_{i=1}^2 F^{y,s,(i)} \psi_s$$

The dynamics of the optimal consumption process for the individual agent are (using equation (19))

$$\frac{dc_t^s}{c_{t-}^s} = \left( \frac{1-\gamma-\rho}{\rho\gamma} \psi_{t-}(\lambda, n) - \frac{1-\gamma}{\rho\gamma} \beta \right) dt - \frac{1}{\gamma} \frac{d\pi_{t-}^{(cp)}}{\pi_{t-}} + \frac{1+\gamma}{\gamma^2} \left( \frac{d\pi_{t-}^{(cp)}}{\pi_{t-}} \right)^2 + \frac{c_t^s - c_{t-}^s}{c_{t-}^s} |dS_t|$$

and  $\frac{c_t^s - c_{t-}^s}{c_{t-}^s}$  is given by

$$\frac{c_t^s - c_{t-}^s}{c_{t-}^s} = \left( \frac{\psi_t}{\psi_{t-}} \right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left( \frac{\pi_t}{\pi_{t-}} \right)^{-\frac{1}{\gamma}} - 1 \quad (26)$$

From equation (26) and the fact that on the aggregate consumption and output are smooth,  $C_t = C_{t-}$  and  $Y_t = Y_{t-}$  (no discontinuities), it follows that the pricing kernel process must have a jump component inherent, and in particular, it must hold

$$\begin{aligned} \frac{\pi_t}{\pi_{t-}} &= \left( \frac{\psi_t}{\psi_{t-}} \right)^{-\frac{1-\gamma-\rho}{\rho}} \\ \frac{d\pi_{t-}}{\pi_{t-}} - \frac{d\pi_{t-}^{(cp)}}{\pi_{t-}} &= \left( \left( \frac{\psi_t}{\psi_{t-}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) |dS_t| \end{aligned} \quad (27)$$

Finally, imposing market clearing in the consumption good market as before ( $dY_t = dC_t$ ) and solving for the SDF yields

$$\frac{d\pi_t^{(cp)}}{\pi_t} = -\gamma \frac{dY_t}{Y_t} + \left[ -\frac{1-\gamma}{\rho} \beta - \gamma \lambda_t + \gamma n_t \sum_{i=1}^2 F^{y,t,(i)} \psi_t + \frac{1-\gamma-\rho}{\rho} \psi_t \right] dt + \frac{1+\gamma}{2\gamma} \left( \frac{d\pi_t^{(cp)}}{\pi_t} \right)^2$$

Adding the jump component leaves us with the quantities

$$\begin{aligned} r_{t-} &= \beta + (1-\rho) \left[ \mu_{t-}^{(Y)} + \lambda_{t-} - n_{t-} \sum_{i=1}^2 F^{y,t-,(i)} \psi_{t-} \right] - \frac{1-\gamma-\rho}{1-\gamma} \lambda_{t-} - \frac{\gamma(2-\rho)}{2} (\sigma^{(A)})^2 \\ &\quad - \bar{\theta}_{t-} \left( \left( \frac{\psi_t}{\psi_{t-}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) + \frac{1-\gamma-\rho}{1-\gamma} \bar{\theta}_{t-} \left( \left( \frac{\psi_t}{\psi_{t-}} \right)^{-\frac{1-\gamma}{\rho}} - 1 \right) \end{aligned}$$

and

$$\begin{aligned} \kappa_t^{(cp)} &= \gamma \sigma^{(A)} \\ \kappa_{t-}^{(J)} &= - \left( \left( \frac{\psi_t}{\psi_{t-}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \end{aligned}$$

with

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \kappa_t^{(cp)} d\widetilde{W}_t - \kappa_t^{(J)} |d\widehat{S}_t|$$



and

$$\begin{aligned}
\psi_{s-} &= \frac{1}{1-\rho}\beta + \frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\lambda_{s-} - \frac{\rho}{1-\rho}r_{s-} - \frac{\rho}{1-\rho}\frac{\gamma}{2}\left(\sigma^{(A)}\right)^2 \\
&\quad - \frac{\rho}{1-\rho}\bar{\theta}_{s-}\left(\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) - \frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\bar{\theta}_{s-}\left(\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right) \\
&= \beta - \rho\mu_{s-}^{(Y)} + \rho n_{s-}\sum_{i=1}^2 F^{y,s-, (i)}\psi_{s-} + \frac{\gamma\rho}{1-\gamma}\lambda_{s-} + \frac{\gamma\rho}{2}\left(\sigma^{(A)}\right)^2 - \frac{\rho}{1-\gamma}\bar{\theta}_{s-}\left(\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right) \\
F^{y,s-, (i)} &= \frac{1}{r_{s-} - \mu_{s-}^{(Y)} + \gamma\left(\sigma^{(A)}\right)^2 + (1 + \delta_i)n_{s-} + \bar{\theta}_{s-}\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma-\rho}{\rho}}\left(1 - \frac{F^{y,s-, (i)}}{F^{y,s-, (i)}}\right)} \frac{aB_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}}
\end{aligned}$$

In the qualitative analysis in the main text I assume that there are no TFP shocks ( $\sigma^{(A)} = 0$ ) for illustrative purposes. In this case there is no market price of risk for TFP shocks but only for demographic shocks and I write  $\kappa_t^{(J)} = \kappa_t$  to keep notation simple. ■

## 5.7 Proofs of Lemmas

**Proof of Lemma 1.** From market clearing in financial markets it follows immediately that

$$P_t = \int_{-\infty}^t W_t^s n_s N_s e^{-\int_s^t \lambda_v dv} ds$$

From the static budget constraint it follows the expression for financial wealth

$$W_t^s = E_t \left[ \int_t^\infty e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} (c_u^s - y_u^s) du \right]$$

The constraint is binding at optimum because of local non-satiation (utility is increasing in consumption). Plugging in yields

$$\begin{aligned}
P_t &= \int_{-\infty}^t E_t \left[ \int_t^\infty e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} c_u^s du \right] n_s N_s e^{-\int_s^t \lambda_v dv} ds \\
&\quad - \int_{-\infty}^t E_t \left[ \int_t^\infty e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} y_u^s du \right] n_s N_s e^{-\int_s^t \lambda_v dv} ds
\end{aligned}$$

For the first term I have

$$\begin{aligned}
&\int_{-\infty}^t E_t \left[ \int_t^\infty e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} c_u^s du \right] n_s N_s e^{-\int_s^t \lambda_v dv} ds \\
&= \int_{-\infty}^t \widehat{W}_t^s n_s N_s e^{-\int_s^t \lambda_v dv} ds = \int_{-\infty}^t \frac{\widehat{W}_t^s}{c_t^s} c_t^s n_s N_s e^{-\int_s^t \lambda_v dv} ds \\
&= \int_{-\infty}^t \frac{1}{\psi_t(\lambda, n)} c_t^s n_s N_s e^{-\int_s^t \lambda_v dv} ds = \frac{1}{\psi_t(\lambda, n)} C_t
\end{aligned}$$

For the second term it holds

$$\begin{aligned}
& \int_{-\infty}^t E_t \left[ \int_t^{\infty} e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} y_u^s du \right] n_s N_s e^{-\int_s^t \lambda_v dv} ds \\
&= \int_{-\infty}^t E_t \left[ \int_t^{\infty} e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} a \frac{Y_u}{G_u} \left( \sum_{i=1}^2 B_i e^{-\delta_i \int_s^t n_v dv} e^{-\delta_i \int_t^u n_v dv} \right) du \right] n_s N_s e^{-\int_s^t \lambda_v dv} ds \\
&= \int_{-\infty}^t \sum_{i=1}^2 e^{-\delta_i \int_s^t n_v dv} E_t \left[ \int_t^{\infty} e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} a \frac{Y_u}{G_u} B_i e^{-\delta_i \int_t^u n_v dv} du \right] n_s N_s e^{-\int_s^t \lambda_v dv} ds \\
&= \int_{-\infty}^t \sum_{i=1}^2 \frac{Y_t}{N_t} e^{-\delta_i \int_s^t n_v dv} F^{y,t,(i)}(\lambda, n, t) n_s N_s e^{-\int_s^t \lambda_v dv} ds \\
&= Y_t \sum_{i=1}^2 F^{y,t,(i)}(\lambda, n, t) \int_{-\infty}^t n_s e^{-(1+\delta_i) \int_s^t n_v dv} ds = Y_t \sum_{i=1}^2 \frac{F^{y,t,(i)}(\lambda, n, t)}{1 + \delta_i}
\end{aligned}$$

Combining and imposing market clearing in the consumption goods market ( $Y_t = C_t$ ) gives

$$P_t = Y_t \left[ \frac{1}{\psi_t(\lambda, n)} - \sum_{i=1}^2 \frac{F^{y,t,(i)}(\lambda, n, t)}{1 + \delta_i} \right]$$

■

**Proof of Lemma 2.** Following the definition of the price of an asset that pays dividends  $D_t$ , I can write

$$\begin{aligned}
P_t &= E_t \left[ \int_t^T \frac{\pi_s}{\pi_t} D_s ds + \frac{\pi_T}{\pi_t} P_T \right] \\
E_t [P_0] &= E_t \left[ \int_0^T \frac{\pi_s}{\pi_0} D_s ds + \frac{\pi_T}{\pi_0} P_T \right] = P_t \frac{\pi_t}{\pi_0} + \int_0^t \frac{\pi_s}{\pi_0} D_s ds
\end{aligned}$$

Noticing that the  $E_t [P_0]$  is a martingale (according to the tower property of conditional expectations), it follows immediately

$$\begin{aligned}
0 &= E_t [d(E_t [P_0])] = E_t \left[ dP_t \frac{\pi_t}{\pi_0} + P_t \frac{d\pi_t}{\pi_0} + dP_t \frac{d\pi_t}{\pi_0} + \frac{\pi_t}{\pi_0} D_t dt \right] \\
E_t \left[ \frac{dP_t + D_t}{P_t} \right] - r_t &= -\frac{dP_t}{P_t} \frac{d\pi_t}{\pi_t}
\end{aligned}$$

■

**Proof of Lemma 3.** First note that I often use the notation  $\Phi_t(r) = \frac{N_t c_t^i(x)}{C_t} = \sum_{i=1}^2 F_t^{y,(i)} \psi_t$  to describe the ratio between consumption of a new born agent and per capita GDP. Let  $\bar{\rho}^{(r)} = \min_{\rho \in \{\Upsilon^{(r)} \cup 0\}} \{\rho\}$  with  $\Upsilon^{(r)} = \{\rho : v^{(r)}(\rho) = 0, \rho < 0\}$  and  $v^{(r)}(\rho) = a(n - \lambda) - \frac{\gamma}{1-\gamma} \lambda - n\Phi(r(\rho))$ . I show that the condition  $\rho < \bar{\rho}^{(r)}$  (or  $EIS < \overline{EIS}^{(r)} = \frac{1}{1-\bar{\rho}^{(r)}}$ ) suffices for the interest rate in an OLG economy to be smaller than the rate in an equivalent economy populated by a representative infinitely-lived agent ( $r < r_*$ ). Moreover, I show that for  $B_2 = 0$  or  $\delta_1 = \delta_2$ , the function  $v^{(r)}(\rho)$  is monotonically decreasing in  $-\rho$  (for  $\rho < 0$ ), and if  $\lim_{\rho \nearrow 0} [v^{(r)}(\rho)] < 0$ , then the set  $\Upsilon^{(r)}$  is single valued, and otherwise empty. It follows that for  $B_2 = 0$  or  $\delta_1 = \delta_2$  there exists no  $\rho > \bar{\rho}^{(r)}$  that satisfies  $v^{(r)}(\rho) < 0$ . In the general case ( $B_2 \neq 0$  and  $\delta_1 \neq \delta_2$ ) there might exist  $\rho > \bar{\rho}^{(r)}$  that satisfies  $v^{(r)}(\rho) < 0$ . I need the technical conditions  $r - \frac{\gamma}{1-\gamma} \lambda - \beta, r - \mu^{(Y)} > 0$ ,

and  $\mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - na \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$ . Given  $\gamma > 1$ , I have

$$(1-\rho)[a(n-\lambda) + \lambda - n\Phi(r)] - \frac{1-\gamma-\rho}{1-\gamma}\lambda < (1-\rho) \left[ a(n-\lambda) - \frac{\gamma}{1-\gamma}\lambda - n\Phi(r) \right]$$

The expression in equation (7) is negative and it holds  $r < r_*$ , if the sufficient condition

$$n\Phi(r) > a(n-\lambda) - \frac{\gamma}{1-\gamma}\lambda \quad (28)$$

is satisfied. Because  $\psi$  is constant across cohorts, it holds

$$\frac{c_t^t}{c_t^s} = \frac{\widehat{W}_t^t}{\widehat{W}_t^s} = \frac{Y_t N_s \widehat{W}_s^s}{Y_s N_t \widehat{W}_t^s} = e^{\int_s^t \mu^{(Y)} - (n-\lambda) - (r+\lambda-\psi) du} = e^{\int_s^t \mu^{(Y)} - n - r + \psi du}$$

and

$$\begin{aligned} \Phi(r) &= \frac{c_t^t}{\frac{c_t^t}{N_t}} = \frac{c_t^t}{\int_{-\infty}^t c_t^s n_s \frac{N_s}{N_t} e^{-\int_s^t \lambda du} ds} = \frac{1}{\int_{-\infty}^t \frac{c_t^s}{c_t^t} n e^{-\int_s^t n du} ds} \\ &= \frac{1}{\int_{-\infty}^t e^{\int_s^t r - \mu^{(Y)} - \psi du} n ds} = \frac{\psi + \mu^{(Y)} - r}{n} \end{aligned}$$

with  $\psi + \mu^{(Y)} - r > 0$  since  $\psi + \mu^{(Y)} - r = n\Phi(r)$ . I look at how condition (28) behaves in the limit when the *EIS* approaches zero. Taking the limit of  $\rho$  approaching  $-\infty$ , I get for my key variables

$$\lim_{\rho \rightarrow -\infty} \frac{r}{1-\rho} = \mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - n \lim_{\rho \rightarrow -\infty} \Phi(r)$$

and

$$\lim_{\rho \rightarrow -\infty} \frac{\psi}{1-\rho} = \lim_{\rho \rightarrow -\infty} \frac{r}{1-\rho}$$

and  $\forall i \in \{1, 2\}$

$$\lim_{\rho \rightarrow -\infty} (1-\rho) F^{y,(i)} = \frac{1}{\lim_{\rho \rightarrow -\infty} \frac{r}{1-\rho}} \frac{aB_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}}$$

Suppose  $|\lim_{\rho \rightarrow -\infty} \frac{r}{1-\rho}| < \infty$  and  $\lim_{\rho \rightarrow -\infty} \frac{r}{1-\rho} \neq 0$ , I get

$$\lim_{\rho \rightarrow -\infty} \Phi(r) = a \frac{B_1 + B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}}$$

Indeed  $|\lim_{\rho \rightarrow -\infty} \frac{r}{1-\rho}| = \mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - na \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} < \infty$  and in general  $\lim_{\rho \rightarrow -\infty} \frac{r}{1-\rho} = \mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - na \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$ . In the case of  $B_2 = 0$  or  $\delta_1 = \delta_2$ ,  $\lim_{\rho \rightarrow -\infty} \Phi(r)$  simplifies to

$$\lim_{\rho \rightarrow -\infty} \Phi(r) = a(1+\delta_1)$$

In the limit as the *EIS* approaches zero condition (28) is satisfied if  $\frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} > 1 - \frac{1+\frac{1-a}{1-\gamma}\lambda}{1-\gamma} \frac{\lambda}{n}$ . For  $B_2 = 0$  or  $\delta_1 = \delta_2$ , the condition boils down to  $\delta_1 > -\frac{1+\frac{1-a}{1-\gamma}\lambda}{1-\gamma} \frac{\lambda}{n}$ . Using  $\gamma > 1$ ,  $r - \frac{\gamma}{1-\gamma}\lambda - \beta$  and the conditions of

Lemma 3 ( $r - \mu^{(Y)} > 0$ ), and taking the derivative of  $r$  with respect to  $-\rho$  gives

$$\begin{aligned}
\frac{\partial r}{\partial(-\rho)} &= \frac{\mu^{(Y)} + \lambda - n\Phi(r) - n\frac{1}{1-\rho}\left(r - \beta - \frac{\gamma}{1-\gamma}\lambda\right) \sum_{i=1}^2 F^{y,(i)} - \frac{1}{1-\gamma}\lambda}{1 + (1-\rho)n\Phi'(r)} \\
&= \frac{\mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - n\Phi(r) - \left(\frac{1}{1-\rho}r - \frac{1}{1-\rho}\beta - \frac{1}{1-\rho}\frac{\gamma}{1-\gamma}\lambda\right) \frac{1}{\psi}n\Phi(r)}{1 + (1-\rho)n\Phi'(r)} \\
&= \frac{\frac{1}{\psi}(\psi - n\Phi(r))\left(\mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - n\Phi(r)\right)}{1 + (1-\rho)n\Phi'(r)} \\
&= \frac{\frac{1}{\psi}(r - \mu^{(Y)})\left(r - \frac{\gamma}{1-\gamma}\lambda - \psi\right)}{1 + (1-\rho)n\Phi'(r)} = \frac{\frac{1}{\psi}(r - \mu^{(Y)})\frac{1}{1-\rho}\left(r - \frac{\gamma}{1-\gamma}\lambda - \beta\right)}{1 + (1-\rho)n\Phi'(r)} > 0
\end{aligned}$$

Assuming  $\rho < 0$  and taking the derivative of  $\psi$  with respect to  $-\rho$  yields

$$\begin{aligned}
\frac{\partial \psi}{\partial(-\rho)} &= \frac{1}{(1-\rho)^2}\left(r - \frac{\gamma}{1-\gamma}\lambda - \beta\right) - \frac{\rho}{1-\rho}\left(-\frac{\partial r}{\partial\rho}\right) \\
&= \frac{1}{(1-\rho)^2}\left(r - \frac{\gamma}{1-\gamma}\lambda - \beta\right)\left(1 - \rho\frac{1}{\psi}\frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)}\right) > 0
\end{aligned}$$

Analyzing the function  $\Phi(r)$ , I get

$$\frac{\partial \Phi(r)}{\partial r} = -\sum_{i=1}^2 \left( \frac{1}{r - \mu^{(Y)} + (1 + \delta_i)n} F^{y,(i)}\psi + F^{y,(i)}\frac{\rho}{1-\rho} \right)$$

and

$$\begin{aligned}
\frac{\partial(\Phi(r))}{\partial(-\rho)} &= \frac{1}{n}\frac{\partial(\psi + \mu^{(Y)} - r)}{\partial(-\rho)} = \frac{1}{n}\left(\frac{\partial\psi}{\partial(-\rho)} - \frac{\partial r}{\partial(-\rho)}\right) \\
&= \frac{1}{n}\frac{1}{(1-\rho)^2}\left(r - \frac{\gamma}{1-\gamma}\lambda - \beta\right)\left(1 - \frac{\frac{1}{\psi}(r - \mu^{(Y)})}{1 + (1-\rho)n\Phi'(r)}\right) \\
&= \frac{1}{(1-\rho)^2}\frac{1}{\psi}\frac{r - \frac{\gamma}{1-\gamma}\lambda - \beta}{1 + (1-\rho)n\Phi'(r)}\left(\frac{\psi + \mu^{(Y)} - r}{n} + (1-\rho)\Phi'(r)\psi\right) \\
&= \frac{1}{(1-\rho)^2}\frac{1}{\psi}\frac{r - \frac{\gamma}{1-\gamma}\lambda - \beta}{1 + (1-\rho)n\Phi'(r)} \\
&\quad \cdot \left(\Phi(r) - (1-\rho)\psi\sum_{i=1}^2\left(\frac{1}{r - \mu^{(Y)} + (1 + \delta_i)n} F^{y,(i)}\psi + F^{y,(i)}\frac{\rho}{1-\rho}\right)\right) \\
&= \frac{1}{1-\rho}\frac{1}{\psi}\frac{r - \frac{\gamma}{1-\gamma}\lambda - \beta}{1 + (1-\rho)n\Phi'(r)}\left(\Phi(r) - \psi\sum_{i=1}^2\frac{1}{r - \mu^{(Y)} + (1 + \delta_i)n} F^{y,(i)}\psi\right) \\
&= \frac{1}{1-\rho}\frac{1}{\psi}\frac{r - \frac{\gamma}{1-\gamma}\lambda - \beta}{1 + (1-\rho)n\Phi'(r)}\sum_{i=1}^2 F^{y,(i)}\psi\left(1 - \psi\frac{1}{r - \mu^{(Y)} + (1 + \delta_i)n}\right) \\
&= \frac{1}{1-\rho}\frac{1}{\psi}\frac{r - \frac{\gamma}{1-\gamma}\lambda - \beta}{1 + (1-\rho)n\Phi'(r)}n\sum_{i=1}^2\frac{F^{y,(i)}\psi}{r - \mu^{(Y)} + (1 + \delta_i)n}(1 + \delta_i - \Phi(r))
\end{aligned}$$

$\frac{\partial(\Phi(r))}{\partial(-\rho)} > 0$  holds and  $\Phi(r)$  is strictly increasing in  $-\rho$  if  $\sum_{i=1}^2\frac{F^{y,(i)}\psi}{r - \mu^{(Y)} + (1 + \delta_i)n}(1 + \delta_i - \Phi(r)) > 0$ . In general

it is hard to tell whether this condition is satisfied. However, if  $B_2 = 0$  or  $\delta_1 = \delta_2$ , then

$$\frac{\partial(\Phi(r))}{\partial(-\rho)} = \frac{1}{1-\rho} \frac{1}{\psi} \frac{r - \frac{\gamma}{1-\gamma}\lambda - \beta}{1 + (1-\rho)n\Phi'(r)} n \frac{\Phi(r)}{r - \mu^{(Y)} + (1 + \delta_1)n} (1 + \delta_1 - \Phi(r))$$

and  $a(1 + \delta_1) > \Phi(r)$  and  $\frac{\partial(\Phi(r))}{\partial(-\rho)} > 0$  must hold for  $\rho < 0$ . I proof this statement as follows. Suppose  $\Phi(r) = \Phi^* > 1 + \delta_1$  was true for some  $\hat{\rho} < 0$ . Then, since  $\Phi(r)$  is a continuous function and  $1 + \delta_1 > (<) \Phi(r)$  implies  $\frac{\partial(\Phi(r))}{\partial(-\rho)} > (<) 0$ ,  $1 + \delta_1 > \Phi(r)$  can never occur for  $\rho < \hat{\rho}$  and  $\Phi(r)$  will converge (from above) to  $\hat{\Phi} \in [1 + \delta_1, \Phi^*)$  as  $\rho$  approaches  $-\infty$ . But this contradicts  $\lim_{\rho \rightarrow -\infty} \Phi(r) = a(1 + \delta_1)$ . Similar, suppose  $\Phi(r) = \Phi^* \in (a(1 + \delta_1), 1 + \delta_1)$  for some  $\hat{\rho} < 0$ . Then, since  $\Phi(r)$  is a continuous function and  $1 + \delta_1 > (<) \Phi(r)$  implies  $\frac{\partial(\Phi(r))}{\partial(-\rho)} > (<) 0$ ,  $\Phi(r)$  is approaching the limit  $\hat{\Phi} \in (\Phi^*, 1 + \delta_1]$  (from below) as  $\rho$  approaches  $-\infty$ , which contradicts  $\lim_{\rho \rightarrow -\infty} \Phi(r) = a(1 + \delta_1)$ . The only possibility is that  $\Phi(r)$  is strictly increasing and approaches the limit  $a(1 + \delta_1)$  (from below) as  $\rho$  approaches  $-\infty$ . Finally, since  $\Phi(r)$  is a continuous function in  $\rho$  (for  $\rho < 0$ ),  $\rho < \bar{\rho}^{(r)}$  for  $\bar{\rho}^{(r)} = \min_{\rho \in (\Upsilon^{(r)} \cup \{0\})} \{\rho\}$  with  $\Upsilon^{(r)} \equiv \{\rho : v^{(r)} = 0, \rho \leq 0\}$  and  $v^{(r)} = a(n - \lambda) - \frac{\gamma}{1-\gamma}\lambda - n\Phi(r(\rho))$  satisfies condition (28), if  $\frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} > 1 - \frac{1 + \frac{1-a}{1-\gamma}\gamma}{1-\gamma} \frac{\lambda}{n}$ . Moreover, in the case of  $B_2 = 0$  or  $\delta_1 = \delta_2$  and  $\delta_1 > -\frac{1 + \frac{1-a}{1-\gamma}\gamma}{1-\gamma} \frac{\lambda}{n}$ , if condition (28) is not satisfied for  $\rho \nearrow 0$ , then  $\Upsilon^{(r)}$  is single valued (as  $v^{(r)}$  is monotonic), and if condition (28) is satisfied for  $\rho \nearrow 0$ , then  $\Upsilon^{(r)}$  is empty.

A similar result can be achieved following the argument in Garleanu and Panageas (2010). Provided the (sufficient) conditions

$$n\Phi(r_*) > a(n - \lambda) + \frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \lambda \quad (29)$$

and

$$\frac{\gamma\rho}{1-\gamma} \lambda + \beta - \rho\mu^{(Y)} > 0 \quad (30)$$

and

$$\beta - a(n - \lambda) - \rho\mu_*^{(Y)} > 0 \quad (31)$$

the equilibrium interest rate in the OLG economy is lower than the respective interest rate in an economy populated by an infinitely-lived representative agent ( $r < r_*$ ). I define  $r_* = (1 - \rho)\mu_*^{(Y)} + \beta$  (interest rate in infinitely-lived representative agent economy). Let the function  $f^{(r)}(x) = (1 - \rho)\mu^{(Y)} + \frac{\gamma\rho}{1-\gamma}\lambda + \beta - (1 - \rho)n\Phi(x) - x$ . First, I note that condition (31) implies that  $r_* > \mu^{(Y)}$ . Next, condition (29) implies  $0 > f^{(r)}(r_*)$ , and condition (30) implies  $0 < f^{(r)}(\mu^{(Y)})$ . Since the function  $f^{(r)}(\cdot)$  is continuous, then by the intermediate value theorem, there exists  $\{r : r \in (\mu^{(Y)}, r_*), f^{(r)}(r) = 0\}$ . This means that there exists an equilibrium interest rate in the OLG economy that is larger than the growth rate in GDP but smaller than the rate in an equivalent economy populated by an infinitely-lived agent. As pointed out by Garleanu and Panageas (2010), condition (29) can be interpreted as a requirement on life-cycle earnings to be sufficiently strong declining in age. Assuming the special parameterization of  $G(0, t) = B_1 e^{-\delta_1 n t}$  ( $B_2 = 0$ ), it becomes

$$\delta_1 > \frac{\left(a(n - \lambda) + \frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \lambda\right) \left(\beta - \rho\mu_*^{(Y)} - a(n - \lambda)\right)}{n \left(a\beta - a\rho\mu_*^{(Y)} - a(n - \lambda) - (1 - a) \frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \lambda\right)} - 1$$

It requires  $\delta_1$  to be large enough and life-cycle earnings to decrease fast enough as an agent age. Condition (30) is implied by  $\gamma > 1$ . This is because condition (30) it is implied by condition (ii) of Proposition 1 if  $\rho > 0$  and  $r > \mu^{(Y)}$  (which is an implication of the just discussed intermediate value theorem), or it is satisfied for  $\rho < 0$  and  $\gamma \notin \left(\frac{\mu^{(Y)}}{\mu^{(Y)} + \lambda}, 1\right)$ . The condition (31) requires the *EIS* to be small enough,  $\rho < \frac{\beta - a(n - \lambda)}{\mu^{(Y)} - a(n - \lambda)}$  (given

$\mu_*^{(Y)} > 0$ ).

The argument of both discussed proofs are interdependent and complement each other. I need a sufficiently decreasing life-cycle earnings profile and a strong enough consumption smoothing motive. The difference is that once I explore the magnitude of the *EIS* and once I focus on the labor income path. ■

**Proof of Lemma 4.** Let  $\bar{\rho}^{(n)} = \min_{\rho \in (\Upsilon^{(n)} \cup \{0\})} \{\rho\}$  with  $\Upsilon^{(n)} = \{\rho : v^{(n)}(\rho) = 0, \rho < 0\}$  and  $v^{(n)}(\rho) = \Phi(r(\rho)) - a \frac{r(\rho) - \mu^{(Y)} + (1 + \delta_1)n}{r(\rho) - \mu^{(Y)} + an}$ . I show that the condition  $\rho < \bar{\rho}^{(n)}$  (or  $EIS < \overline{EIS}^{(n)} \equiv \frac{1}{1 - \bar{\rho}^{(n)}}$ ) suffices for  $\frac{\partial r}{\partial n} < 0$  to hold. Moreover, I show that for  $B_2 = 0$  or  $\delta_1 = \delta_2$ , the function  $v^{(n)}(\rho)$  is monotonically increasing in  $-\rho$  (for  $\rho < 0$ ), and if  $\lim_{\rho \nearrow 0} [v^{(n)}(\rho)] < 0$ , then the set  $\Upsilon^{(n)}$  is single valued, and otherwise  $\Upsilon^{(n)}$  is empty. It follows that for  $B_2 = 0$  or  $\delta_1 = \delta_2$  there exists no  $\rho > \bar{\rho}^{(n)}$  that satisfies  $v^{(n)}(\rho) > 0$ . In the general case ( $B_2 \neq 0$  and  $\delta_1 \neq \delta_2$ ) there might exist  $\rho > \bar{\rho}^{(n)}$  that satisfies  $v^{(n)}(\rho) > 0$ . I need the technical conditions  $B_1 > \frac{1 + \frac{1}{\delta_1}}{1 + \frac{1}{\delta_2}} |B_2|$ ,  $\delta_1 > 0$ ,  $r - \frac{\gamma}{1 - \gamma} \lambda - \beta > 0$ ,  $\mu^{(Y)} - \frac{\gamma}{1 - \gamma} \lambda - na \frac{B_1 + B_2}{\frac{1}{1 + \delta_1} + \frac{1}{1 + \delta_2}} \neq 0$  and the conditions of Proposition 1 and Lemma 3 to hold. For  $\frac{\partial r}{\partial n} < 0$  to hold, I need  $a - \Phi(r) + n \sum_{i=1}^2 \frac{-a + 1 + \delta_i}{r - \mu^{(Y)} + (1 + \delta_i)n} F^{y,(i)} \psi < 0$ . Using the assumptions  $\delta_1 > 0$  and Lemma 3 ( $r - \mu^{(Y)} > 0$ ), I get

$$\begin{aligned} a - \Phi(r) + n \sum_{i=1}^2 \frac{-a + 1 + \delta_i}{r - \mu^{(Y)} + (1 + \delta_i)n} F^{y,(i)} \psi &= a - \Phi(r) + \sum_{i=1}^2 \left( 1 - \frac{r - \mu^{(Y)} + an}{r - \mu^{(Y)} + (1 + \delta_i)n} \right) F^{y,(i)} \psi \\ &= a - \sum_{i=1}^2 \frac{r - \mu^{(Y)} + an}{r - \mu^{(Y)} + (1 + \delta_i)n} F^{y,(i)} \psi \\ &< a - \frac{r - \mu^{(Y)} + an}{r - \mu^{(Y)} + (1 + \delta_1)n} \Phi(r) \end{aligned}$$

which implies that

$$\Phi(r) > a \frac{r - \mu^{(Y)} + (1 + \delta_1)n}{r - \mu^{(Y)} + an} \quad (32)$$

suffices for  $\frac{\partial r}{\partial n} < 0$  to hold. First, I look at how condition (32) behaves in the limit when the *EIS* approaches zero. Following the result in Lemma 3, I get

$$\lim_{\rho \rightarrow -\infty} \Phi(r) = a \frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}}$$

and for  $B_2 = 0$  or  $\delta_1 = \delta_2$ ,  $\lim_{\rho \rightarrow -\infty} \Phi(r)$  simplifies to

$$\lim_{\rho \rightarrow -\infty} \Phi(r) = a(1 + \delta_1)$$

Moreover,

$$\lim_{\rho \rightarrow -\infty} a \frac{r - \mu^{(Y)} + (1 + \delta_1)n}{r - \mu^{(Y)} + an} = \lim_{\rho \rightarrow -\infty} a \frac{\frac{r}{1 - \rho} + \frac{-\mu^{(Y)} + (1 + \delta_1)n}{1 - \rho}}{\frac{r}{1 - \rho} + \frac{-\mu^{(Y)} + an}{1 - \rho}} = a$$

In the limit as the *EIS* approaches zero condition (32) is satisfied ( $\lim_{\rho \rightarrow -\infty} \Phi(r) > \lim_{\rho \rightarrow -\infty} a \frac{r - \mu^{(Y)} + (1 + \delta_1)n}{r - \mu^{(Y)} + an}$ )

if  $\frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} > 1$  or equivalently  $B_1 > \frac{1 + \frac{1}{\delta_1}}{1 + \frac{1}{\delta_2}} |B_2|$ . In the case of  $B_2 = 0$  or  $\delta_1 = \delta_2$  condition  $\frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} > 1$

boils down to  $\delta_1 > 0$ . Next, I note that  $a \frac{r - \mu^{(Y)} + (1 + \delta_1)n}{r - \mu^{(Y)} + an} \in (a, 1 + \delta_1)$ . Using  $\gamma > 1$ ,  $r - \frac{\gamma}{1 - \gamma} \lambda - \beta$ ,  $1 + \delta_1 > a$  and the conditions of Lemma 3 ( $r - \mu^{(Y)} > 0$ ), I see that the term  $a \frac{r - \mu^{(Y)} + (1 + \delta_1)n}{r - \mu^{(Y)} + an}$  is strictly decreasing in

$-\rho$  until it approaches  $a$  in the limit where  $\rho$  approaches  $-\infty$ , because

$$\frac{\partial \left( a \frac{r - \mu^{(Y)} + (1 + \delta_1)n}{r - \mu^{(Y)} + an} \right)}{\partial(-\rho)} = a \frac{\partial \left( 1 + \frac{(1 + \delta_1)n - an}{r - \mu^{(Y)} + an} \right)}{\partial r} \frac{\partial r}{\partial(-\rho)} = -an \frac{1 + \delta_1 - a}{(r - \mu^{(Y)} + an)^2} \frac{\partial r}{\partial(-\rho)} < 0$$

Following Lemma 3, I also know that  $\frac{\partial(\Phi(r))}{\partial(-\rho)} > 0$  holds and  $\Phi(r)$  is strictly increasing in  $-\rho$  if

$$\sum_{i=1}^2 \frac{F^{y,(i)}\psi}{r - \mu^{(Y)} + (1 + \delta_i)n} (1 + \delta_i - \Phi(r)) > 0$$

It is hard to tell whether this condition is satisfied in general. However, if  $B_2 = 0$  or  $\delta_1 = \delta_2$ , then

$$\frac{\partial(\Phi(r))}{\partial(-\rho)} = \frac{1}{1 - \rho} \frac{1}{\psi} \frac{r - \frac{\gamma}{1 - \gamma}\lambda - \beta}{1 + (1 - \rho)n\Phi'(r)} n \frac{\Phi(r)}{r - \mu^{(Y)} + (1 + \delta_1)n} (1 + \delta_1 - \Phi(r))$$

and  $a(1 + \delta_1) > \Phi(r)$  and  $\frac{\partial(\Phi(r))}{\partial(-\rho)} > 0$  must hold for  $\rho < 0$  (as shown in Lemma 3). In conclusion, since  $\Phi(r)$  and  $a \frac{r - \mu^{(Y)} + (1 + \delta_1)n}{r - \mu^{(Y)} + an}$  are continuous functions in  $\rho$  (for  $\rho < 0$ ),  $\rho < \bar{\rho}^{(n)}$  satisfies condition (32) and  $\frac{\partial r}{\partial n} < 0$  (if  $B_1 > \frac{1 + \delta_1}{1 + \delta_2} |B_2|$ ). Moreover, in the case of  $B_2 = 0$  or  $\delta_1 = \delta_2$ , if condition (32) is not satisfied for  $\rho \nearrow 0$ , then  $\Upsilon^{(n)}$  is single valued (as  $v^{(n)}$  is monotonic), and if condition (32) is satisfied for  $\rho \nearrow 0$ , then  $\Upsilon^{(n)}$  is empty. ■

**Proof of Lemma 5.** Let  $\bar{\rho}^{(\lambda)} = \min_{\rho \in (\Upsilon^{(\lambda)} \cup \{0\})} \{\rho\}$  with  $\Upsilon^{(\lambda)} = \{\rho : v^{(\lambda)}(\rho) = 0, \rho < 0\}$  and  $v^{(\lambda)}(\rho) = 1 - a - \left( -\frac{\gamma}{1 - \gamma} \frac{1}{\psi(\rho)} - \frac{a}{r(\rho) - \mu^{(Y)} + (1 + \delta_1)n} \right) na(1 + \delta_1) - \frac{1 - \gamma - \rho}{(1 - \gamma)(1 - \rho)}$ . I show that the condition  $\rho < \bar{\rho}^{(\lambda)}$  (or  $EIS < \overline{EIS}^{(\lambda)} \equiv \frac{1}{1 - \bar{\rho}^{(\lambda)}}$ ) suffices for  $\frac{\partial r}{\partial \lambda} > 0$  to hold. Moreover, I show that for  $B_2 = 0$  or  $\delta_1 = \delta_2$ , the function  $v^{(\lambda)}(\rho)$  is monotonically increasing in  $-\rho$  (for  $\rho < 0$ ), and if  $\lim_{\rho \nearrow 0} [v^{(\lambda)}(\rho)] < 0$ , then the set  $\Upsilon^{(\lambda)}$  is single valued, and otherwise  $\Upsilon^{(\lambda)}$  is empty. It follows that for  $B_2 = 0$  or  $\delta_1 = \delta_2$  there exists no  $\rho > \bar{\rho}^{(\lambda)}$  that satisfies  $v^{(\lambda)}(\rho) > 0$ . In the general case ( $B_2 \neq 0$  and  $\delta_1 \neq \delta_2$ ) there might exist  $\rho > \bar{\rho}^{(\lambda)}$  that satisfies  $v^{(\lambda)}(\rho) > 0$ . The technical conditions needed are the same as in Lemma 4. For  $\frac{\partial r}{\partial \lambda} > 0$  to hold, I need  $(1 - \rho) \left( 1 - a + \sum_{i=1}^2 \frac{an}{r - \mu^{(Y)} + (1 + \delta_i)n} F^{y,(i)}\psi - n \frac{\rho}{1 - \rho} \frac{\gamma}{1 - \gamma} \sum_{i=1}^2 F^{y,(i)} \right) - \frac{1 - \gamma - \rho}{1 - \gamma} > 0$ . Suppose that  $\rho < 0$ . Using  $\gamma > 1$ , and  $\Phi(r) < a(1 + \delta_1)$  (result of Lemma 3), I note that

$$\begin{aligned} & (1 - \rho) \left( 1 - a + \sum_{i=1}^2 \frac{an}{r - \mu^{(Y)} + (1 + \delta_i)n} F^{y,(i)}\psi - n \frac{\rho}{1 - \rho} \frac{\gamma}{1 - \gamma} \sum_{i=1}^2 F^{y,(i)} \right) - \frac{1 - \gamma - \rho}{1 - \gamma} \\ & > (1 - \rho) \left( 1 - a - \left( \frac{\rho}{1 - \rho} \frac{\gamma}{1 - \gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n} \right) n\Phi(r) \right) - \frac{1 - \gamma - \rho}{1 - \gamma} \\ & > (1 - \rho) \left( 1 - a - \left( -\frac{\gamma}{1 - \gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n} \right) n\Phi(r) \right) - \frac{1 - \gamma - \rho}{1 - \gamma} \\ & > (1 - \rho) \left( 1 - a - \left( -\frac{\gamma}{1 - \gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n} \right) na(1 + \delta_1) \right) - \frac{1 - \gamma - \rho}{1 - \gamma} \end{aligned}$$

Condition

$$(1 - \rho) \left( 1 - a - \left( -\frac{\gamma}{1 - \gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n} \right) na(1 + \delta_1) \right) - \frac{1 - \gamma - \rho}{1 - \gamma} > 0 \quad (33)$$

suffices for  $\frac{\partial r}{\partial \lambda} > 0$ . For the case when the *EIS* approaches zero condition (33) is satisfied since

$$\lim_{\rho \rightarrow -\infty} \left( -\frac{\gamma}{1-\gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n} \right) = 0$$

and

$$\lim_{\rho \rightarrow -\infty} (1 - \rho) \left( 1 - a - \left( -\frac{\gamma}{1-\gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n} \right) na(1 + \delta_1) \right) - \frac{1 - \gamma - \rho}{1 - \gamma} > 0$$

The term  $-\frac{1-\gamma-\rho}{1-\gamma}$  is increasing in  $-\rho$  (since  $\gamma > 1$ ). The expression  $-\frac{\gamma}{1-\gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n}$  is decreasing in  $-\rho$  if  $\sum \frac{F^{y,(i)}\psi}{r - \mu^{(Y)} + (1 + \delta_i)n} (1 + \delta_i - \Phi(r)) > 0$ , because

$$\begin{aligned} & \frac{\partial \left( -\frac{\gamma}{1-\gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n} \right)}{\partial(-\rho)} \\ &= \frac{\gamma}{1-\gamma} \frac{1}{\psi^2} \frac{\partial \psi}{\partial(-\rho)} + \frac{a}{(r - \mu^{(Y)} + (1 + \delta_1)n)^2} \frac{\partial r}{\partial(-\rho)} \\ &= \frac{\gamma}{1-\gamma} \frac{1}{\psi^2} \frac{1}{(1-\rho)^2} \left( r - \frac{\gamma}{1-\gamma} \lambda - \beta \right) \left( 1 - \rho \frac{1}{\psi} \frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)} \right) \\ &+ \frac{a}{(r - \mu^{(Y)} + (1 + \delta_1)n)^2} \frac{\frac{1}{\psi} (r - \mu^{(Y)}) \frac{1}{1-\rho} \left( r - \frac{\gamma}{1-\gamma} \lambda - \beta \right)}{1 + (1-\rho)n\Phi'(r)} \\ &= \left( r - \frac{\gamma}{1-\gamma} \lambda - \beta \right) \frac{1}{(1-\rho)^2} \left[ \frac{\frac{\gamma}{1-\gamma} \frac{1}{\psi^2} \left( 1 - \rho \frac{1}{\psi} \frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)} \right)}{\left( r - \mu^{(Y)} + (1 + \delta_1)n \right)^2} (1-\rho) \frac{1}{\psi} \frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)} \right] \\ &< \left( r - \frac{\gamma}{1-\gamma} \lambda - \beta \right) \frac{1}{(1-\rho)^2} \frac{1}{\psi^2} \left[ \frac{\frac{\gamma}{1-\gamma} \left( 1 - \rho \frac{1}{\psi} \frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)} \right)}{+ a (1-\rho) \frac{1}{\psi} \frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)}} \right] \\ &< \frac{\gamma}{1-\gamma} \left( r - \frac{\gamma}{1-\gamma} \lambda - \beta \right) \frac{1}{(1-\rho)^2} \frac{1}{\psi^2} \left[ \frac{\left( 1 - \rho \frac{1}{\psi} \frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)} \right)}{- (1-\rho) \frac{1}{\psi} \frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)}} \right] \\ &= \frac{\gamma}{1-\gamma} \left( r - \frac{\gamma}{1-\gamma} \lambda - \beta \right) \frac{1}{(1-\rho)^2} \frac{1}{\psi^2} \left[ 1 - \frac{1}{\psi} \frac{r - \mu^{(Y)}}{1 + (1-\rho)n\Phi'(r)} \right] \\ &= \frac{\gamma}{1-\gamma} \frac{1}{1-\rho} \frac{1}{\psi^3} \frac{r - \frac{\gamma}{1-\gamma} \lambda - \beta}{1 + (1-\rho)n\Phi'(r)} n^2 \sum_{i=1}^2 \frac{F^{y,(i)}\psi}{r - \mu^{(Y)} + (1 + \delta_i)n} (1 + \delta_i - \Phi(r)) \end{aligned}$$

where the first inequality follows from  $\psi < r - \mu^{(Y)} + (1 + \delta_1)n$  and the second one from  $\frac{-\gamma}{1-\gamma} > a$ . As in the discussion in Lemma 4, it is hard to tell whether  $\sum \frac{F^{y,(i)}\psi}{r - \mu^{(Y)} + (1 + \delta_i)n} (1 + \delta_i - \Phi(r)) > 0$  is satisfied in general. However, if  $B_2 = 0$  or  $\delta_1 = \delta_2$ , then  $a(1 + \delta_1) > \Phi(r)$  and  $\frac{\partial(\Phi(r))}{\partial(-\rho)} > 0$  must hold for  $\rho < 0$ . Since  $-\frac{\gamma}{1-\gamma} \frac{1}{\psi} - \frac{a}{r - \mu^{(Y)} + (1 + \delta_1)n}$  and  $-\frac{1-\gamma-\rho}{1-\gamma}$  are continuous function in  $\rho$  (for  $\rho < 0$ ), it follows that  $\rho < \bar{\rho}$  satisfies condition (33) and  $\frac{\partial r}{\partial \lambda} > 0$  (if  $B_1 > \frac{1+\delta_1}{1+\frac{1}{\delta_2}} |B_2|$ ). Moreover, in the case of  $B_2 = 0$  or  $\delta_1 = \delta_2$ , if condition (33) is not satisfied for  $\rho \nearrow 0$ , then  $\Upsilon^{(\lambda)}$  is single valued (as  $v^{(\lambda)}$  is monotonic), and if condition (33) is satisfied for  $\rho \nearrow 0$ , then  $\Upsilon^{(\lambda)}$  is empty. ■

**Proof of Lemma 6.** Let  $\bar{\rho}_1^{(n)} = \min_{\rho \in \{\Upsilon_1^{(n)}, 0\}} \{\rho\}$  with  $\Upsilon_1^{(n)} = \left\{ \rho : v_1^{(n)}(\rho) = 0, \rho < 0 \right\}$  and  $v_1^{(n)}(\rho) = r_L^{(n)}(\rho) - r_H^{(n)}(\rho)$ . I show that for  $\rho < \bar{\rho}_1^{(n)}$  (or  $EIS < \overline{EIS}_1^{(n)} = \frac{1}{1-\bar{\rho}_1^{(n)}}$ ), the interest rate during a period characterized by a high birth rate (baby boom) is lower than the rate during times of a low birth rate



(baby bust),  $r_H^{(n)} < r_L^{(n)}$ . This is a sufficient condition and there might exist some  $\rho > \bar{\rho}_1^{(n)}$  that satisfies  $r_H^{(n)} < r_L^{(n)}$ . I need the technical conditions  $B_1 > \frac{1+\delta_1}{1+\delta_2}|B_2|$ ,  $\delta_1 > 0$ ,  $\mu_L^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_L a \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$  and  $\mu_H^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_H a \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$ . The conditions  $\rho < 0$  and  $\gamma \in (1, 1-\rho)$  imply  $\frac{1-\gamma-\rho}{\rho} < 0$ , and  $\psi_H^{(n)} < (>) \psi_L^{(n)}$  is true if  $r_H^{(n)} < (>) r_L^{(n)}$  holds. I can show this using a proof by contradiction. Suppose  $\psi_H^{(n)} > \psi_L^{(n)}$  and  $r_H^{(n)} < r_L^{(n)}$ . I have

$$\begin{aligned} \psi_H^{(n)} - \psi_L^{(n)} &= \underbrace{-\frac{\rho}{1-\rho} \left( r_H^{(n)} - r_L^{(n)} \right)}_{<0} - \underbrace{\frac{\rho}{1-\rho} \bar{\theta}_H^{(n)} \left( \left( \frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right)}_{<0} \\ &\quad - \underbrace{\frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \bar{\theta}_H^{(n)} \left( \left( \frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{-\frac{1-\gamma}{\rho}} - 1 \right)}_{<0} + \underbrace{\frac{\rho}{1-\rho} \bar{\theta}_L^{(n)} \left( \left( \frac{\psi_H^{(n)}}{\psi_L^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right)}_{<0} \\ &\quad + \underbrace{\frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \bar{\theta}_L^{(n)} \left( \left( \frac{\psi_H^{(n)}}{\psi_L^{(n)}} \right)^{-\frac{1-\gamma}{\rho}} - 1 \right)}_{<0} \end{aligned}$$

which contradicts the assumption  $\psi_H^{(n)} > \psi_L^{(n)}$ . Hence, if there exists a solution, then  $\psi_H^{(n)} < \psi_L^{(n)}$  and  $r_H^{(n)} < r_L^{(n)}$  must hold. The same line of argument holds for  $r_H^{(n)} > r_L^{(n)}$  and  $\psi_H^{(n)} > \psi_L^{(n)}$ .

Next, I look at the difference between the interest rate during a baby bust and the rate during a baby boom. To proof the Lemma I have to find conditions such that the  $r_L^{(n)} - r_H^{(n)} > 0$  holds. I explore the behavior of  $r_L^{(n)} - r_H^{(n)}$  under the limit when  $\rho$  approaches  $-\infty$ . I first suppose that  $\forall j \in \{L, H\}$ ,  $i \in \{1, 2\}$ ,  $|\lim_{\rho \rightarrow -\infty} \frac{r_j^{(n)}}{1-\rho}| < \infty$ ,  $\lim_{\rho \rightarrow -\infty} \frac{r_j^{(n)}}{1-\rho} \neq 0$ ,  $|\lim_{\rho \rightarrow -\infty} \frac{\psi_j^{(n)}}{1-\rho}| < \infty$ ,  $\lim_{\rho \rightarrow -\infty} \frac{\psi_j^{(n)}}{1-\rho} \neq 0$ ,  $|\lim_{\rho \rightarrow -\infty} (1-\rho) F_j^{y,(i),(n)}| < \infty$ , and  $\lim_{\rho \rightarrow -\infty} (1-\rho) F_j^{y,(i),(n)} \neq 0$  hold, and verify these assumptions in the end. For the interest rate I have  $\forall (j, h) \in \{(L, H), (H, L)\}$

$$\lim_{\rho \rightarrow -\infty} \frac{r_j^{(n)}}{1-\rho} = \mu_j^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_j \lim_{\rho \rightarrow -\infty} \left( \sum_{i=1}^2 F_j^{y,(i),(n)} \psi_j^{(n)} \right)$$

For the consumption to wealth ratio I get  $\forall (j, h) \in \{(L, H), (H, L)\}$

$$\lim_{\rho \rightarrow -\infty} \frac{\psi_j^{(n)}}{1-\rho} = \lim_{\rho \rightarrow -\infty} \frac{r_j^{(n)}}{1-\rho}$$

For the function  $F_j^{y,i,(n)}$ ,  $\forall i \in \{1, 2\}$ ,  $(j, h) \in \{(L, H), (H, L)\}$  it holds

$$\lim_{\rho \rightarrow -\infty} (1-\rho) F_j^{y,(i),(n)} = \frac{1}{\lim_{\rho \rightarrow -\infty} \frac{r_j^{(n)}}{1-\rho}} \frac{aB_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}}$$

The consumption share of the new born cohort is  $\forall j \in \{L, H\}$

$$n_j \lim_{\rho \rightarrow -\infty} \left( \sum_{i=1}^2 F_j^{y,(i),(n)} \psi_j^{(n)} \right) = n_j a \frac{B_1 + B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}}$$

Plugging the last expression into the equation of  $\lim_{\rho \rightarrow -\infty} \frac{r_j^{(n)}}{1-\rho}$ , it follows that my assumptions are indeed

true. I can now compare how  $r_H^{(n)}$  and  $r_L^{(n)}$  behave in the limit,

$$\lim_{\rho \rightarrow -\infty} \left( \frac{r_L^{(n)}}{1-\rho} - \frac{r_H^{(n)}}{1-\rho} \right) = a(n_H - n_L) \left( \frac{\sum_{i=1}^2 B_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} - 1 \right)$$

As a result, in the limit as  $\rho$  approaches  $-\infty$ ,  $\lim_{\rho \rightarrow -\infty} (r_L^{(n)} - r_H^{(n)}) > 0$  is satisfied, if  $\frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} > 1$  or equivalently  $B_1 > \frac{1+\frac{1}{\delta_1}}{1+\frac{1}{\delta_2}} |B_2|$  holds. This is the same condition as in the static case (Lemma 4), and requires that  $\frac{B_1}{|B_2|}$  is large enough and  $(\delta_2 - \delta_1)$  is small enough. In the case of  $B_2 = 0$  or  $\delta_1 = \delta_2$ , the condition becomes  $\delta_1 > 0$ . In conclusion, since the functions  $r_L^{(n)}(\rho)$  and  $r_H^{(n)}(\rho)$  are continuous in  $\rho$ , the condition  $\rho < \bar{\rho}_1^{(n)}$  ensures that  $r_L^{(n)} > r_H^{(n)}$  holds.

It is straightforward that  $|\kappa_L^{(n)}| < |\kappa_H^{(n)}|$  must hold given  $r_H^{(n)} < r_L^{(n)}$ ,  $\psi_H^{(n)} < \psi_L^{(n)}$  and  $\frac{1-\gamma-\rho}{\rho} < 0$ . It is true that

$$0 < \left[ \left( \psi_H^{(n)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} - \left( \psi_L^{(n)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} \right]^2$$

and rearranging yields

$$\left( \psi_L^{(n)} \right)^{-\frac{1-\gamma-\rho}{\rho}} \left( \psi_H^{(n)} \right)^{-\frac{1-\gamma-\rho}{\rho}} - \left( \psi_H^{(n)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} < \left( \psi_L^{(n)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} - \left( \psi_L^{(n)} \right)^{-\frac{1-\gamma-\rho}{\rho}} \left( \psi_H^{(n)} \right)^{-\frac{1-\gamma-\rho}{\rho}}$$

and dividing both sides by  $\left( \psi_L^{(n)} \right)^{-\frac{1-\gamma-\rho}{\rho}} \left( \psi_H^{(n)} \right)^{-\frac{1-\gamma-\rho}{\rho}}$  gives

$$\underbrace{\left( \left( \frac{\psi_H^{(n)}}{\psi_L^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right)}_{|\kappa_L^{(n)}|} < \underbrace{\left( \frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1}_{|\kappa_H^{(n)}|}$$

■  
**Proof of Lemma 7.** Let  $\bar{\rho}_2^{(n)} = \min_{\rho \in \{\Upsilon_2^{(n)}, 0\}} \{\rho\}$  with  $\Upsilon_2^{(n)} = \left\{ \rho : v_2^{(n)}(\rho) = 0, \rho < 0 \right\}$  and  $v_2^{(n)}(\rho) = \frac{1}{\psi_H^{(n)}(\rho)} - \frac{1}{\psi_L^{(n)}(\rho)} - \sum_{i=1}^2 \frac{F_H^{y,(i),(n)}(\rho) - F_L^{y,(i),(n)}(\rho)}{1+\delta_i}$ . I show that for  $\rho < \bar{\rho}_2^{(n)}$  (or  $EIS < \overline{EIS}_2^{(n)} = \frac{1}{1-\bar{\rho}_2^{(n)}}$ ), the equity premium is positive in both states of the world. This is a sufficient condition and there might exist some  $\rho > \bar{\rho}_2^{(n)}$  which is consistent with a positive equity premium in both states of the world. I need the technical conditions  $\mu_L^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_L a \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$  and  $\mu_H^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_H a \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$ . Condition  $\frac{1-\gamma-\rho}{\rho} < 0$  implies (independent of  $\psi_L^{(n)} > \psi_H^{(n)}$  or  $\psi_L^{(n)} < \psi_H^{(n)}$ )  $\forall (i, j) \in \{(L, H), (H, L)\}$

$$-\bar{\theta}_i^{(n)} \frac{Y_t}{F_t^{(n_i)}} \frac{1}{\psi_i^{(n)}} \left( \left( \frac{\psi_j^{(n)}}{\psi_i^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \left( \left( \frac{\psi_j^{(n)}}{\psi_i^{(n)}} \right)^{-1} - 1 \right) > 0$$

To ensure that  $E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt > 0, \forall n_t \in \{n_L, n_H\}$ , it is sufficient to show that the two inequalities

$$\begin{aligned} & -\bar{\theta}_i^{(n)} \frac{Y_t}{P_t^{(n_i)}} \frac{1}{\psi_i^{(n)}} \left( \left( \frac{\psi_j^{(n)}}{\psi_i^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \left( \left( \frac{\psi_j^{(n)}}{\psi_i^{(n)}} \right)^{-1} - 1 \right) \\ & > |\bar{\theta}_i^{(n)} \frac{Y_t}{P_t^{(n_i)}} \left( \left( \frac{\psi_j^{(n)}}{\psi_i^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \sum_{k=1}^2 \frac{F_j^{y,(k),(n)} - F_i^{y,(k),(n)}}{1 + \delta_k}| \end{aligned}$$

$\forall (i, j) \in \{(L, H), (H, L)\}$ , hold. Note that  $(1 - \rho) P_t^{(n_L)}$  and  $(1 - \rho) P_t^{(n_H)}$  are positive and finite for  $\rho < 0$ . Rewriting both inequalities yields the single condition

$$(1 - \rho) \left| \frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}} \right| \geq (1 - \rho) \left| \sum_{i=1}^2 \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1 + \delta_i} \right| \quad (34)$$

From Lemma 6 it follows that

$$\begin{aligned} & \lim_{\rho \rightarrow -\infty} (1 - \rho) \left| \frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}} \right| - \lim_{\rho \rightarrow -\infty} (1 - \rho) \left| \sum_{i=1}^2 \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1 + \delta_i} \right| \\ & = (1 - a) \left| \frac{1}{\lim_{\rho \rightarrow -\infty} \frac{r_H^{(n)}}{1-\rho}} - \frac{1}{\lim_{\rho \rightarrow -\infty} \frac{r_L^{(n)}}{1-\rho}} \right| > 0 \end{aligned}$$

In the limit as  $\rho$  approaches  $-\infty$ , condition (34) is satisfied. Since the function  $v_2^{(n)}(\rho)$  is continuous, the condition  $\rho < \bar{\rho}_2^{(n)}$  ensures that condition (34) is satisfied and  $E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt > 0, \forall n_t \in \{n_L, n_H\}$ .

■ **Proof of Lemma 8.** Let  $\bar{\rho}_3^{(n)} = \min_{\rho \in \{\Upsilon_1^{(n)}, \Upsilon_2^{(n)}, \Upsilon_3^{(n)}, 0\}} \{\rho\}$  with  $\Upsilon_3^{(n)} = \left\{ \rho : v_3^{(n)}(\rho) = 0, \rho < 0 \right\}$ ,

$$v_3^{(n)}(\rho) = \frac{\bar{\theta}_H^{(n)}}{\bar{\theta}_L^{(n)}} - \frac{1 - \sum_{i=1}^2 \frac{F_H^{y,(i),(n)}(\rho) \psi_H^{(n)}(\rho)}{1 + \delta_i}}{1 - \sum_{i=1}^2 \frac{F_L^{y,(i),(n)}(\rho) \psi_L^{(n)}(\rho)}{1 + \delta_i}} \left( \frac{\psi_L^{(n)}(\rho)}{\psi_H^{(n)}(\rho)} \right)^{\frac{1-\gamma}{\rho}}$$

and  $\Upsilon_1^{(n)}$  and  $\Upsilon_2^{(n)}$  as defined in Lemma 6 and 7. I show that the condition  $\rho < \bar{\rho}_3^{(n)}$  (or  $EIS < \overline{EIS}_3^{(n)} \equiv \frac{1}{1 - \bar{\rho}_3^{(n)}}$ ) ensures that the equity premium is larger (lower) during a period characterized by a high birth rate (baby boom) than the premium during times of a low birth rate (baby bust). This is a sufficient condition and there might exist some  $\rho > \bar{\rho}_2^{(n)}$  which is consistent with the result of the Lemma. To give proof I have to show that for  $\bar{\theta}_H^{(n)} > (<) \bar{\theta}_L^{(n)}$ , it holds

$$E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [n_t = n_H] - E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [n_t = n_L] > (<) 0$$

or plugging in the expression for the equity premium

$$\begin{aligned} 0 & < (>) \bar{\theta}_H^{(n)} \frac{Y_t}{P_t^{(n_H)}} \left( \frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}} - \sum_{i=1}^2 \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1 + \delta_i} \right) \\ & \cdot \left[ \left( \left( \frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) + \frac{\bar{\theta}_L^{(n)} P_t^{(n_H)}}{\bar{\theta}_H^{(n)} P_t^{(n_L)}} \left( \left( \frac{\psi_H^{(n)}}{\psi_L^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \right] \end{aligned}$$

Since  $\psi_H^{(n)} < \psi_L^{(n)}$  (by Lemma 6),  $\left| \frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}} \right| > \left| \sum_{i=1}^2 \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1 + \delta_i} \right|$  (by Lemma 7), and  $\rho < 0$ ,  $\gamma \in (1, 1 - \rho)$  (by Lemma 6 and 7), it suffices to show that the last term in square brackets is positive

(negative),

$$\frac{\bar{\theta}_H^{(n)}}{\bar{\theta}_L^{(n)}} > (<) \frac{P_t^{(n_H)}}{P_t^{(n_L)}} \left( \frac{\psi_H^{(n)}}{\psi_L^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} = \frac{1 - \sum_{i=1}^2 \frac{F_H^{y,(i),(n)} \psi_H^{(n)}}{1+\delta_i}}{1 - \sum_{i=1}^2 \frac{F_L^{y,(i),(n)} \psi_L^{(n)}}{1+\delta_i}} \left( \frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{\frac{1-\gamma}{\rho}} \quad (35)$$

In the limit as the *EIS* approaches zero condition (35) is satisfied if

$$\frac{\bar{\theta}_H^{(n)}}{\bar{\theta}_L^{(n)}} > (<) \lim_{\rho \rightarrow -\infty} \left[ \frac{1 - \sum_{i=1}^2 \frac{F_H^{y,(i),(n)} \psi_H^{(n)}}{1+\delta_i}}{1 - \sum_{i=1}^2 \frac{F_L^{y,(i),(n)} \psi_L^{(n)}}{1+\delta_i}} \left( \frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{\frac{1-\gamma}{\rho}} \right] = 1$$

Since the function  $v_3^{(n)}(\rho)$  is continuous in  $\rho < 0$ , the condition  $\rho < \bar{\rho}_3^{(n)}$  ensures that condition (35) is satisfied and  $E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [n_t = n_H] > (<) E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [n_t = n_L]$ . ■

**Proof of Lemma 9.** Let  $\bar{\rho}_1^{(\lambda)} = \min_{\rho \in \{\Upsilon_1^{(\lambda)}, 0\}} \{\rho\}$  with  $\Upsilon_1^{(\lambda)} = \{\rho : v_1^{(\lambda)}(\rho) = 0, \rho < 0\}$  and  $v_1^{(\lambda)}(\rho) = r_H^{(\lambda)}(\rho) - r_L^{(\lambda)}(\rho)$ . I show that for  $\rho < \bar{\rho}_1^{(\lambda)}$  (or  $EIS < \overline{EIS}_1^{(\lambda)} \equiv \frac{1}{1-\bar{\rho}_1^{(\lambda)}}$ ), the interest rate during a period characterized by a high death rate is higher than the rate during times of a low mortality,  $r_H^{(\lambda)} > r_L^{(\lambda)}$ . This is a sufficient condition and there might exist some  $\rho > \bar{\rho}_1^{(\lambda)}$  which is consistent with the result of the Lemma. The proof follows the same line of argument as the proof of Lemma 6. I need the technical conditions  $\mu_L^{(Y,\lambda)} - \frac{\gamma}{1-\gamma} \lambda_L - na \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$  and  $\mu_H^{(Y,\lambda)} - \frac{\gamma}{1-\gamma} \lambda_H - na \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$ . The conditions  $\rho < 0$  and  $\gamma \in (1, 1-\rho)$  imply  $\frac{1-\gamma-\rho}{\rho} < 0$ , and  $\psi_H^{(\lambda)} > \psi_L^{(\lambda)}$  is implied by  $r_H^{(\lambda)} > r_L^{(\lambda)}$  holds. I can show this using a proof by contradiction. Suppose  $\psi_H^{(\lambda)} < \psi_L^{(\lambda)}$  and  $r_H^{(\lambda)} > r_L^{(\lambda)}$ . I have

$$\begin{aligned} \psi_H^{(\lambda)} - \psi_L^{(\lambda)} &= \underbrace{\frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} (\lambda_H - \lambda_L)}_{>0} - \underbrace{\frac{\rho}{1-\rho} (r_H^{(\lambda)} - r_L^{(\lambda)})}_{>0} \\ &\quad - \underbrace{\frac{\rho}{1-\rho} \bar{\theta}_H^{(\lambda)} \left( \left( \frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right)}_{>0} + \underbrace{\frac{\rho}{1-\rho} \bar{\theta}_L^{(\lambda)} \left( \left( \frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right)}_{>0} \\ &\quad - \underbrace{\frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \bar{\theta}_H^{(\lambda)} \left( \left( \frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}} \right)^{-\frac{1-\gamma}{\rho}} - 1 \right)}_{>0} + \underbrace{\frac{\rho}{1-\rho} \frac{\gamma}{1-\gamma} \bar{\theta}_L^{(\lambda)} \left( \left( \frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}} \right)^{-\frac{1-\gamma}{\rho}} - 1 \right)}_{>0} \end{aligned}$$

which contradicts the assumption  $\psi_H^{(\lambda)} < \psi_L^{(\lambda)}$ . Hence, if there exists a solution, then  $\psi_H^{(\lambda)} > \psi_L^{(\lambda)}$  and  $r_H^{(\lambda)} > r_L^{(\lambda)}$  holds.

Next, I look at the difference between the interest rate in a high death rate state and the rate in a low death rate state. To proof the Lemma I have to find conditions such that the  $r_H^{(\lambda)} - r_L^{(\lambda)} > 0$  holds. In the limit as *EIS* goes to zero my key quantities are essentially the same as derived in Lemma 6, for  $j \in \{L, H\}$ ,

$k \in \{1, 2\}$

$$\begin{aligned}
\lim_{\rho \rightarrow -\infty} \frac{r_j^{(\lambda)}}{1-\rho} &= \mu_j^{(Y,\lambda)} - \frac{\gamma}{1-\gamma} \lambda_j - n \lim_{\rho \rightarrow -\infty} \left( \sum_{i=1}^2 F_j^{y,(i),(\lambda)} \psi_j^{(\lambda)} \right) \\
\lim_{\rho \rightarrow -\infty} \frac{\psi_j^{(\lambda)}}{1-\rho} &= \lim_{\rho \rightarrow -\infty} \frac{r_j^{(\lambda)}}{1-\rho} \\
\lim_{\rho \rightarrow -\infty} (1-\rho) F_j^{y,(k),(\lambda)} &= \frac{1}{\lim_{\rho \rightarrow -\infty} \frac{r_j^{(\lambda)}}{1-\rho}} \frac{aB_k}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \\
n \lim_{\rho \rightarrow -\infty} \left( \sum_{i=1}^2 F_j^{y,(i),(\lambda)} \psi_j^{(\lambda)} \right) &= na \frac{B_1 + B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}}
\end{aligned}$$

and

$$\lim_{\rho \rightarrow -\infty} \left( \frac{r_H^{(\lambda)}}{1-\rho} - \frac{r_L^{(\lambda)}}{1-\rho} \right) = \left( -\frac{\gamma}{1-\gamma} - a \right) (\lambda_H - \lambda_L)$$

Thus, for  $\gamma > 1$ ,  $\lim_{\rho \rightarrow -\infty} (r_H^{(\lambda)} - r_L^{(\lambda)}) > 0$  is satisfied. In conclusion, since the functions  $r_L^{(\lambda)}(\rho)$  and  $r_H^{(\lambda)}(\rho)$  are continuous in  $\rho < 0$ , the condition  $\rho < \bar{\rho}_1^{(\lambda)}$  ensures that  $r_H^{(\lambda)} > r_L^{(\lambda)}$  holds.

It is straightforward that  $|\kappa_L^{(\lambda)}| > |\kappa_H^{(\lambda)}|$  must hold given  $r_H^{(\lambda)} > r_L^{(\lambda)}$ ,  $\psi_H^{(\lambda)} > \psi_L^{(\lambda)}$  and  $\frac{1-\gamma-\rho}{\rho} < 0$ . It is true that

$$0 < \left[ \left( \psi_H^{(\lambda)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} - \left( \psi_L^{(\lambda)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} \right]^2$$

and rearranging yields

$$\left( \psi_H^{(\lambda)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} - \left( \psi_L^{(\lambda)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} \left( \psi_H^{(\lambda)} \right)^{-\frac{1-\gamma-\rho}{\rho}} > \left( \psi_L^{(\lambda)} \right)^{-2\frac{1-\gamma-\rho}{\rho}} \left( \psi_H^{(\lambda)} \right)^{-\frac{1-\gamma-\rho}{\rho}} - \left( \psi_L^{(\lambda)} \right)^{-2\frac{1-\gamma-\rho}{\rho}}$$

and dividing both sides by  $\left( \psi_L^{(\lambda)} \right)^{-\frac{1-\gamma-\rho}{\rho}} \left( \psi_H^{(\lambda)} \right)^{-\frac{1-\gamma-\rho}{\rho}}$  gives

$$\underbrace{\left( \frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1}_{|\kappa_L^{(\lambda)}|} > - \underbrace{\left( \left( \frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right)}_{|\kappa_H^{(\lambda)}|}$$

■  
**Proof of Lemma 10.** Let  $\bar{\rho}_2^{(\lambda)} = \min_{\rho \in \{\Upsilon_2^{(\lambda)}, 0\}} \{\rho\}$  with  $\Upsilon_2^{(\lambda)} = \left\{ \rho : v_2^{(\lambda)}(\rho) = 0, \rho < 0 \right\}$  and  $v_2^{(\lambda)}(\rho) = \frac{1}{\psi_L^{(\lambda)}(\rho)} - \frac{1}{\psi_H^{(\lambda)}(\rho)} - \sum_{i=1}^2 \frac{F_L^{y,(i),(\lambda)}(\rho) - F_H^{y,(i),(\lambda)}(\rho)}{1+\delta_i}$ . I show that for  $\rho < \bar{\rho}_2^{(\lambda)}$  (of  $EIS < \overline{EIS}_2^{(\lambda)} \equiv \frac{1}{1-\bar{\rho}_2^{(\lambda)}}$ ), the equity premium is positive in both states of the world. This is a sufficient condition and there might exist  $\rho > \bar{\rho}_2^{(\lambda)}$  which is consistent with a positive equity premium in both states of the world. The proof is the same as the proof of Lemma 7. I need the technical conditions  $\mu_L^{(Y,\lambda)} - \frac{\gamma}{1-\gamma} \lambda_L - na \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$  and  $\mu_H^{(Y,n)} - \frac{\gamma}{1-\gamma} \lambda_H - na \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$ . As in Lemma 7, under the condition  $\frac{1-\gamma-\rho}{\rho} < 0$  I only have to

show that  $v_2^{(\lambda)}(\rho) > 0$ . Using the results of Lemma 9, I have

$$\begin{aligned} & \lim_{\rho \rightarrow -\infty} (1 - \rho) \left( \left| \frac{1}{\psi_L^{(\lambda)}} - \frac{1}{\psi_H^{(\lambda)}} \right| - \left| \sum_{i=1}^2 \frac{F_L^{y,(i),(\lambda)} - F_H^{y,(i),(\lambda)}}{1 + \delta_i} \right| \right) \\ &= (1 - a) \left| \frac{1}{\lim_{\rho \rightarrow -\infty} \frac{r_L^{(\lambda)}}{1 - \rho}} - \frac{1}{\lim_{\rho \rightarrow -\infty} \frac{r_H^{(\lambda)}}{1 - \rho}} \right| > 0 \end{aligned}$$

It follows that in the limit as  $\rho$  approaches  $-\infty$ , the equity premium is positive in both states of the world. Since the function  $v_2^{(\lambda)}(\rho)$  is continuous in  $\rho < 0$ , the condition  $\rho < \bar{\rho}_2^{(\lambda)}$  ensures  $E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt > 0$ ,  $\forall \lambda_t \in \{\lambda_L, \lambda_H\}$ . ■

**Proof of Lemma 11.** Let  $\bar{\rho}_3^{(\lambda)} = \min_{\rho \in \{\Upsilon_1^{(\lambda)}, \Upsilon_2^{(\lambda)}, \Upsilon_3^{(\lambda)}, 0\}} \{\rho\}$  with  $\Upsilon_3^{(\lambda)} = \{\rho : v_3^{(\lambda)}(\rho) = 0, \rho < 0\}$ ,

$$v_3^{(\lambda)}(\rho) = \frac{\bar{\theta}_L^{(\lambda)}}{\bar{\theta}_H^{(\lambda)}} - \frac{1 - \sum_{i=1}^2 \frac{F_L^{y,(i),(\lambda)} \psi_L^{(\lambda)}}{1 + \delta_i}}{1 - \sum_{i=1}^2 \frac{F_H^{y,(i),(\lambda)} \psi_H^{(\lambda)}}{1 + \delta_i}} \left( \frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}} \right)^{\frac{1-\gamma}{\rho}}$$

and  $\Upsilon_1^{(\lambda)}$  and  $\Upsilon_2^{(\lambda)}$  as defined in Lemma 9 and 10. I show that the condition  $\rho < \bar{\rho}_3^{(\lambda)}$  (or  $EIS < \overline{EIS}_3^{(\lambda)} \equiv \frac{1}{1 - \bar{\rho}_3^{(\lambda)}}$ ) ensures that the equity premium is larger (lower) during a period characterized by a low death rate than the premium in times of high mortality. This is a sufficient condition and there might exist some  $\rho > \bar{\rho}_3^{(\lambda)}$  which is consistent with the result of the Lemma. The proof follows the same argument as the proof of Lemma 8. I have to show that for  $\bar{\theta}_L^{(n)} > (<) \bar{\theta}_H^{(n)}$

$$E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [\lambda_t = \lambda_L] - E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [\lambda_t = \lambda_H] > (<) 0$$

or plugging in the expression for the equity premium

$$\begin{aligned} 0 &< (>) \bar{\theta}_L^{(\lambda)} \frac{Y_t}{P_t^{(\lambda_L)}} \left( \frac{1}{\psi_L^{(\lambda)}} - \frac{1}{\psi_H^{(\lambda)}} - \sum_{i=1}^2 \frac{F_L^{y,(i),(\lambda)} - F_H^{y,(i),(\lambda)}}{1 + \delta_i} \right) \\ &\cdot \left[ \left( \left( \frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) + \frac{\bar{\theta}_H^{(\lambda)} P_t^{(\lambda_L)}}{\bar{\theta}_L^{(\lambda)} P_t^{(\lambda_H)}} \left( \left( \frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \right] \end{aligned}$$

Since  $\psi_H^{(\lambda)} > \psi_L^{(\lambda)}$  (by Lemma 9),  $\left| \frac{1}{\psi_L^{(\lambda)}} - \frac{1}{\psi_H^{(\lambda)}} \right| - \left| \sum_{i=1}^2 \frac{F_L^{y,(i),(\lambda)} - F_H^{y,(i),(\lambda)}}{1 + \delta_i} \right|$  (by Lemma 12), and  $\rho < 0$ ,  $\gamma \in (1, 1 - \rho)$  (by Lemma 9 and 10), it suffices to show that

$$\frac{\bar{\theta}_L^{(\lambda)}}{\bar{\theta}_H^{(\lambda)}} > (<) \frac{P_t^{(\lambda_L)}}{P_t^{(\lambda_H)}} \left( \frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} = \frac{1 - \sum_{i=1}^2 \frac{F_L^{y,(i),(\lambda)} \psi_L^{(\lambda)}}{1 + \delta_i}}{1 - \sum_{i=1}^2 \frac{F_H^{y,(i),(\lambda)} \psi_H^{(\lambda)}}{1 + \delta_i}} \left( \frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}} \right)^{\frac{1-\gamma}{\rho}} \quad (36)$$

In the limit as the *EIS* approaches zero condition (36) is satisfied,

$$\frac{\bar{\theta}_L^{(\lambda)}}{\bar{\theta}_H^{(\lambda)}} > (<) \lim_{\rho \rightarrow -\infty} \frac{1 - \sum_{i=1}^2 \frac{F_L^{y,(i),(\lambda)} \psi_L^{(\lambda)}}{1 + \delta_i}}{1 - \sum_{i=1}^2 \frac{F_H^{y,(i),(\lambda)} \psi_H^{(\lambda)}}{1 + \delta_i}} \left( \frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}} \right)^{\frac{1-\gamma}{\rho}} = 1$$

Since the function  $v_3^{(\lambda)}(\rho)$  is continuous in  $\rho < 0$ , the condition  $\rho < \bar{\rho}_3^{(\lambda)}$  ensures that condition (36) is satisfied and  $E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [\lambda_t = \lambda_L] > (<) E_t \left[ \frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [\lambda_t = \lambda_H]$ . ■

**Proof of Lemma 12.** I need the technical conditions  $B_1 > \frac{1 + \frac{1}{\delta_1}}{1 + \frac{1}{\delta_2}} |B_2|$  and  $\mu_t^{(Y)} - \frac{\gamma}{1-\gamma} \lambda_t - n_t a \frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \neq 0$ . Suppose for now that  $\lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \neq 0$ ,  $|\lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho}| < \infty$ ,  $|\lim_{\rho \rightarrow -\infty} \frac{\mu_t^{(\psi)}}{\psi_t}| < \infty$ ,

$|\lim_{\rho \rightarrow -\infty} \frac{\sigma_t^{(\psi)}}{\psi_t}| < \infty$ ,  $|\lim_{\rho \rightarrow -\infty} (1 - \rho) \mu_t^{(F^{y,(i)})}| < \infty$ , and  $|\lim_{\rho \rightarrow -\infty} (1 - \rho) \sigma_t^{(F^{y,(i)})}| < \infty$ . I will later verify that these assumptions are indeed true. In the limit as the *EIS* approaches zero I have

$$\begin{aligned}
\lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} &= \mu_t^{(Y)} - \frac{\gamma}{1 - \gamma} \lambda_t - n_t \lim_{\rho \rightarrow -\infty} \left[ \sum_{i=1}^2 F^{y,t,(i)}(\lambda, n, t) \psi_t(\lambda, n) \right] \\
&= \mu_t^{(Y)} - \frac{\gamma}{1 - \gamma} \lambda_t - n_t a \frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \\
\lim_{\rho \rightarrow -\infty} \frac{\partial}{\partial \lambda} \left( \frac{r_t}{1 - \rho} \right) &= -\frac{\gamma}{1 - \gamma} - a > 0 \\
\lim_{\rho \rightarrow -\infty} \frac{\partial}{\partial n} \left( \frac{r_t}{1 - \rho} \right) &= a \left( 1 - \frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \right) < 0 \\
\lim_{\rho \rightarrow -\infty} \frac{\psi_t(\lambda, n)}{1 - \rho} &= \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \\
\lim_{\rho \rightarrow -\infty} \frac{\sigma_t^{(\psi)}}{1 - \rho} &= \left( -\frac{\gamma}{1 - \gamma} - a \right) \lambda_t \sigma^{(\lambda)} + a \left( 1 - \frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \right) n_t \sigma^{(n)} \\
\lim_{\rho \rightarrow -\infty} \frac{\mu_t^{(\psi)}}{1 - \rho} &= \left( -\frac{\gamma}{1 - \gamma} - a \right) \lambda_t \mu^{(\lambda)} + a \left( 1 - \frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \right) n_t \mu^{(n)} \\
\lim_{\rho \rightarrow -\infty} (1 - \rho) F^{y,t,(i)}(\lambda, n, t) &= \frac{1}{\lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho}} \frac{a B_i}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \\
\lim_{\rho \rightarrow -\infty} (1 - \rho) \sigma_t^{(F^{y,(i)})} &= -\frac{\frac{\partial}{\partial \lambda} \left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)}{\left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)^2} \frac{a B_i}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \lambda_t \sigma^{(\lambda)} - \frac{\frac{\partial}{\partial n} \left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)}{\left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)^2} \frac{a B_i}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} n_t \sigma^{(n)} \\
\lim_{\rho \rightarrow -\infty} (1 - \rho) \mu_t^{(F^{y,(i)})} &= \frac{a B_i}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}} \left[ \begin{aligned} &\frac{\lim_{\rho \rightarrow -\infty} \frac{\partial}{\partial \lambda} \frac{r_t}{1 - \rho} \lambda_t \mu^{(\lambda)}}{\left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)^2} + \frac{\lim_{\rho \rightarrow -\infty} \frac{\partial}{\partial n} \frac{r_t}{1 - \rho} n_t \mu^{(n)}}{\left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)^2} \\ &- \frac{\left( \lim_{\rho \rightarrow -\infty} \frac{\partial}{\partial \lambda} \frac{r_t}{1 - \rho} \right)^2 \left( \lambda_t \sigma^{(\lambda)} \right)^2}{\left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)^3} - \frac{\left( \lim_{\rho \rightarrow -\infty} \frac{\partial}{\partial n} \frac{r_t}{1 - \rho} \right)^2 \left( n_t \sigma^{(n)} \right)^2}{\left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)^3} \\ &- \frac{\left( \lim_{\rho \rightarrow -\infty} \frac{\partial}{\partial \lambda} \frac{r_t}{1 - \rho} \right) \left( \lim_{\rho \rightarrow -\infty} \frac{\partial}{\partial n} \frac{r_t}{1 - \rho} \right) \lambda_t n_t \sigma^{(\lambda)} \left( \sigma^{(n)} \right)^T}{\left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right)^3} \end{aligned} \right] \\
\lim_{\rho \rightarrow -\infty} \sum_{i=1}^2 F^{y,t,(i)}(\lambda, n, t) \psi_t(\lambda, n) &= a \frac{B_1 + B_2}{\frac{B_1}{1 + \delta_1} + \frac{B_2}{1 + \delta_2}}
\end{aligned}$$

Obviously, all assumptions are satisfied. The equity premium is positive in the limit because

$$\lim_{\rho \rightarrow -\infty} E_t \left[ \frac{dP_t + D_t dt}{P_t} - r_t \right] = -\frac{1 - \gamma - \rho}{\rho} \left( \frac{\frac{\partial}{\partial \lambda} \left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right) \lambda_t \sigma^{(\lambda)} + \frac{\partial}{\partial n} \left( \lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho} \right) n_t \sigma^{(n)}}{\lim_{\rho \rightarrow -\infty} \frac{r_t}{1 - \rho}} \right)^2 > 0$$

Thus,  $\lim_{\rho \rightarrow -\infty} \frac{\partial r_t}{\partial n} < 0$ ,  $\lim_{\rho \rightarrow -\infty} \frac{\partial r_t}{\partial \lambda} > 0$ , and  $\lim_{\rho \rightarrow -\infty} E_t \left[ \frac{dP_t + D_t}{P_t} - r_t \right] > 0$ . Since  $r_t$ ,  $\psi_t$ ,  $F^{y,t,(i)}$ ,  $\mu_t^{(\psi)}$ ,  $\sigma_t^{(\psi)}$ ,  $\mu_t^{(F^{y,(i)})}$ , and  $\sigma_t^{(F^{y,(i)})}$  are continuous in  $\rho$  (for  $\rho < 0$ ), there exists  $\bar{\rho}(n_t, \lambda_t)$  such that  $\rho < \bar{\rho}(n_t, \lambda_t)$  (or  $EIS < \overline{EIS}(n_t, \lambda_t) \equiv \frac{1}{1 - \bar{\rho}(n_t, \lambda_t)}$ ) ensures  $\frac{\partial r_t}{\partial n} < 0$ ,  $\frac{\partial r_t}{\partial \lambda} > 0$ , and  $E_t \left[ \frac{dP_t + D_t}{P_t} - r_t \right] > 0$ . Note that  $\bar{\rho}(n_t, \lambda_t)$  ( $\overline{EIS}(n_t, \lambda_t)$ ) depends crucially on the current level of  $n_t$  and  $\lambda_t$ . ■