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Middle-School Students' Understanding of the Equal Sign: The Books They Read Can't Help

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This study examined how 4 middle school textbook series (2 skills-based, 2 standards-based) present equal signs. Equal signs were often presented in standard operations-equals-answer contexts (e.g., 3 + 4 = 7) and were rarely presented in nonstandard *operations on both sides* contexts (e.g., 3 + 4 = 5 + 2). They were, however, presented in other nonstandard contexts (e.g., 7 = 7). Two follow-up experiments showed that students' interpretations of the equal sign depend on the context. The other nonstandard contexts were better than the operations-equals-answer context at eliciting a relational understanding of the equal sign, but the *operations on both sides* context was best. Results suggest that textbooks rarely present equal signs in contexts most likely to elicit a relational interpretation—an interpretation critical to success in algebra.

The equal sign (=) is ubiquitous in mathematics, and a sophisticated concept of the symbol is essential for understanding many topics in mathematics (e.g., algebraic equations). However, over 20 years of research in developmental psychology and mathematics education has indicated that many elementary school students (ages 7 to 11) have an inadequate understanding of the equal sign (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Carpenter, Franke, & Levi, 2003; Kieran, 1981; McNeil & Alibali, 2005a; Rittle-Johnson & Alibali, 1999). Instead of interpreting it as a relational symbol of mathematical equivalence, most students interpret the equal sign as an operational symbol meaning "find the total" or "put the

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answer." Students not only provide operational interpretations when asked to define the equal sign, but also rate operational interpretations such as "the total" and "the answer" as smarter than relational interpretations such as "equal to" or "two amounts are the same" (McNeil & Alibali, 2005a).

Far less is known about students' understanding of the equal sign beyond the elementary school years. It might be assumed that, despite the aforementioned research, students will acquire a relational understanding of the equal sign by the time they reach middle school. Unlike students in elementary school, students in middle school (ages 11 to 14) possess many of the general cognitive structures and functions thought to be necessary for learning higher-level mathematics. For example, according to Piaget and colleagues (Inhelder & Piaget, 1955/1958; Oleron, Piaget, Inhelder, & Greco, 1963/1995), children in this age range have developed the logical structures necessary for coordinating relationships of equivalence and detecting complex relational similarities. Children in this age range also have a mature working memory system (Gathercole, 1999), which is thought to be necessary for solving complex arithmetic problems (Hitch, 1978) and processing complex relations (Halford, Wilson, & Phillips, 1997). Thus, from a developmental perspective, students in middle school should be more likely than students in elementary school to have a relational understanding of the equal sign, and students in elementary school should not be ready to learn the relational concept.

Although developmental factors may contribute to students' ideas about the equal sign, Baroody and Ginsburg (1983) and Carpenter et al. (2003) provided evidence that age alone cannot account for students' operational interpretation of the symbol. When first- through sixth-grade students were asked what number should be placed in the box to make the number sentence $8 + 4 = \_\_ + 5$ true, Carpenter et al. found that fewer than 10% in any grade gave the correct answer and that performance did not improve with age. Baroody and Ginsburg likewise found a deeply ingrained operational understanding among second- and third-grade students who experienced traditional instruction. Both Baroody and Ginsburg and Carpenter et al. found, however, that provided the appropriate experiences—such as viewing the equal sign in nonoperational contexts (e.g., $8 = 8$)—even first grade students were capable of gaining an understanding of the equal sign as a relational symbol.

In light of this evidence, some researchers (e.g., Baroody & Ginsburg, 1983; Carpenter et al., 2003; McNeil & Alibali, 2005b; Seo & Ginsburg, 2003) have argued that the operational interpretation of the equal sign is a byproduct of students' experiences with the symbol in elementary school mathematics. In elementary school, students often encounter the equal sign in the context of equations that have operations on the left side of the equal sign and the answer blank on the right side (e.g., $3 + 4 = \_\_$, $12 - 4 + 2 = \_\_$). To solve these standard operations-equals-answer equations correctly, it is not necessary for students to interpret the equal sign as a symbol of equivalence. Instead, students only need to be able to operate on the numbers to get an answer. As a result, they may focus on performing the arithmetic operations to get a final answer and they may come to associate the equal sign with those operations—in essence interpreting the equal sign as a signal to perform the operations preceding it. Unfortunately, this operational interpretation of the equal sign is reinforced year after year as students gain more and more experience with traditional arithmetic equations. Because the operational interpretation is well established by middle school, it may be far from trivial for students in middle school to acquire a relational understanding of the equal sign. Compounding this problem is the fact that little, if any, instructional time is explicitly spent on the equal sign in the middle grades.

Theories that focus on children's general developmental limitations clearly have different implications for mathematics instruction than do theories that focus on children's experience (rather than age, per se). For example, if children's difficulties with the equal sign are due to developmental cognitive limitations (e.g., lack of domain general logical structures or an immature working memory system), then children may not be developmentally ready to learn the relational concept before a certain age. If this is the case, then why should teachers spend valuable class time trying to teach children something that they are not developmentally ready to learn? Instead, they should just wait until children are old enough to learn the relational concept and teach it to them at that point. This view is consistent with the way that mathematics has historically been taught in the United States—in elementary school (and, in particular, arithmetic), students learn to reason about operations as procedures to follow, and they do not learn to reason about operations as expressions of quantitative relationships until middle school (and, in particular, prealgebra). One possible exception is magnitude comparison problems, in which children must evaluate relationships between quantities as greater than, less than, or equal to one another (e.g., $32 = 32$; $3{,}421 > 1{,}620$; $.65 < .9$, $1/2 = 3/6$). Such problems are typically introduced in elementary school. These problems may help to focus children's attention on the relational meaning of the equal sign; however, such problems do not typically involve arithmetic operations (e.g., $23 + 52 < 2 \times 40$) until middle school.

In contrast, if difficulties with the equal sign are due to knowledge built from early experience with arithmetic, then students' ability to acquire the relational concept of the equal sign may depend on the learning context. If this is the case, then teachers can work to improve aspects of the learning context to promote the relational concept. For example, instead of always presenting the equal sign in the standard operations-equals-answer equation context (e.g., $3 + 4 = 7$), teachers could present it in nonstandard contexts that highlight the equal sign as expressing an equivalence relationship between the quantities on each side of an equation (e.g., $3 + 4 = 5 + 2$). This view is consistent with some recent efforts to reform the way elementary school mathematics is taught in the United States, especially efforts to make algebra a K–12 strand (e.g., Blanton & Kaput, 2003; Carpenter et al., 2003).

It seems reasonable to suggest that the contexts in which teachers (and curricula) present the equal sign play a major role in the development of students' understanding of the equal sign. Indeed, a number of researchers have argued that knowledge of concepts may be *context dependent* (e.g., Barsalou, 1982; Munakata, McClelland, Johnson, & Siegler, 1997; Thelen & Smith, 1994). That is, individuals may exhibit knowledge of a concept in some contexts, but not in others. This may be especially true of newly emerging concepts, such as the relational understanding of the equal sign. As Barsalou (1982) argued, newly emerging ways of thinking tend to be activated only in a limited range of contexts, whereas well established ways of thinking tend to be activated in a wider range of contexts.

If newly emerging ways of thinking are activated only in a limited range of contexts, then children in middle school should not be expected to exhibit a relational understanding of the equal sign across contexts, despite being developmentally ready to do so. The operational interpretation is very well established throughout the elementary school years, so it should be activated in most contexts. The relational interpretation, in contrast, is not well established, so it may need a great deal of contextual support to be activated. Activating the relational interpretation may require presenting the equal sign in nonstandard contexts. Such equations necessitate a relational understanding of the equal sign, and thus, may provide enough contextual support to activate that understanding.

In a recent study, McNeil and Alibali (2005a) showed that students in middle school do not tend to exhibit a relational understanding of the equal sign unless they have the necessary contextual support. Seventh-grade students were randomly assigned to view the equal sign in one of three contexts: (a) alone (=), (b) in an operations-equals-answer equation (3 + 4 + 5 + 3 = __), and (c) in an *operations on both sides* equation (3 + 4 + 5 = 3 + __). Very few students in the alone and operations-equals-answer contexts (11% and 25%, respectively) exhibited a relational understanding of the equal sign. In contrast, most students in the *operations on both sides* context (88%) exhibited a relational understanding of the equal sign. Results suggest that students in seventh grade do not interpret the equal sign as a relational symbol of equivalence in general, but they are able to interpret the equal sign as a relational symbol in the context of an equation with operations on both sides of the equal sign.

Given that students' interpretations of the equal sign are likely to be shaped by context, it is important for researchers to examine the contexts in which students typically see the equal sign. Moreover, researchers should try to determine which contexts, if any, are most likely to activate a relational interpretation of the equal sign. To date, Seo and Ginsburg (2003) provided the only systematic examination of the contexts in which students actually see the equal sign. They conducted a case study of a second-grade classroom, which included an analysis of two mathematics textbooks used by students in the classroom. They found that the equal sign was nearly always presented in the operations-equals-answer context (e.g., 3 + 4 = __). This finding is in line with the hypothesis that students' understanding of the equal sign can be explained by their experiences. However, it is also possible that textbook authors choose the operations-equals-answer context frequently because they think it is a developmentally appropriate context. In that case, students' understanding of the equal sign would be driving their experience, rather than vice versa.

Seo and Ginsburg (2003) focused on second grade students' experience with the equal sign, and it is unclear whether their results generalize beyond the elementary school years. Examining middle-school students' experiences with the equal sign seems particularly important because middle school is often thought to mark the transition between arithmetic and algebra. To prepare for success in algebra, students need to develop a relational understanding of the equal sign. Indeed, studies have shown that students are more likely to solve algebraic equations correctly if they have a relational understanding of the equal sign (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Knuth, Stephens, McNeil, & Alibali, in press).

In this study, we extended the work of Seo and Ginsburg (2003) by analyzing the presentation of the equal sign in several popular middle-school textbook series. Similar to Seo and Ginsburg, we examined the proportion of equal sign instances presented in the standard operations-equals-answer equation context because this context is thought to promote an operational interpretation of the equal sign. We also were interested in the extent to which the textbooks present the equal sign in the "operations on both sides" equation context (e.g., 3 + 4 + 5 = 3 + __), because this context has been shown to elicit the relational interpretation of the equal sign in middle-school students (McNeil & Alibali, 2005a). To foreshadow the results of the textbook analysis, we found very few instances of the equal sign in the "operations on both sides" context, but many instances in other nonstandard contexts (e.g., 7 = 3 + 4). We subsequently conducted two experiments to examine whether other nonstandard equal sign contexts are as effective as the "operations on both sides" context at eliciting a relational interpretation of the equal sign. In the following sections, we report the textbook analysis, followed by the two experiments examining the effect of equal sign context on students' interpretations of the equal sign.

TEXTBOOK ANALYSIS

Method

Materials. We examined four middle-school textbook series (Grades 6 to 8): (a) *Saxon Math* (Hake & Saxon, 2004), (b) *Prentice Hall Mathematics* (Charles, Branch-Boyd, Illingworth, Mills, & Reeves, 2004), (c) *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), and (d) *Mathematics in Context*

(Romberg et al., 1998). It should be noted that *Prentice Hall Mathematics* (Charles et al., 2004) is the latest edition in the Pearson Education line of middle-school mathematics textbook series (earlier editions of *Prentice Hall Mathematics* (Charles et al., 2004) include Scott Foresman-Addison Wesley *Middle School Math* and Prentice Hall *Middle Grades Math*.

The textbook series were selected based primarily on two criteria: (a) They were currently being used in middle schools, and (b) they could be categorized as either *skills-based* or *standards-based*. By skills-based, we mean textbooks that have an emphasis on developing skills (e.g., exercises requiring only arithmetic or algebraic computation). In contrast, by standards-based, we mean textbooks whose creation was supported by National Science Foundation funding and whose design was guided by the National Council of Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). Standards-based textbooks emphasize conceptual understanding, with problems often situated in realistic contexts. It is important to note that all four textbook series focus on both skills and concepts to some degree; however, for the sake of comparison and conciseness, we refer to *Saxon Math* (Hake & Saxon, 2004) and *Middle School Mathematics* as skills-based textbook series, and *Connected Mathematics* (Lappan et al., 1998) and *Mathematics in Context* (Romberg et al., 1998) as Standards-based textbook series.

Coding. We examined every instance of the equal sign on a randomly selected 50% sample of the pages in every book. Each instance of the equal sign was coded as being in an operations-equals-answer context (e.g., 3 + 4 = 7) or a nonstandard context (e.g., 7 = 3 + 4). An *operations-equals-answer* context was defined as any equation containing operations on the left-hand side of the equal sign, and either one number (e.g., 3 + 4 = 7, 2x + 5 = 7) or an unknown quantity (e.g., 3 + 4 = __, 3 + 4 = x) on the right-hand side of the equal sign. Fractions (e.g., 1/2) were considered to be numbers, not operations. A nonstandard context was defined as any equal sign not in the operations-equals-answer context (e.g., 7 = 3 + 4, 7 = 7, 1 ft = 12 in., y = x). The nonstandard context code was further divided into one of two subcategories: (a) equations with operations on both sides of the equal sign (e.g., 5 + 2 = 3 + 4, 3x + 6 = 2x), or (b) other nonstandard contexts (e.g., 7 = 3 + 4, 7 = 7, y = 2x). Finally, the other nonstandard context code was further divided into one of three subcategories: (a) equations with operations on the right side of the equal sign (e.g., 7 = 3 + 4, y = 2x), (b) equations without explicit operations on either side (e.g., 7 = 7, 12 in. = 1 ft, x = y), or (c) no equation (e.g., "Use <, =, or > to complete each statement.").

Reliability was established by having a second coder evaluate a randomly selected 20% sample. Agreement between coders was 96% for coding whether an equal sign was in an operations-equals-answer context or a nonstandard context. Agreement between coders was 99% for coding whether or not an equation had operations on both sides of the equal sign. Agreement between coders was 96% for coding whether an equal sign was in an *operations on the right side*, *no explicit operations on either side*, or *no equation* context.

Results and Discussion

Table 1 presents the total number of equal sign instances found in each textbook's 50% sample, along with the number of pages in the 50% sample and the average number of instances per page. As can be seen in the table, the number of equal sign instances is greater in eighth grade than in sixth grade for all four textbook series. This is true whether one considers the total number of instances, or the average number per page. Averaging across the three grade levels, the skills-based textbooks present far more equal sign instances than do the Standards-based textbooks.

Table 2 displays the proportion of equal sign instances in each grade level and textbook series presented in an operations-equals-answer context. We used logistic regression to examine the log of the odds that an equal sign would be presented in an operations-equals-answer context. Predictor variables were grade level (6–8) and textbook series (*Saxon Math*, Hake & Saxon, 2004; *Prentice Hall Mathematics*, Charles et al., 2004; *Connected Mathematics*, Lappan et al., 1998; *Mathematics in Context*, Romberg et al., 1998). Three contrast codes were used to represent the three degrees of freedom of textbook series: (a) the two skills-based series (*Saxon Math*, Hake & Saxon, 2004; *Prentice Hall Mathematics*, Charles et al., 2004) versus the two standards-based series (*Connected Mathematics*, Lappan et

TABLE 1
Number of Equal Sign Instances, Number of Pages in Sample, and Average Number of Equal Sign Instances Per Page in Each Grade Level and Textbook Series

Grade	*Textbook Series*	*Instances*	*Pages*	*Instances per Page*
6	*Saxon Math*	280	317	0.88
	Prentice Hall Mathematics	1,150	333	3.45
	Connected Mathematics	46	314	0.15
	Mathematics in Context	51	215	0.24
7	*Saxon Math*	558	327	1.71
	Prentice Hall Mathematics	1,801	354	5.09
	Connected Mathematics	315	327	0.96
	Mathematics in Context	106	216	0.49
8	*Saxon Math*	999	424	2.36
	Prentice Hall Mathematics	1,629	364	4.47
	Connected Mathematics	707	285	2.48
	Mathematics in Context	245	228	1.07

TABLE 2
Proportion of Equal Sign Instances in Each Grade Level and Textbook Series Presented in the "Operations Equals Answer" Context

Textbook Series	*6*	*7*	*8*
Saxon Math	.70	.690	.45
Prentice Hall Mathematics	.49	.390	.27
Connected Mathematics	.24	.095	.16
Mathematics in Context	.65	.280	.30

al., 1998; and *Mathematics in Context*, Romberg et al., 1998), (b) *Saxon Math* (Hake & Saxon, 2004) versus *Prentice Hall* (Charles et al., 2004), (c) *Connected Mathematics* (Lappan et al., 1998) versus *Mathematics in Context* (Romberg et al., 1998).

The likelihood of seeing an equal sign in an operations-equals-answer context was higher in the skills-based textbook series than in the Standards-based textbook series, $\hat{\beta} = -0.95$, $z = -12.85$, *Wald* $(1, N = 7887) = 164.23$, $p < .001$. Considering only the skills-based textbook series, the likelihood of seeing an equal sign in an operations-equals-answer context was higher in *Saxon Math* (Hake & Saxon, 2004) than in *Prentice Hall* (Charles et al., 2004), $\hat{\beta} = -0.94$, $z = -15.90$, *Wald* $(1, N = 7887) = 256.93$, $p < .001$. Considering only the Standards-based textbook series, the likelihood of seeing an equal sign in an operations-equals-answer context was higher in *Mathematics in Context* (Romberg et al., 1998) than in *Connected Mathematics* (Lappan et al., 1998), $\hat{\beta} = 1.08$, $z = 7.77$, *Wald* $(1, N = 7887) = 60.81$, $p < .001$. Thus, the variability among textbook series was quite high. Controlling for textbook series, the likelihood of seeing an equal sign in an operations-equals-answer context decreased with grade level, $\hat{\beta} = -0.50$, $z = -15.15$, *Wald* $(1, N = 7887) = 231.57$, $p < .001$.

Instances of the operations-equals-answer context decreased across grade levels for all four textbook series. There were not, however, a substantial number of equal signs presented in the "operations on both sides" context. Table 3 presents the proportion of equal sign instances in each grade level and textbook series presented in an equation with operations on both sides of the equal sign. As shown in the table, the "operations on both sides" context (e.g., 3 + 4 = 5 + 2) accounted for only a very small proportion (0.05) of the equal signs on average.

We examined the log of the odds that an equal sign would be presented in the "operations on both sides" context; predictor variables were identical to those used in the previous analysis. The likelihood of seeing an equal sign in an "operations on both sides" context did not differ statistically for the Standards-based and skills-based textbook series, $\hat{\beta} = -0.19$, $z = -1.37$, *Wald* $(1, N = 7887) = 1.87$, $p = .17$. Considering only the skills-based textbook series, the likelihood of seeing an equal sign in an "operations on both sides" context was higher in *Prentice Hall*

TABLE 3
Proportion of Equal Sign Instances in Each Grade Level and Textbook Series Presented in the "Operations on Both Sides" Context

Textbook Series	*6*	*7*	*8*
Saxon Math	.032	.0230	.080
Prentice Hall Mathematics	.076	.0630	.085
Connected Mathematics	.000	.0032	.090
Mathematics in Context	.020	.0660	.057

Mathematics (Charles et al., 2004) than in *Saxon Math* (Hake & Saxon, 2004), $\hat{\beta}$ = 0.39, z = 3.28 *Wald* (1, N = 7887) = 10.75, p = .001. Considering only the Standards-based textbook series, the likelihood of seeing an equal sign in an "operations on both sides" context did not differ statistically in *Mathematics in Context* (Romberg et al., 1998) and *Connected Mathematics* (Lappan et al., 1998), $\hat{\beta}$ = –0.080, z = –0.32 *Wald* (1, N = 7887) = 0.10, p = .75. Controlling for textbook series, the likelihood of seeing an equal sign in an "operations on both sides" context increased with grade level, $\hat{\beta}$ = 0.28, z = 4.41, *Wald* (1, N = 7887) = 19.48, p < .001.

Although it is evident that all of the textbook series present more "operations on both sides" contexts as they progress from sixth to eighth grade, very few "operations on both sides" contexts were found in any of the textbooks (5% of equal signs on average, with a maximum of only 9%). This seems problematic, given that this context has been shown to activate a relational understanding of the equal sign in middle-school students (McNeil & Alibali, 2005a). However, there are other types of equation contexts to consider. Specifically, as indicated by the proportion of equal sign instances not accounted for in Tables 2 and 3, all of the textbooks contained other types of nonstandard equations (e.g., 7 = 3 + 4, 7 = 7). Indeed, these other nonstandard contexts accounted for a majority of the equal sign instances presented to students in all textbook series except *Saxon Math* (Hake & Saxon, 2004).

The other nonstandard equal sign instances fell into one of three categories: (a) equations with operations on the right side of the equal sign (e.g., 7 = 3 + 4, y = 2x), (b) equations without explicit operations on either side (e.g., 7 = 7, 12 in. = 1 ft, x = y), or (c) no equation (e.g., "Use <, =, or > to complete each statement."). Table 4 presents the proportion of other nonstandard equal sign instances in each grade level and textbook series that fell into each of these categories. As shown in the table, the "equations without explicit operations on either side" context (e.g., 7 = 7) was the most frequent nonstandard equation context, especially in sixth grade.

It is unclear whether other types of nonstandard equations would elicit a relational understanding of the equal sign as well as the "operations on both sides" context because McNeil and Alibali (2005a) did not include other types of nonstandard equations in their study. Thus, in the two experiments that follow, we

TABLE 4
Proportion of "Other Nonstandard" Equal Sign Instances in Each Grade Level and Textbook Series Presented in the "Operations on Right Side," "No Explicit Operations on Either Side," and "No Equation" Contexts

Grade	*Textbook Series*	*Operations on Right Side*	*No Explicit Operations on Either Side*	*No Equation*
6	*Saxon Math*	.25	.75	.0000
	Prentice Hall Mathematics	.30	.65	.0480
	Connected Mathematics	.00	.91	.0860
	Mathematics in Context	.39	.61	.0000
7	*Saxon Math*	.27	.71	.0190
	Prentice Hall Mathematics	.38	.60	.0190
	Connected Mathematics	.53	.47	.0000
	Mathematics in Context	.46	.54	.0000
8	*Saxon Math*	.37	.61	.0170
	Prentice Hall Mathematics	.47	.52	.0100
	Connected Mathematics	.69	.31	.0000
	Mathematics in Context	.47	.52	.0064

tested whether other nonstandard contexts are effective at eliciting a relational understanding of the equal sign in middle-school students.

EXPERIMENT 1

Method

Participants. Participants were 110 sixth-grade students (44 boys, 66 girls), 119 seventh-grade students (57 boys, 62 girls), and 93 eighth-grade students (48 boys, 45 girls) recruited from a public middle school in the Midwest. The middle school used a Standards-based math curriculum,

Connected Mathematics (Lappan et al., 1998). The racial–ethnic makeup of the school was 24% African American, 7% Asian, 6% Hispanic, and 63% White. Approximately 37% of students received free or reduced-price lunch.

Procedure. Students were randomly assigned to view the equal sign in one of three contexts. In the operations-equals-answer context, the equal sign was presented in a typical addition equation: $3 + 4 = 7$. In the *operations on right side* context, the equal sign was presented in an equation with the addends on the right side of the equal sign: $7 = 3 + 4$. In the reflexive context, the equal sign was presented in a reflexive equation: $7 = 7$. In all three contexts, there was an arrow pointing to the equal sign followed by two questions: (a) "The arrow above points to a symbol.

What is the name of the symbol?" and (b) "What does the symbol mean?" These questions were modeled after tasks used in previous work examining students' understanding of the equal sign (e.g., McNeil & Alibali, 2000, 2005a; Rittle-Johnson & Alibali, 1999). The questions were part of a larger paper and pencil assessment designed to assess middle-school students' understanding of algebraic concepts. Students' mathematics teachers administered the test during regular school hours.

Coding. Students' interpretations of the equal sign were coded according to whether they expressed a relational understanding of the equal sign (e.g., "two amounts are the same," "equivalent to"), an operational interpretation of the equal sign (e.g., "the answer," "the total," "add up all the numbers"), a vague unspecified equal interpretation of the equal sign (e.g., "equal"), or some other interpretation of the equal sign (e.g., "is," "I don't know"). Examples of students' interpretations along with their respective codes are presented in Table 5. Reliability for coding equal sign interpretations was established by having a second coder evaluate the interpretations of a randomly selected 20% sample. Agreement between coders was 97%.

Results and Discussion

Logistic regression was used to examine the relationship between equal sign context and the likelihood of exhibiting a relational understanding of the equal sign. Predictor variables included equal sign context (operations-equals-answer, operations on right side, or reflexive), grade level (6 to 8), and gender as a control variable (*girl* = 0, *boy* = 1). Two Helmert contrast codes were used to represent the two degrees of freedom of equal sign context: (a) operations-equals-answer context versus the two nonstandard contexts and (b) *operations on right side* context versus reflexive context.

TABLE 5
Examples of Students' Equal Sign Interpretations

Interpretation	*Code*
"It means time for answer."	Operational
"Equals means the sum of something when you +, ×, –, or ÷, and then you get the answer."	Operational
"On either side of the equal sign are the same."	Relational
"It means the two equations on each side are equivalent. Three plus four is the same as 7."	Relational
"What something equals."	Unspecified equal
"The symbol means equal."	Unspecified equal
"On the computer it could be the eyes of a smiley face. What's up =)"	Other
"The equation of the math problem."	Other

TABLE 6
Proportion of Middle-School Students in Each Context Who Gave Each Interpretation

Context	*Relational*	*Operational*	*Unspecified Equal*	*Other*
3 + 4 = 7 (operations equals answer)	.30	.54	.120	.036
7 = 3 + 4 (operations on right side)	.47	.39	.076	.067
7 = 7 (reflexive)	.45	.38	.100	.067

Table 6 displays the proportion of students who exhibited each type of equal sign understanding in each equal sign context averaging across grade level. Students in the operations-equals-answer context were less likely than those in one of the nonstandard contexts to exhibit a relational understanding of the equal sign (34 of 112 vs. 96 of 210), $\hat{\beta} = -0.64$, $z = -2.52$, *Wald* $(1, N = 322) = 6.37$, $p = .01$. Students in the *operations on right side* context and students in the reflexive context were equally likely to exhibit a relational understanding (47 of 105 vs. 49 of 105), $\hat{\beta} = 0.078$, $z = 0.27$, *Wald* $(1, N = 322) = 0.075$, $p = .78$. The likelihood of exhibiting a relational understanding of the equal sign increased with grade level, $\hat{\beta} = 0.62$, $z = 4.07$, *Wald* $(1, N = 322) = 16.45$, $p < .001$, and it did not differ between boys and girls, $\hat{\beta} = 0.17$, $z = 0.72$, *Wald* $(1, N = 322) = 0.52$, $p = .47$.

Results suggest that nonstandard equations are more effective than operations-equals-answer equations at eliciting a relational understanding of the equal sign in middle-school students. Such equations may, thus, be valuable resources for teachers who wish to promote a relational understanding of the equal sign in their classrooms. It should be noted, however, that only 44% of seventh-grade students in this study exhibited a relational understanding averaging across the two nonstandard equation contexts. This is far from the 88% who exhibited a relational understanding of the equal sign when it was presented in an *operations on both sides* context in McNeil and Alibali (2005a). Thus, it may be the case that equations with operations on both sides of the equal sign are the best at eliciting a relational understanding of the equal sign. We cannot be sure because the samples in the two studies differed on a number of dimensions, including school district, location of testing, ethnic–racial diversity, and socioeconomic status. Despite these differences, however, both studies showed the operations-equals-answer context to be the least effective at eliciting the relational interpretation of the equal sign. Because neither study directly compared the *operations on both sides* context to other nonstandard contexts, we cannot say if equations with operations on both sides are better than other nonstandard equations at eliciting a relational understanding of the equal sign. This issue is important, given that the textbook analysis showed very few equal signs presented in an *operations on both sides* context. Thus, in the next experiment, we directly compared equal sign understanding in one of the other nonstandard contexts to equal sign understanding in an *operations on both sides* context.

EXPERIMENT 2

Method

Participants. Participants were 97 sixth-grade students (55 boys, 42 girls), 107 seventh-grade students (42 boys, 65 girls), and 106 eighth-grade students (50 boys, 56 girls) recruited from the same public middle school as described in Experiment 1.

Procedure. The procedure was identical to that of Experiment 1, with the following two exceptions. First, students were randomly assigned to view the equal sign in one of two contexts (instead of three). In the *operations on right side* context, the equal sign was presented in an equation with the addends on the right side of the equal sign: 7 = 3 + 4. In the *operations on both sides* context, the equal sign was presented in an equation with operations on both sides of the equal sign: 5 + 2 = 3 + 4. Second, the question was presented by itself as a paper and pencil test for students in sixth and seventh grade; the question was part of a larger paper and pencil test of algebraic concepts administered to students in eighth grade.

Coding. The coding procedure was identical to that in Experiment 1. Reliability for coding equal sign interpretations was established by having a second coder evaluate the interpretations of a randomly selected 20% sample. Agreement between coders was 100%.

Results and Discussion

Logistic regression was used to examine the relationship between equal sign context and the likelihood of exhibiting a relational understanding of the equal sign. Predictor variables included equal sign context (*operations on both sides* = 0, *operations on right side* = 1), grade level (6 to 8), and gender as a control variable (*girl* = 0, *boy* = 1).

Table 7 displays the proportion of students who exhibited each type of equal sign understanding in each equal sign context averaging across grade level. Students in the *operations on both sides* context were more likely than those in the *operations on right side* context to exhibit a relational understanding of the equal sign (85 of 154 vs. 64 of 156), $\hat{\beta} = 0.57$, $z = 2.46$, $Wald\,(1, N = 310) = 6.06$, $p = .01$. The likelihood of exhibiting a relational understanding of the equal sign increased with

TABLE 7
Proportion of Middle-School Students in Each Context Who Gave Each Interpretation

Context	*Relational*	*Operational*	*Unspecified Equal*	*Other*
3 + 4 = 5 + 2 (operations on both sides)	.54	.28	.14	.045
7 = 3 + 4 (operations on right side)	.41	.39	.15	.039

grade level, $\hat{\beta} = 0.39$, $z = 2.72$, $Wald\,(1, N = 310) = 7.40$, $p = .007$, and it did not differ between boys and girls, $\hat{\beta} = 0.004$, $z = 0.017$, $Wald\,(1, N = 310) < 0.001$, $p = .99$.

Results suggest that all nonstandard equation contexts are not equally effective at eliciting a relational understanding of the equal sign. Specifically, equations with operations on both sides of the equal sign are more effective than equations with operations on the right side of the equal sign. It is, therefore, troublesome that the four middle-school textbooks analyzed rarely presented the equal sign in the context of an equation with operations on both sides. Thus, teachers who seek to promote a relational understanding of the equal sign in their classrooms may want to supplement textbooks by including equations with operations on both sides on the equal sign in some of their lessons.

GENERAL DISCUSSION

Results of the textbook analysis revealed that four popular middle-school mathematics textbooks frequently present the equal sign in an operations-equals-answer context, and rarely present the equal sign in an *operations on both sides* context. This practice may reinforce students' operational interpretation of the equal sign. Indeed, these experiments showed that many middle-school students continue to interpret the equal sign as an operational symbol. Findings suggest that middle-school mathematics textbooks may not be optimally designed to help students acquire a relational understanding of the equal sign.

In all four textbook series analyzed herein, the proportion of equal signs presented in an operations-equals-answer context declined across the middle grades. This is not surprising, given the increasing emphasis on algebraic manipulations (and decreased emphasis on arithmetic operations) as students advance from sixth to eighth grade. This shift in emphasis may help to weaken students' operational interpretation as they progress through the middle grades (cf. McNeil & Alibali, 2005a). Nonetheless, even in eighth grade, many students continue to interpret the equal sign as an operational symbol. One contributing factor may be that students are not exposed to a sufficient number of equations with operations on both sides of the equal sign (McNeil & Alibali, 2005a). Indeed, all four textbook series analyzed in this study seldom presented the equal sign in an *operations on both sides* context. Although the textbooks often presented the equal sign in other nonstandard contexts (e.g., 7 = 3 + 4, 7 = 7), our experiments showed that these other nonstandard contexts are not as effective as the *operations on both sides* context at eliciting a relational interpretation of the equal sign.

Averaging across contexts and grade levels in the two experiments, only 44% of middle-school students exhibited a relational understanding of the equal sign. This may be surprising from a developmental standpoint, given that children in this age range are thought to possess many of the cognitive structures and functions deemed necessary for learning higher-level mathematics. However, it is less surprising if early mathematics experience is a primary factor behind students' misconceptions about the equal sign (as suggested by Baroody & Ginsburg, 1983; Carpenter et al., 2003; McNeil & Alibali 2005b; Seo & Ginsburg, 2003). Students in the United States often learn arithmetic in a highly procedural fashion, with little or no explicit reference to the equal sign as a statement of mathematical equivalence, and they typically see the equal sign in the operations-equals-answer context (Seo & Ginsburg, 2003). The operational interpretation of the equal sign is well entrenched by middle school and, thus, it may be difficult to overcome without the contextual support of equations with operations on both sides of the equal sign.

Do students who offer an operational interpretation of the equal sign really fail to understand the equal sign in a relational way? Some evidence suggests that this may be the case. For example, elementary school students who offer an operational interpretation of the equal sign are less likely to solve equations with operations on both sides of the equal sign (e.g., 3 + 4 + 5 = 3 + __) correctly (Rittle-Johnson & Alibali, 1999). Similarly, middle-school students who offer an operational interpretation are less likely to solve algebraic equations (e.g., 4x + 10 = 70) correctly (Knuth et al., in press). However, this is far from being an open-and-shut case. A view of knowledge as context dependent implies that students may possess both relational and operational understandings of the equal sign, and different understandings may be activated in different contexts (McNeil & Alibali, 2005a). Thus, it is possible that a greater number of students actually understand the equal sign in a relational way, but they may be unable to demonstrate that understanding in an equation-solving situation because the equations they typically encounter in school frequently elicit the operational interpretation. Once elicited, the operational interpretation interferes with equation-solving performance (cf. McNeil & Alibali, 2005b). These results suggest that students may be better able to demonstrate their knowledge of the equal sign as a relational symbol (i.e., display their competence) when they are given proper contextual support.

In this study, the *operations on both sides* context was most effective in eliciting a relational understanding of the equal sign. For this reason, it is tempting to assume that educators could improve students' understanding of the equal sign simply by giving students more experience with equations that have operations on both sides of the equal sign. In fact, some research has suggested that such experiences do serve to promote a relational understanding of the equal sign among elementary school students (Carpenter et al., 2003). However, it is possible that middle-school students who are given more experience with the *operations on both sides* context would bolster their relational understanding of the equal sign in the *operations on both sides* context only and not transfer their understanding to other more mathematically sophisticated equation contexts (e.g., $3x + 7 = 16$). Moreover, even if students do transfer their understanding to algebraic equations, they may never fully appreciate that the relational interpretation applies to all mathematical equations, including typical arithmetic problems. The evidence in support of this hypothesis is mixed. In a study by McNeil and Alibali (2005a), a small number of college students, who presumably had years of experience with operations on both sides of the equal sign, continued to interpret the equal sign as an operational symbol in the context of a typical arithmetic problem (e.g., $3 + 4 = 7$). However, most undergraduates in the study interpreted the equal sign as a relational symbol in the context of a typical arithmetic problem, and all of the physics graduate students in the study interpreted the equal sign as a relational symbol regardless of context. Thus, it is not inevitable that students will continue to interpret the equal sign as an operational symbol in the context of typical arithmetic problems. Future studies should investigate whether more experience with the *operations on both sides* context leads students to construct a relational concept of the equal sign that generalizes broadly.

Although the jury is still out, we argue that middle-school students would benefit from seeing more equal signs in an *operations on both sides* context. Granted, students may view those contexts as exceptions at first and keep their operational interpretation in general (as did the seventh-grade students in McNeil & Alibali, 2005a). However, as students encounter an increasing number of nonstandard equations (including those with operations on both sides), the relational interpretation will apply in more and more contexts over time. Eventually, the relational interpretation may supersede the original, operational interpretation. Although the original way of thinking may not ever be erased, it may become obsolete because the relational interpretation can be applied in all contexts, whereas the operational interpretation cannot (McNeil & Alibali, 2005a; Siegler, 1999).

As the RAND Mathematics Study Panel (2003) report suggested, "the notion of 'equal' is complex and difficult for students to comprehend, and it is also a central mathematical idea within algebra" (p. 53). Improving students' understanding of the equal sign, and thus their preparation for algebra, may require changes in teachers' instructional practices as well as changes in elementary and middle school mathematics curricula. We concur with Stacey and MacGregor's (1997) recommendation that teachers should present students with statements of equality in different ways to further develop students' notions of equivalence. Moreover, given the complex links between teachers' instructional practices and the curricula they implement (and between students' learning and the curricula they encounter), it also is important that curricula reflect research-based information such as that presented here. It is our hope that attending to such recommendations with respect to the equal sign will ultimately pay significant dividends in terms of students' success in algebra—an important consideration given algebra's role as a gatekeeper to future educational and employment opportunities (Moses & Cobb, 2001).

This study highlights the importance of curricula analyses, not simply in terms of the scope and sequence of topics, but rather in terms of the particulars of problem contexts and formats. We contend that it is important for educators to pay attention to the contexts and formats in which they are presenting problems because small differences in how problems are presented can influence what students come to understand about the associated concepts. This has been shown not only for children's understanding of the equal sign (as in this study), but also for children's understanding of variables (McNeil, Weinberg, Alibali, & Knuth, 2005) and simple algebra problems involving two operations and one unknown (Koedinger & Nathan, 2004).

At a more general level, our work contributes to cognitive developmental theory. We have shown that many middle-school students (ages 11 to 14) continue to interpret the equal sign as an operational symbol, despite being developmentally ready to interpret it as a relational symbol. This finding suggests that developmental factors alone cannot always account for children's misunderstandings. We have further shown that middle-school students are more likely to interpret the equal sign as a relational symbol when it is presented in an equation with operations on both sides. This finding emphasizes that children's understanding is inextricably linked to the context. These general lessons about developmental readiness and context dependence challenge traditional developmental accounts that focus on what children know at different ages averaged across contexts. If the ultimate goal is to understand how knowledge is constructed and organized over time, then it will be essential for researchers to pay close attention to the contexts in which knowledge is elicited and used.

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