6. THE ELECTRICAL SYSTEM OF THE BODY

6.1. THE PHYSICS OF THE NERVOUS SYSTEM
All body functions are controlled by the electrical system! information transfer in the central and autonomous nervous system, operation of the brain and spinal cord, operation of the muscle functions operation of the body organs to control and maintain system billions of electrical signals have to be generated in the human nervous system the source of the electrical signals are electrochemical potentials in the nerve cells

⇒ measurement of electrical signals and electrical potential in nerve transmission (As well as body response) allows to obtain useful clinical information.

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<th>Electromyogram</th>
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The nervous system of the body is structured in two parts: the central nervous system controls the voluntary body functions

the autonomous nervous system controls the involuntary body functions

The central nervous system consists of brain, spinal cord and the peripheral nerves Neurons transfer information (obtained by the acoustical, optical, temperature, odor, etc sensitive sensor system of the body) to the brain and the spinal cord. They also transmit response information from the brain to the appropriate muscles.

The autonomous nervous system concentrates on controlling and maintaining the functions of all the inner organs, heart and intestines. It cannot be controlled voluntarily. The information transfer is also controlled by neurons.
The basis structural unit of the nervous system is the **neuron**. It allows the transmission and reception of electrical pulses which are the basis for the information transfer.

The basis structure of neurons is similar in all living creatures. *Most of the available information about neuron structure has been obtained by studying the neurons of squids.*

The main difference is the number of neurons and the complexity of the neuron interconnections which forms a net of potential information links. *For example a lobster has only a few thousand neurons, the brain of a human operates with \( \approx 100 \text{ billion} \ (10^{11}) \) neurons. Each of the neurons is linked to an average of \( 10^5 \) other neurons which gives a total of \( 10^{16} \) connections.*

The figure gives an impression about the connections between the neurons. The contact points between the neurons are called the synapses.
The single neuron is a fairly complex nerve fiber. The figure shows the structure of a motor neuron, which connects directly to a muscle.

The basic parts of a neuron cell are axons, nucleus, dendrites, and synapses. The nucleus is the core of the cell body which contains the DNA genetic information to control the operation of the cell. Dendrites are receiver antennas of neurons, synapses are contact points to other neurons. The axon is a long fiber which allows to transmit electrical signals to the final receiver.
The neuron cell operates unidirectional, the receiving end is at the
dentrites either through direct contact to the sensors or through synaptic
connections with other neurons. The transmitting axis is the axon which
is connected to other neurons or to the receptor. The dentrites have a
threshold for incoming signals, only electrical signals with an amplitude
above the threshold are accepted and the signal is transmitted further
along the axon. The axon in human neurons has a diameter of $\approx 10-20$
$\mu$m and can have a length of $\approx 1$m.

in the human it extends from the brain to low in the spinal cord or
from the spinal cord to the finger etc

The electrical pulse along the axon travels with a speed of $\approx 0.6 - 100$
m/s. The conductivity depends mainly on the membrane of the axon be-
cause the neuron operates mainly on the basis of an electrical potential
difference between the interior and the exterior side of the membrane.
This potential difference is caused by different concentration of negative
and positive charged ions on either side.

\[
\begin{align*}
\text{Inside of axon} & & \text{Extracellular fluid} \\
\left[\text{Na}^+\right] = 15 & & \left[\text{Na}^+\right] = 145 \quad 9.7 \\
\left[\text{K}^+\right] = 150 & & \left[\text{K}^+\right] = 5 \quad 0.03 \\
\left[\text{Cl}^-\right] = 9 & & \left[\text{Cl}^-\right] = 125 \quad 13.9 \\
\left[\text{Misc}^-\right] = 156 & & \left[\text{Misc}^-\right] = 30 \quad 0.2 \\
v = -70 \text{ mV} & & v = 0 
\end{align*}
\]

The achieved potential difference is between $\Delta V = 60 - 90$ mV.
This is the resting potential of the axon, when no stimulating pulse has
been received.
Potential difference is achieved due to selective diffusion of positive ions through the membrane which causes an enrichment of positive charges outside the cell. This causes the potential difference $\Delta V \approx 70$ mV and therefore an electrical field $E$.

The equilibrium will be established for a certain concentration ratio which is described by the Nernst equation:

$$\frac{C_i}{C_o} = e^{-\frac{z e (V_i - V_o)}{k T}}$$

With the electrical charge $e = 1.6 \times 10^{-19}$ C, and the Boltzmann constant $k = 1.38 \times 10^{-23}$ J/K, and $z$ as the valence of the ion

$C_i/C_o = 13.7$ for positive ions

$C_i/C_o = 0.073$ for negative ions

The thickness of the cell membrane is $x \approx 5-6$ nm, if it is covered with myelin it is 2 $\mu$m.

$$E = \frac{-dV}{dx} = -\frac{70 \times 10^{-3} V}{6 \times 10^{-9} m} = 1.17 \times 10^7 V/m$$

The accumulated charge $Q$ at the surface $S$ of the membrane is:

(for a dielectrical constant $\kappa \approx 7$, the electrical permittivity $\epsilon_o = 8.85 \times 10^{-12} \text{ C}^2/N \cdot m^2$)

$$Q = \kappa \epsilon_o \cdot S \cdot E$$
The intracellular and extracellular liquid are two conductive fluids, the membrane is an insulator

\[ \Rightarrow \text{axon membrane is a cylindrical capacitor} \]

The capacitance for a membrane of thickness \( b \) is:

\[
C = \frac{Q}{V} = \frac{Q \kappa \varepsilon_0 \cdot S}{Q \cdot b} = \frac{\kappa \varepsilon_0 \cdot S}{b}
\]

The capacitance per surface area is:

\[
\frac{C}{S} = \frac{Q}{V} = \frac{\kappa \varepsilon_0}{b} \approx 0.01 \frac{F}{m^2}
\]

and the charge density \( \sigma \) at the surface of the membrane is:

\[
\sigma = \frac{Q}{S} = \frac{C \cdot V}{S} \approx 7 \cdot 10^{-4} \frac{C}{m^2}
\]

For myelinated fibers the thickness of the membrane is much larger which reduces the capacity/area by a factor of 300.
EXAMPLE

Calculate the capacity of an unmyelinated axon with a length of \( L = 1 \text{ m} \), a membran thickness of \( b = 10 \text{ nm} \), and a radius of \( a = 2.5 \mu\text{m} \). Calculate the total amount of charge on the axon surface!

The capacity of the axon is:

\[
C = \frac{k \cdot \epsilon_0 \cdot S}{b} = \frac{k \cdot \epsilon_0}{b} \cdot (2\pi \cdot a \cdot L)
\]

\[
C = 9.7 \cdot 10^{-8} \frac{C^2}{N \cdot m} \approx 0.1 \mu F
\]

For a voltage of \( V = 70 \text{ mV} \) across the membrane of the axon the total charge is:

\[
Q = C \cdot V = 6.8 \text{ nC}
\]

For a myelinated axon the length of the myelinated section is \( D = 1.4 \text{ mm} \) and its thickness is \( b = 2 \mu\text{m} \). This yields for the capacity \( C \) and the total charge of a myelinated sector:

\[
C = 6.8 \cdot 10^{-13} \text{ F} \quad Q = 4.8 \cdot 10^{-14} \text{ C}
\]

This yields a total capacity and charge for a 1 m long axon of:

\[
C = 4.9 \cdot 10^{-8} \text{ F} = 49 \text{ nF} \quad Q = 3.4 \cdot 10^{-11} \text{ C} = 34 \text{ pC}
\]

capacity and charge are much smaller than for the unmyelinated axon.
leakage current across cell membrane

The membrane is not a perfect insulator therefore a leakage current across the membrane will occur. The leakage current is determined by the resistance $R_m$ of the membrane material.

\[
R_m = \frac{\rho_m \cdot b}{S}
\]

with $\rho_m \approx 1.6 \cdot 10^7 \ \Omega \cdot m$ as electrical conductivity of the membrane material.

The leakage current $I_m$ discharges the capacitance, the time is determined by the time constant $\tau = R \cdot C$.

\[
\frac{dQ}{dt} = -I_m = -\frac{V}{R_m}
\]

\[
\frac{dQ}{dt} = C \cdot \frac{dV}{dt}
\]

This yields a differential equation for the potential difference $V$:

\[
\frac{dV}{dt} = -\frac{V}{R_m \cdot C}
\]

with the solution

\[
V(t) = V_0 \cdot e^{-t/\tau}
\]

The time constant $\tau = R_m \cdot C_m = \kappa \varepsilon_0 \rho_m \approx 10^{-3} \text{s}$
The resistance and the capacitance of a portion of the axon membrane are expressed in terms of the axon radius $r$ and length $L$ using $S = 2\pi aL$.

$$C = \frac{\kappa \varepsilon_0 2\pi aL}{b}$$

$$R_m = \frac{\rho_m b}{2\pi aL}$$

The capacitance increases with length, the resistance decreases with length.

**EXAMPLE**

Calculate the membrane resistance $R_m$ for the unmyelinated and myelinated axon of the previous example.

For the unmyelinated 1 m long axon the resistance is:

$$R_m = \frac{\rho_m b}{2\pi a \cdot L} = \frac{1.6 \cdot 10^7 \ \Omega m}{2\pi \cdot 2.5 \cdot 10^{-6} \ m} = 10^4 \ \Omega$$

For the 1.4 mm long myelinated axon section the resistance is:

$$R_m = \frac{1.6 \cdot 10^7 \ \Omega m}{2\pi \cdot 2.5 \cdot 10^{-6} \ m} \cdot \frac{2 \cdot 10^{-6} \ m}{1.4 \cdot 10^{-3} \ m} = 1.5 \cdot 10^9 \ \Omega$$

This indicates a much higher resistance than for the unmyelinated axon.

The time constant $\tau$ is not affected because it is independent of the axon dimensions.

$$\tau = \kappa \cdot \varepsilon_0 \cdot \rho_m = 1.0 \cdot 10^{-3} \ s$$
resistance along the axon

Because the external and internal fluids are conductive, a current can flow inside or outside of the axon. The interior fluid has a certain resistance, which is determined by the conductivity $\rho_i$ of the fluid and by the axon radius $a$ and length $L$.

$$R_i = \frac{\rho_i L}{\pi a^2}$$

$\rho_i$ is the conductance of the axoplasm:

$$\rho_i \approx 0.5 \ \Omega \ m$$

The internal resistance increases with decreasing radius $a$. The resistance per unit length is:

$$r_i = \frac{R}{L} = \frac{\rho_i}{\pi a^2}$$
When a section of the membrane is excited by electrical current or external stimulus, the membrane permeability changes, ion exchange takes place across the membrane causing a rapid change in potential according to the Nernst equation:

\[
\frac{C_i}{C_o} = e^{-z \cdot \varepsilon (V_i - V_o)/kT}
\]

This changes the polarity because positively charged ions enter the interior of the axon.

The polarisation time scale is determined by the time constant \( \tau = R_m \cdot C \). Because of the potential difference along the outer surface of the axon, electrical currents are induced in the internal conductive fluids. Therefore the depolarization moves along the axon.
The process can be described by application of Kirchhoff's law for the current conditions in the axon.

- current across the memran: \( I_m = \frac{dQ}{dt} = C_m \cdot \frac{dV}{dt} \)
- current along the inside of the axon: \( I_i = \frac{\Delta V}{R_i} \)

(Ohms law, \( \Delta V \) is the potential change along the interior of the axon due to depolarization)

\[
I_i = \frac{1}{r_i} \frac{dV}{dx}
\]

Applying Kirchhoff's Law that at a node point the sum of the incoming currents is equal to the sum of the outgoing currents:

\[
I_1 + I_3 + I_4 = I_2
\]

\[
I_i(x) - I_i(x+\delta x) - I_m = \Delta I - I_m = C_m \cdot \frac{dV}{dt}
\]

dividing the entire equation by the surface area element \( \Delta S = 2\pi adx \)

yields the cable equation

\[
\frac{C_m}{\Delta S} \cdot \frac{dV}{dt} = - \frac{I_m}{\Delta S} + \frac{1}{2\pi ar_i} \frac{d^2V}{dx^2}
\]
which describes the propagation of a pulse along a 'conductive' material.

The change of the potential with time is a function of the leakage current through the surface of the membrane and the change of voltage along the axis.

The speed of the pulse is determined by the capacitance $C$ and by the internal resistance $R_i$.

As larger the capacitance, as longer it takes the membrane to discharge (time constant), therefore a smaller propagation speed.

However as larger the internal resistance, as smaller the axial current $I_i$.

The resistance is inverse to the axon radius, as larger the radius, as higher the propagation velocity.
EXAMPLE Longitudinal current as a function of position

The voltage along the axis is shown at some instant time.
The axon radius is $a = 10\ \mu m$, the resistivity of the axoplasm is $\rho_i = 0.5\Omega m$. The longitudinal current $I_i$ can then be expressed as a function of position

\[
\begin{align*}
\frac{\delta V}{\delta x} &= \frac{110 \times 10^{-3}}{1 \times 10^{-3}} = 110\ \text{V m}^{-1} \\
\frac{\delta V}{\Delta x} &= \frac{-110 \times 10^{-3}}{(0.5 \times 10^{-3})} = -220\ \text{V m}^{-1}
\end{align*}
\]

\[
I_i(x) = -\frac{1}{R_i}\frac{dV}{dx}
\]

Left hand side: $dV/dx = 0.11\ \text{V/10}^{-3}\text{m} = 110\ \text{V/m}$
Right hand side: $dV/dx = -0.11\ \text{V/0.5}\cdot10^{-3}\text{m} = -220\ \text{V/m}$

resistance per unit length $r_i$

\[
r_i(x) = \frac{\rho}{S} = \frac{\rho}{\pi a^2} = 1.6 \cdot 10^9\Omega/m
\]

Left hand side current: $I_i = -7 \cdot 10^{-8}\ \text{A}$
Right hand side current: $I_i = 14 \cdot 10^{-8}\ \text{A}$
The cable equation can be reformulated:

$$\frac{dV}{dt} = -\frac{I_m}{C_m} + \frac{\Delta S}{2\pi \cdot a \cdot \frac{1}{r_i C_m}} \frac{d^2V}{dx^2}$$

with $\frac{C}{\Delta S} = \frac{\kappa \varepsilon_0}{b}$, $r_i = \frac{\rho_i}{\pi \cdot a^2}$ and $\Delta V = V_{peak} - V_{rest} \approx 70$ mV:

$$\frac{dV}{dt} = \frac{\Delta V}{\tau} + \frac{b \cdot a}{2 \rho_i \cdot \kappa \cdot \varepsilon_0} \frac{d^2V}{dx^2}$$

multiplying with the time constant $\tau = \kappa \cdot \varepsilon_0 \rho_m$ yields:

$$\tau \cdot \frac{dV}{dt} = \Delta V + \frac{\tau \cdot b \cdot a}{2 \rho_i \cdot \kappa \cdot \varepsilon_0} \frac{d^2V}{dx^2}$$

$$= \Delta V + \frac{b \cdot a \cdot \rho_m}{2 \rho_i} \frac{d^2V}{dx^2}$$

introducing the space constant $\lambda = \sqrt{\frac{\rho_m \cdot a \cdot b}{2 \rho_i}}$ and using the time constant $\tau = \kappa \cdot \varepsilon_0 \cdot \rho_m$ yields the final formulation for the motion of the signal along the axon with time,

$$\Delta V = V_{peak} - V_{rest} = \lambda^2 \frac{d^2V}{dx^2} - \tau \frac{dV}{dt}$$

for an unmyelinated axon with $a=2.5$ μm and with $b=6$ nm and a myelinated axon with $b=2$ μm, respectively the space constants $\lambda_{um}$ and $\lambda_m$ are calculated to:

$$\lambda_{um} = 4.9 \cdot 10^{-4} m \quad \lambda_m = 8.9 \cdot 10^{-3} m$$

with a time constant of $\tau = 10^{-3}$ s
The remaining question is:

What is the speed of the pulse moving along the axon?

The speed of the pulse $v$ in an axon fiber can be determined from the ratio of space constant and time constant,

$$ v = \frac{\lambda}{\tau} = \sqrt{\frac{a \cdot b}{2 \cdot \kappa^2 \cdot \epsilon_0^2 \cdot \rho_i \cdot \rho_m}} $$

The speed of a signal in an unmyelinated axon fiber with $b = 6$ nm is:

$$ v = \sqrt{\frac{6 \cdot 10^{-9}}{2 \cdot 49 \cdot (8.5 \cdot 10^{-12})^2 \cdot 0.5 \cdot 1.6 \cdot 10^7}} \cdot \sqrt{a} = 313 \cdot \sqrt{a} \text{ m/s} $$

The speed increases proportionally with the square-root of the radius.
This are severe limitations since the radius of unmyelinated human axons are typically smaller than 1 $\mu$m.

**EXAMPLE**

for $a \leq 1 \mu$m $\rightarrow v \leq 0.3$ m/s
This indicates slow response times of $\tau_{resp}(L=1 \text{ m}) \geq 3.2$ s

Even for larger sized unmyelinated axons the speeds remain slow and the response time large.

for $a = 2.5 \mu$m $\rightarrow v = 0.5$ m/s
this indicates a relatively slow response, $\tau_{resp}(L=1 \text{ m}) = 2$ s.

for $a = 1 \text{ mm}$ (squid) $\rightarrow v = 10$ m/s
fast response time, $\tau_{resp}(L=1 \text{ m}) = 0.1$ s.
For a myelinated axon fiber there is a fixed relation between the radius of the axon $a$ and the thickness of the myelin layer $b$: $b \approx 0.4a$. This simplifies the expression for the space constant $\lambda$;

$$
\lambda = \sqrt{\frac{a \cdot b}{2} \cdot \frac{\rho_m}{\rho_i}} = \sqrt{0.2 \cdot a^2 \cdot \frac{\rho_m}{\rho_i}}
$$

$$
\lambda = \sqrt{0.2 \cdot \frac{\rho_m}{\rho_i} \cdot a} = \sqrt{0.2 \cdot \frac{1.6 \cdot 10^7}{0.5} \cdot a} \approx 2530 \cdot a
$$

this yields for the speed of the signal in an myelinated axon:

$$
v = \frac{\lambda}{\tau} = \frac{2530 \cdot a}{\kappa \cdot \varepsilon_0 \cdot \rho_m} = 2.6 \cdot 10^6 \cdot a \text{ m/s}
$$

The speeds increases linearly with the axon radius $a$. 

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EXAMPLE

For \( a = 0.015 \text{ mm} = 15 \mu\text{m} \rightarrow \quad v = 3.9 \times 10^4 \text{ m/s} \)

fast response time, \( \tau_{r_{esp}}(L=1 \text{ m}) = 26 \mu\text{s} \).

The response time is significantly faster than in the case of unmyelinated axons. Even for thin axon fibers the speed is significant.

For \( a = 0.2 \mu\text{m} \rightarrow \quad v = 5.1 \times 10^{-1} \text{ m/s} \)

this results in a response time, \( \tau_{r_{esp}}(L=1 \text{ m}) = 2 \text{ s} \).

Properties of unmyelinated and myelinated axons of the same radius.

<table>
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<tr>
<th>Quantity</th>
<th>Unmyelinated</th>
<th>Myelinated</th>
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<tbody>
<tr>
<td>Axon inner radius, ( a )</td>
<td>( 5 \times 10^{-6} \text{ m} )</td>
<td>( 5 \times 10^{-6} \text{ m} )</td>
</tr>
<tr>
<td>Membrane thickness ( b' )</td>
<td>( 6 \times 10^{-9} \text{ m} )</td>
<td>( 2 \times 10^{-6} \text{ m} )</td>
</tr>
<tr>
<td>Myelin thickness ( b )</td>
<td>( 6.2 \times 10^{-11} \text{ s}^{-1} \text{ } \Omega^{-1} \text{ m}^{-1} )</td>
<td>( 6.2 \times 10^{-11} \text{ s}^{-1} \text{ } \Omega^{-1} \text{ m}^{-1} )</td>
</tr>
<tr>
<td>Axoplasm resistivity ( p_i )</td>
<td>( 1.1 \text{ } \Omega \text{ m} )</td>
<td>( 1.1 \text{ } \Omega \text{ m} )</td>
</tr>
<tr>
<td>Membrane (resting) or myelin resistivity ( p_m )</td>
<td>( 10^7 \text{ } \Omega \text{ m} )</td>
<td>( 10^7 \text{ } \Omega \text{ m} )</td>
</tr>
<tr>
<td>Time constant ( \tau = \kappa L p_m )</td>
<td>( 6.2 \times 10^{-4} \text{ s} )</td>
<td>( 6.2 \times 10^{-4} \text{ s} )</td>
</tr>
<tr>
<td>Space constant ( \lambda )</td>
<td>( \lambda = \sqrt{\frac{ab p_m}{2p_i}} )</td>
<td>( \lambda = \sqrt{\frac{ab p_m}{2p_i}} = \sqrt{\frac{0.4a^2p_m}{2p_i}} )</td>
</tr>
<tr>
<td></td>
<td>( = 0.165 \sqrt{a} )</td>
<td>( = a \sqrt{\frac{0.4p_m}{2p_i}} )</td>
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<tr>
<td></td>
<td>( = 370 \times 10^{-6} \text{ m} )</td>
<td>( = 1350a )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = 6.8 \times 10^{-3} \text{ m} )</td>
</tr>
<tr>
<td>Node spacing ( D )</td>
<td>( D = 280a )</td>
<td>( = 1.4 \times 10^{-3} \text{ m} )</td>
</tr>
<tr>
<td>Conduction speed from model</td>
<td>( u_{unmyelinated} = \frac{\lambda}{D} \approx 270 \sqrt{a} )</td>
<td>( u_{myelinated} = \frac{\lambda}{D} \approx 2.2 \times 10^6 \text{ a} )</td>
</tr>
<tr>
<td>Conduction speed, empirical</td>
<td>( u_{unmyelinated} \approx 1800 \sqrt{a} )</td>
<td>( u_{myelinated} \approx 4.5 \times 10^5 \text{ a} )</td>
</tr>
<tr>
<td>Ratio of empirical to model conduction speed</td>
<td>6.7</td>
<td>7.2 or 38</td>
</tr>
<tr>
<td>Space constant using thick membrane model</td>
<td>( \lambda = a \sqrt{\frac{\ln(1+b/a)p_m}{2p_i}} )</td>
<td>( \lambda = a \sqrt{\frac{\ln(1.4)p_m}{2p_i}} = 1240a )</td>
</tr>
</tbody>
</table>