Multivariable Zero-Free Transfer Functions with applications in Economic Modelling

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Global Motivation

- Bill's work in multivariable transfer functions was inspirational
  - At the time
  - In performing the work reported today
- This talk mentions an important applications area
- The talk highlights some properties of an important class of multivariable transfer functions

Outline

- Background and Applications
- Tall Transfer Functions and Zeros
- Left Invertibility, Impulse Responses and MFDs
- Canonical Forms
- Second Order Properties

Factor Models

- Factor models were used in social science to capture the following idea.
- Suppose we measure for *m* people *N* attributes with m and N large. Can we use the data to infer that there exists a small number, *q* say, of meta-attributes (*factors)* such that

$$y_k = Ax_k + n_k \quad k = 1, 2, \ldots m$$
$$\text{where } y_k \in R^N, x_k \in R^q, n_k \in R^N, A \in R^{N \times q}$$

- Original application: IQ, with q=1.
- Here n_k denotes zero mean noise with diagonal or almost diagonal covariance
- Key issues: identifying *q*, identifying A, identifying the factors x_k, using model for forecasting.

Dynamic Factor Models

Econometricians do things like:

- Measure 150+ quarterly variables relating to production, consumption, employment, interest rates, price inflation, in different sectors or states
- Build a dynamic model which has one to four inputs, called factor variables, plus output noise:

$$y_k = W(z)u_k + n_k$$

- When the covariance of n_k is diagonal, the model is a ***dynamic factor model.*** When the covariance is not diagonal, it is a ***generalized dynamic factor model (GDFM)***

Models *like* this appear in multi-sensor signal processing too

Applications areas

- Business cycle analysis
- Economy-wide and global shock identification
- Indexing and Forecasting
- Conduct of Monetary Policy
- Arbitrage Pricing
- Risk Management

Key questions

- How to construct models from data
  - Nontrivial result 1: when the dimension of the output variable N goes to infinity, one can ***eliminate the effects of noise***. Intuition: the 1 to 4 exciting variables affect all the outputs so the ‘signal power’ grows with the output dimension, while the ‘output noise power’ is bounded. It is as if one has measurements for the following model with white noise input

$$y_k = W(z)u_k$$

  - Nontrivial result 2: Knowing the spectrum of y_k, one can recover $W(z)$ uniquely to within inessential right multiplication.
- How to predict: use $W(z)$ and standard theory.

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Tall transfer functions

- From now on we look at transfer functions

$$W(z) = D + C(zI - A)^{-1}B$$

where *W(z)* is $p \times m$, McMillan degree *n*, and *A*,*B*,*C*,*D* is a **minimal realization**. Further:

$$p > m$$

- We will now forget economics and talk about such transfer functions.

Tall Transfer function zeros

- Let $W(z)=D+C(zI-A)^{-1}B$ be tall with full column rank. Zeros are values of z at which following matrix drops normal rank:

$$M = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}$$

- Let $W(z) = U_1(z)V(z)U_2(z)$ where $U_i(z)$ are polynomial with constant nonzero determinant and $V(z)=\text{diag}[n_i(z)/d_i(z)\ \ 0]^T$ and n_i divides n_{i+1} and d_{i+1} divides d_i. Finite zeros are zeros of the n_i.
- Let $W(z)=M^{-1}(z)N(z)$ with M,N coprime polynomials. Then finite zeros are where $N(z)$ drops full rank

Tall Transfer function zeros

- Let $W(z)=D+C(zI-A)^{-1}B$ be tall with full column rank. Zeros are values of z at which following matrix drops normal rank:

$$M = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}$$

- Infinite zeros arise iff D fails to have full column rank.
- For Smith-McMillan or MFDs, let

$$\tilde{W}(q) = W(\frac{aq+b}{q+c})$$

where a is not a pole of W. Then W has an infinite zero if and only if $\tilde{W}$ has no zero at $q=-c$.

Tall Transfer function zeros

- Let $W(z)=D+C(zI-A)^{-1}B$ be tall with full column rank. Zeros are values of z at which following matrix drops normal rank:

$$M = \begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix}$$

- Infinite zeros arise iff D fails to have full column rank.

Theorem: Let $W(z)=D+C(zI-A)^{-1}B$ be tall with generic A,B,C,D. Then $W(z)$ has full column rank and is zero-free.

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Left Invertibility

- A system is termed left invertible with unknown initial state if given a sufficiently long (possibly infinite) output sequence $y_0, y_1, y_2, \ldots$, the input sequence $u_0, u_1, u_2, \ldots$ and the initial state x_0 can be computed.

Theorem: Consider a square or tall $W(z)$ with minimal realization $D+C(zI-A)^{-1}B$, and with no zeros in the finite complex plane. Then it is left invertible. In fact

- There exists $L \leq n$, such that for arbitrary k, x_k and u_k are computable from $y_k, y_{k+1}, \ldots, y_{k+L-1}$.

Left Invertibility (ctd)

Theorem: Consider a square or tall $W(z)$ with minimal realization $D+C(zI-A)^{-1}B$ and with no zeros in the finite complex plane. Then it is left invertible. In fact

- There exists $L \leq n$, such that for arbitrary k, x_k and u_k are computable from $y_k, y_{k+1}, \ldots, y_{k+L-1}$.

- If also there is no zero at infinity, one can additionally recover $x_{k+1}, x_{k+2}, \ldots x_{k+L}$ and the input segment $u_{k+1}, u_{k+2}, \ldots u_{k+L-1}$.
- Further result in case there is no zero at infinity but there is a zero at zero.
- Main result builds on a Theorem of Moylan of 1977.

Indication of a derivation

- Suppose $W(z)= M^{-1}(z)N(z)$ is a coprime factorization. Zero-free $W(z)$ implies $N(z)$ has full rank for all z and so there exists $N_2(z)$ so that

$$\bar{N}(z) = [N(z) \;\; N_2(z)]$$

is (square and) unimodular. Let $Q(z)$ be an inverse.

- From $y_k=W(z)u_k$, we obtain

$$M(z)y_k = \bar{N}\begin{bmatrix} u_k \\ 0 \end{bmatrix}$$

or

$$u_k = [I \;\; 0]Q(z)M(z)y_k$$

Impulse response description

$$\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+L-1} \end{bmatrix} = \begin{bmatrix} C & D & 0 & 0 & 0 & \dots & 0 \\ CA & CB & D & 0 & 0 & \dots & 0 \\ CA^2 & CAB & CB & D & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ CA^{L-1} & CA^{L-2}B & CA^{L-3}B & CA^{L-4}B & CA^{L-5}B & \dots & D \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ u_{k+1} \\ \vdots \\ u_{k+L-1} \end{bmatrix}$$

N_L

- If $W(z)$ is tall and zero-free, then there exists $L \leq n$ with L at least equal to the observability index of (C,A) such that the matrix N_L has full column rank.
- Variations possible for when $W(z)$ has a zero at infinity, or a zero at $z=0$.
- Now use the fact that x_k is producible from prior u_j.

Impulse response description ctd

$$w_0 = D,\ w_k = CA^{k-1}B \text{ for } k \geq 1$$

$$\begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+L-1} \end{bmatrix} = \begin{bmatrix} w_k & \dots & w_1 & w_0 & 0 & 0 & \dots & 0 \\ w_{k+1} & \dots & w_2 & w_1 & w_0 & 0 & \dots & 0 \\ w_{k+2} & \dots & w_3 & w_2 & w_1 & w_0 & \dots & 0 \\ \vdots & & & & & \vdots & & \\ w_{k+L-1} & \dots & w_l & w_{L-1} & w_{L-2} & w_{L-3} & \dots & w_0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_k \\ u_{k+1} \\ \vdots \\ u_{k+L-1} \end{bmatrix}$$

- $W(z)$ is zero free iff given $y_k, y_{k+1}, \dots y_{k+L-1}$, this equation can be solved to give unique $u_k, \dots, u_{k+L-1}$
- Write the matrix above as $[W_1\ \ W_2]$. Let S be a nonsingular matrix premultiplying W_1 to generate row echelon form:

$$S[W_1\ W_2] = \begin{bmatrix} \tilde{W}_{11} & \tilde{W}_{12} \\ 0 & \tilde{W}_{22} \end{bmatrix}$$

Zero-free is equivalent to $\tilde{W}_{22}$ has full column rank.

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Matrix Fraction Descriptions

- In multivariate time-series analysis, one works often with Autoregressive (AR) and Autoregressive Moving Average (ARMA) models:

$$y_k + A_1 y_{k-1} + A_2 y_{k-2} + \cdots + A_\alpha y_{k-\alpha} = B_0 u_k$$

$$y_k + A_1 y_{k-1} + A_2 y_{k-2} + \cdots + A_\alpha y_{k-\alpha}$$
$$= B_0 u_k + B_1 u_{k-1} + B_2 u_{k-2} + \cdots + B_\beta u_{k-\beta}$$

- The associated transfer function matrix is $A^{-1}(z^{-1})B(z^{-1})$ but it is easier to define $q=z^{-1}$ to recover true matrix polynomials. Thus the system has transfer function $W(q)=A^{-1}(q)B(q)$.
- The zero–free property for $B(q)$ for all finite q means u_k can be recovered from a finite interval of outputs that maybe starts *before* time k.

Matrix Transfer Descriptions ctd

- Assume $W(q)=A^{-1}(q)B(q)$ and the zero free property for $B(q)$ for all finite q.
- Because $B(q)$ is zero free there exists $B_2(q)$ so that $[B(q)\ \ B_2(q)]$ is unimodular, with inverse $Q(q)$.
- Then $W(q)$ can always be assumed to have a *nonstandard AR* description:

$$
\begin{aligned}
W(q) &= A^{-1}(q)[B(q)\ \ B_2(q)]\begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= [Q(q)A(q)]^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= D^{-1}(q)\begin{bmatrix} I \\ 0 \end{bmatrix}
\end{aligned}
$$

Matrix Transfer Descriptions ctd

$$W(q) = D^{-1}(q)\begin{bmatrix} I \\ 0 \end{bmatrix}$$

- There exist unimodular $V(q)$ with $V(q)[\,I \quad 0]^T = [\,I \quad 0]^T$. Hence it makes sense to look for a canonical representation.
- $V(q)$ must have the following form:

$$V = \begin{bmatrix} I_m & V_{12} \\ 0 & V_{22} \end{bmatrix}$$

- It transforms $D(q)$ according to

$$D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \to \begin{bmatrix} D_1 + V_{12}D_2 \\ V_{22}D_2 \end{bmatrix}$$

- What can be done with unimodular V_{22} and arbitrary V_{12}?

Matrix Transfer Descriptions ctd

$$D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \rightarrow \begin{bmatrix} D_1 + V_{12} D_2 \\ V_{22} D_2 \end{bmatrix}$$

- $V_{22}(q)$ reduces D_2 to row (polynomial) echelon form,
  - Then V_{12} is used to minimize the row degrees of $D_1 + V_{12} D_2$ with also degrees in entries above pivot index entries of $V_{22} D_2$ being less than degrees of the pivot entries
  - Gives a unique new D

OR

- $V_{22}(q)$ reduces D_2 to Hermite form; a left square submatrix of D_2 is column proper.
  - Then V_{12} is used to reduce the column degrees of the entries of $D_1 + V_{12} D_2$ above the column proper part of D_2 to below the degree of the diagonal entries of this column proper part.
  - Gives a unique new D

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Second order properties

- Let $\Sigma(z)$ be a power spectrum associated with a tall zeroless rational $W(z)$. Thus

$$\Sigma(z) = W(z)W^T(z^{-1})$$

- The zeroless property on $W(z)$ is reflected in a zeroless property of $\Sigma(z)$.
- Starting with a minimal state-variable description of $\Sigma(z)$, a finite number of rational calculations will determine up to right multiplication by a real orthogonal matrix the unique $W(z)$ satisfying the above equation, and of minimum McMillan degree.

Summary

- For transfer function matrices arising in econometric problems:
  - Very tall means we can get rid of noise
  - Transfer function matrix can be recovered from spectrum
  - Canonical forms can be found
  - Input can be recovered from output

Questions?