Some Notes on Realizations

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Motivation

Consider the transfer function

$$P^0(s) = \begin{bmatrix} \dfrac{2}{(s+1)(s-1)} & \dfrac{1}{s-1} \\ \dfrac{1}{s-1} & \dfrac{1}{s-1} \end{bmatrix}$$

- **The order of a minimal realization is ‘2’**
- **Assume that the plant $P^0(s)$ can be stabilized by a compensator $C^0(s)$ of order q.**

Now consider the plant *$P^0(s)$* perturbed to

$$P_1(s) = P(s,\delta) = \begin{bmatrix} \dfrac{2}{(s+1)(s-1)} & \dfrac{1+\delta}{s-1} \\ \dfrac{1}{s-1} & \dfrac{1}{s-1} \end{bmatrix}$$

- **The closed-loop system with *$C^0(s)$* and *$P_1(s)$* is unstable with a closed-loop pole near *s*=1**

  This is true for every *$C^0(s)$* that stabilizes *$P^0(s)$*

Parametrized Plants

Let

$$P(s,\mathbf{p}) = P^{-}(s,\mathbf{p}) + P^{+}(s,\mathbf{p})$$

and let the McMillan degrees be

$$\nu^{+}(\mathbf{p}) := \nu\left[P^{+}(s,\mathbf{p})\right] = \nu^{+}\left[P(s,\mathbf{p})\right]$$
$$\nu^{-}(\mathbf{p}) := \nu\left[P^{-}(s,\mathbf{p})\right] = \nu^{-}\left[P(s,\mathbf{p})\right]$$

Example

$$P_1(s,\mathbf{a}) = \begin{bmatrix} \dfrac{2+a_1}{(s-1+a_2)(s+1)} & \dfrac{1+a_1}{s-1+a_2} \\ \dfrac{1+a_1}{s-1+a_2} & \dfrac{1+a_1}{s-1+a_2} \end{bmatrix}, \mathbf{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}, \mathbf{a}^0 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$P_2(s,\mathbf{b}) = \begin{bmatrix} \dfrac{2+b_1}{(s-1+b_5)(s+1)} & \dfrac{1+b_2}{s-1+b_5} \\ \dfrac{1+b_3}{s-1+b_6} & \dfrac{1+b_4}{s-1+b_6} \end{bmatrix}, \quad \begin{aligned} \mathbf{b} &= \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix} \\ \mathbf{b}^0 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$P_3(s,\mathbf{c}) = \begin{bmatrix} \dfrac{2+c_1}{(s-1+c_5)(s+1)} & \dfrac{1+c_2}{s-1+c_5} \\ \dfrac{1+c_3}{s-1+c_5} & \dfrac{1+c_4}{s-1+c_6} \end{bmatrix}, \quad \begin{aligned} \mathbf{c} &= \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} \\ \mathbf{c}^0 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$P_4(s,\mathbf{d}) = \begin{bmatrix} \dfrac{2+d_1}{(s-1+d_5)(s+1)} & \dfrac{1+d_2}{s-1+d_6} \\ \dfrac{1+d_3}{s-1+d_7} & \dfrac{1+d_4}{s-1+d_8} \end{bmatrix}, \quad \begin{aligned} \mathbf{d} &= \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 \end{bmatrix} \\ \mathbf{d}^0 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$P^0(s) = P_1(s, \mathbf{a}^0) = P_2(s, \mathbf{b}^0) = P_3(s, \mathbf{c}^0) = P_4(s, \mathbf{d}^0)$$

Antistable McMillan degrees

$$\nu^+\left[P^0(s)\right] = \nu_1^+(\mathbf{a}^0) = \nu_2^+(\mathbf{b}^0) = \nu_3^+(\mathbf{c}^0) = \nu_4^+(\mathbf{d}^0) = 1$$

Under arbitrary but infinitesimal perturbations,

$$\nu^+(\mathbf{a}) = 2, \quad \nu^+(\mathbf{b}) = 2, \quad \nu^+(\mathbf{c}) = 3, \quad \nu^+(\mathbf{d}) = 4$$

- The McMillan degree $\nu(\mathbf{p}) = \nu[P(s,\mathbf{p})]$ as well as $\nu^{+}(\mathbf{p})$ is a discontinuous function of **p** and in general its value drops on an algebraic variety.

- If $\mathcal{B}$ denotes an arbitrarily small ball in R^{l}, centered at the origin define

$$\nu_{max} = \max_{\delta\mathbf{p}\in\mathcal{B}} \nu(\mathbf{p}+\delta\mathbf{p})$$

and the algebraic variety: $\mathcal{V}$ $\mathcal{V} = \{\mathbf{p} \,:\, \nu(\mathbf{p}) \neq \nu_{max}\}$

Similarly, we write $\mathcal{V}^{+} := \{\mathbf{p} \,:\, \nu^{+}(\mathbf{p}) \neq \nu^{+}{}_{max}\}$

Structurally Stable Stabilization

Theorem 1 *A plant with transfer function $P(s,\mathbf{p}^0)$ can be stabilized by a linear time invariant feedback controller in a structurally stable manner iff*

$$\nu^+(\mathbf{p}^0) = \nu^+_{max}.$$

If $\nu^+(\mathbf{p}^0) \; < \; \nu^+_{max}$ then

- *any stabilizing controller for $P(s,\mathbf{p}^0)$ renders the closed loop is not structurally stable, that is, the closed loop is destabilized by arbitrarily small perturbations. of the parameter $\mathbf{p^0}$.*
- *any controller that stabilizes $P(s,\mathbf{p_1})$ with $\nu^+(\mathbf{p_1}) = \nu^+_{max}$ fails to stabilize the plant $P(s,\mathbf{p^0})$.*

Example 2 Consider the plant with transfer function parametrization:

$$y(s) = P(s,\delta)u(s)$$

where

$$P(s,\delta) = \begin{bmatrix} \frac{2}{(s+1)(s-1)} & \frac{1+\delta}{s-1} \\ \frac{1}{s-1} & \frac{1}{s-1} \end{bmatrix}.$$

With $\delta = 0$,

$$P(s,0) =: P^0(s) = \begin{bmatrix} \frac{2}{(s+1)(s-1)} & \frac{1}{s-1} \\ \frac{1}{s-1} & \frac{1}{s-1} \end{bmatrix}.$$

We consider the stabilizing controller

$$u = -Ky + v$$

with

$$K = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

A minimal realization of $P^0(s)$ is

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} x$$

and the closed loop system is

$$\dot{x} = (A - BKC)x + Bv \qquad (10)$$

is internally stable with the controller (2) with characteristic polynomial

$$s^2 + 9s + 12.$$

Now consider a "small" perturbation of $P^0(s)$ obtained by letting δ be nonzero. A minimal realization of (6) with $\delta \neq 0$ is:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1+\delta \\ 1 & 1 \end{bmatrix} u \qquad (11)$$

$$y = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

and the closed loop system with the previous controller is

$$\dot{x} = \begin{bmatrix} 0 & -1 & -2 \\ 4+3\delta & -3-3\delta & -6-4\delta \\ 4 & -4 & -5 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1+\delta \\ 1 & 1 \end{bmatrix} v. \qquad (12)$$

The characteristic polynomial of (12) is

$$s^3 + (8+3\delta)s^2 + (3+2\delta)s - \delta - 12 \qquad (13)$$

and is seen to be unstable for "small" values of δ, and in this particular case for all values of δ. Moreover, as $\delta \to 0$, a root to (13) tends to $s = 1$.

Concluding Remarks

- This talk establishes the fact that structurally stable stabilization requires that the nominal system have maximal antistable order

- Showed that discontinuity of this order at the given nominal parameter makes structurally stable stabilization incompatible with nominal stabilization

- The discussion clarifies the importance of system order, McMillan degree, and apriori knowledge of internal structure of state space models

- Emphasizes the fundamental differences between state space modelling and transfer function modelling