Zeros and Zero Dynamics for Linear, Time-delay System

Giuseppe Conte Anna Maria Perdon

***Abstract*— The aim of this paper is to discuss a notion of Zero Module and Zero Dynamics for linear, time-delay systems. The Zero Module in the time-delay framework is defined using the correspondence between time-delay systems and system with coefficients in a ring, so to exploit algebraic and geometric methods. By combining the algebraic notion of Zero Module and the geometric structure of the lattice of invariant submodules of the state module, we point out a natural way to study the Zero Dynamics and its properties. In particular, stability of this latter is characterized using, in the ring framework, a formal notion of Hurwitz set. Application to the study of decoupling problems with stability, inversion problems and traking problems for timke-delay systems are illustrated.**

I. INTRODUCTION

The notion of zero of a linear, dynamical system has been investigated and studied by several authors from many different points of view (see [Schrader and Sain, 1989] for a comprehensive discussion of the literature). Among others, the approach based on the notion of Zero Module, introduced in [Wyman and Sain, 1981] and recalled below provides conceptual and practical tools that, besides being useful in the analysis and synthesis of classical linear systems, can be effectively generalized to a much larger class of dynamical systems. In particular, an algebraic notion of zero in terms of Zero Module has been given in [Conte and Perdon, 1984] for linear, dynamical system with coefficients in a ring.

By exploiting the possibility to associate to any linear, time-delay system a system with coefficients in a suitable ring, the algebraic notion of zero introduced in the ring framework can be employed for defining a notion of zeros and of Zero Dynamics for time-delay systems. This idea has been developed in [Conte and Perdon, 2007],[Conte and Perdon, 2008] where properties of the zeros and of the Zero Dynamics for time-delay systems and their role in inversion and matching problems have been studied.

Here, we give a unified presentation of the results of those papers, discussing the basic aspects of the notion of zero and its interpretation from the geometric point of view, as well as the role of the Zero Dynamics in inversion and tracking problems.

The paper is organized as follows. In Section II, dynamic systems with coefficients in a ring, which are instrumental in developing our approach, are considered and the notion of Zero Module in the ring framework, first defined in [Conte and Perdon, 1984], is recalled. Under suitable hypothesis, this give us the possibility to define a notion of Zero Dynamics, which captures structural basic features of the system at issue.

In Section III, the above notions are interpreted in the time-delay framework, using the natural correspondence between systems over rings and time-delay systems.

Issues such as invertibility, the existence of reduced, in a suitable sense, inverses and stability of inverses are studied by introducing, in Section IV, a notion of phase minimality, based on that of Hurwitz sets and formal Hurwitz stability.

The analysis of the Zero Dynamics is then developed using a geometric approach, which exploits the relation between the Zero Dynamics and elements of the lattices of controlled invariant submodules of the state module, in Section V.

Finally, application to control problems concerning decoupling with stability, inversion and tracking are discussed din Section VI.

II. ZEROS AND ZERO DYNAMICS FOR SYSTEMS OVER RINGS

Let R denote a commutative ring. By a system with coefficients in R, or a system over R, we mean a quadruple $\Sigma = (A, B, C, \mathcal{X})$, where $\mathcal{X} = R^n$ is a free R-module of dimension n and A, B, C are, respectively, $n\times n$, $n\times m$, $p\times n$ matrices with entries in R. The evolution of Σ is described by the set of difference equations

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \tag{1}$$

where $t \in I\!N$ is an independents variable, $x(\cdot)$ belongs to the free module $\mathcal{X} = R^n$, $u(\cdot)$ belongs to the free module $\mathcal{U} = R^m$, $y(\cdot)$ belongs to the free module $\mathcal{Y} = R^p$. By analogy with the classical case of linear, dynamical, discrete-time systems with coefficients in the field of real number $\mathbb{R}$, we view the variables x, u and y as, respectively, the state, input and output of Σ.

Remark 1 *By letting the state module $\mathcal{X}$ to be a projective module, instead of a free one, and interpreting A, B, C as R-morphisms one obtains a slightly more general definition of system with coefficients in R. For reasons that will be clear after the next Section, we are mainly interested to the case in which R is a ring of polynomials with real coefficients. Therefore, since in that case projective modules are free (see [Lam, 1978]), we can restrict our attention to the case in which the state module is a free R-module.*

Dipartimento di Ingegneria Informatica, Gestionale e dell'Automazione, Università Politecnica delle Marche, Via Brecce Bianche, Ancona 60100, ITALY, e-mail: gconte@univpm.it

Besides being interesting abstract algebraic objects, systems with coefficient in a ring have been proved to be useful for modeling and studying particular classes of dynamical systems, such as discrete-time systems with integer coefficients, families of parameter dependent systems and time-delay systems. General results concerning the theory of systems with coefficients in a ring and a number of related control problems can be found in [Sontag, 1976], [Sontag,1981], [Brewer et al., 1986], [Kamen, 1991], [Conte and Perdon, 2000a], [Perdon and Anderlucci, 2006] and the references therein.

In the following, we will generally assume that the considered rings are Noetherian rings, that is rings in which non decreasing chains of ideals are stationary, having no zero divisors. Examples of rings of that kind are the rings of polynomials in one or several variables with real coefficients, that is $\mathbb{R}[\Delta_1, ..., \Delta_k]$, $k \geq 1$, which play a basic role in dealing with time-delay systems. For $k = 1$, $\mathbb{R}[\Delta]$ is also a principal ideal domain (P.I.D.), that is a ring in which any ideal has a single generator (see e.g. [Lang, 1984]).

Introducing the ring $R[z]$ of polynomials in the indeterminate z with coefficients in R and its localization $R(z) = S^{-1}R[z]$ at the multiplicatively closed set $\mathcal{S}$ of all monic polynomials (that is the ring of all rational functions in the indeterminate z with monic denominator), we can associate to any system Σ of the form (1) its transfer function matrix $G(z) = C(zI - A)^{-1}B$, whose elements are in $R(z)$, and the induced $R(z)$-morphism $G : \mathcal{U} \otimes R(z) \to \mathcal{Y} \otimes R(z)$, which is said to be the transfer function of Σ.

Each element $u(z)$ of $\mathcal{U} \otimes R(z)$ can be written as $u(z) = \sum_{t=t_0}^{\infty} u_t z^{-t}$, with $u_t \in \mathcal{U}$, and it can be naturally interpreted as a time sequence, from some time t_0 to ∞, of inputs. Respectively, each element $y(z)$ of $\mathcal{Y} \otimes R(z)$ can be written as $y(z) = \sum_{t=t_0}^{\infty} y_t z^{-t}$, with $y_t \in \mathcal{Y}$, and it can be naturally interpreted as a time sequence, from some time t_0 to ∞, of outputs. Therefore, G can be interpreted as a transfer function between the space of input sequences and the space of output sequences.

In order to simplify our study, we will assume for the rest of the paper, that Σ is a minimal realization of its transfer function. Very roughly, this means that the dimension of the state module cannot be reduced without altering the transfer function. More precisely, minimal representation are characterized by the fact that both the observability matrix $[C^T \;\; C^T A^T .... C^T (A^T)^{n-1}]^T$ and the reachability matrix $[B \;\; AB....A^{n-1}B]$ are full rank. A stronger requirement than minimality is that Σ is a canonical realization of its transfer function. Reachability, in the framework of systems over rings, is a quite strong property, characterized by the fact that the reachability matrix $[B \;\; AB....A^{n-1}B]$ has a right inverse over R. In general, we can have state space representations of the form (1) that are minimal, in the sense explained above, but not reachable and, of course, cannot be transformed into a reachable representation by a change of basis in the state module (see e.g. [Sontag,1981]).

From the point of view we adopt here, following [Wyman and Sain, 1981], the zeros of Σ are determined by the transfer function G in an abstract algebraic way. To this aim, let us recall that the $R[z]$-modules $\mathcal{U} \otimes R[z]$ and $\mathcal{Y} \otimes R[z]$, usually denoted by $\Omega\mathcal{U}$ and by $\Omega\mathcal{Y}$, are naturally embedded into $\mathcal{U} \otimes R(z)$ and into $\mathcal{Y} \otimes R(z)$, respectively. Then, as in [Conte and Perdon, 1984], we can extend to the framework of systems with coefficients in a ring the definition of Zero Module introduced in [Wyman and Sain, 1981].

Definition 1 *(see [Conte and Perdon, 1984] Definition 2.1; [Wyman and Sain, 1981]) Given the system $\Sigma = (A, B, C, X)$ with coefficients in the ring R and transfer function G), the Zero Module of Σ is the $R[z]$-module $\mathcal{Z}$ defined by*

$$\mathcal{Z} = \frac{G^{-1}(\Omega Y) + \Omega U}{Ker\ G + \Omega U}. \tag{2}$$

The reader is referred to [Conte and Perdon, 1984] and [Wyman and Sain, 1981] for a discussion of the above definition. Essentially, in case R is a field, the Zero Module $\mathcal{Z}$ captures the information displayed in the so-called invariant zeros of Σ. Here, we remark that the elements of $\mathcal{Z}$ have a meaningful interpretation in terms of the dynamics of Σ. Any $\zeta \in \mathcal{Z}$ can be written as $\zeta = [u(z)]$, where brackets denote equivalence class and $u(z) = \sum_{t=t_0}^{\infty} u_t z^{-t} \in U(z)$ represents a sequence of inputs whose output $G(u(z)) = \sum_{t=t_0}^{\infty} y_t z^{-t} \in Y(z)$ is such that $y_t = 0$, for $t \geq 1$. Then, writing $u(z) = \sum_{t=t_0}^{0} u_t z^{-t} + \sum_{t=1}^{\infty} u_t z^{-t} = u_1(z) + u_2(z)$ - that is: as the sum of a polynomial part and of a strictly proper one - one has $z = [u(z)] = [u_2(z)]$ and we can say that $\mathcal{Z}$ consists of the strictly proper parts of all the input sequences that generates polynomial outputs.

From another point of view, with the above notations, one can view $u_2(z)$ as representing an input sequence which produces zero outputs if, at $t = 1$, the system is in the state reached from the null state by means of the output $u_1(z)$ over the interval $[t_0, 0]$.

An important property of the zero module $\mathcal{Z}$ is that it turns out to be finitely generated over the ring R (see [Conte and Perdon, 1984], Proposition 2.4). In case $\mathcal{Z}$ is also a free R-module, this implies that we can represent it as a pair (R^m, Z), for some m, where Z is a matrix with entries in R, that defines an R-automorphism $Z : R^m \to R^m$ of the free R-module R^m. Then, we can give the following Definition of the Zero Dynamics of Σ.

Definition 2 *Given a system Σ with coefficients in the ring R, whose zero module is representable as the pair (R^m, D), the Zero Dynamics of Σ is the dynamics induced on R^m by D, that is by the dynamic equation $z(t + 1) = Dz(t)$, for $z \in R^m$.*

According to the above Definition, a Zero Dynamics is represented by the $R[z]$-module structure induced on a free R-module by an R-automorphism. This agrees conceptually with the fact that the dynamics of a systems $\Sigma = (A, B, C, \mathcal{X})$ with coefficients in R is represented by the

$R[z]$-structure induced on the state module $\mathcal{X} = R^n$ by the R-automorphism A. Remark that in case $\mathcal{Z}$ is not a free R-module the Zero Dynamics of Σ is not defined.

Remark 2 *Definition 2 is slightly less general than the one given in [Conte and Perdon, 2008], but more intuitive, and for this reason it has been preferred here. When both definitions apply, they characterize the same concept and differences may exist only in very particular situations.*

To analyze the structure of $\mathcal{Z}$ on the basis of Definition1 may be quite complicated. The task is simplified, in several situations by the relation between the Zero Module and the numerator matrix in polynomial matrix factorizations of $G(z)$, as described in the following Proposition.

Proposition 1 *([Conte and Perdon, 1984] Proposition 2.5) Let $G(z) = D^{-1}(z)N(z)$ be a factorization where $D(z)$ and $N(z)$ are coprime polynomial matrices of suitable dimensions, with $D(z)$ invertible over $R(z)$. Then, the canonical projection $p_N : \Omega\mathcal{Y} \to \Omega\mathcal{Y}/N\Omega\mathcal{U}$ induces an injective $R[z]$-homomorphism $\alpha : \mathcal{Z} \to Tor(\Omega\mathcal{Y}/N\Omega\mathcal{U})$, where $Tor(\Omega\mathcal{Y}/N\Omega\mathcal{U})$ is the so-called torsion submodule of $\Omega\mathcal{Y}/N\Omega\mathcal{U}$.*

The torsion submodule $Tor(\Omega\mathcal{Y}/N\Omega\mathcal{U})$ consists of the elements $[y] \in \Omega\mathcal{Y}/N\Omega\mathcal{U}$, where brackets denote equivalence class and $y = \sum_{t=t_0}^{0} y_t z^{-t} \in \Omega\mathcal{Y}$, such that $ay \in N\Omega\mathcal{U}$ for some $a \neq 0, a \in R$, or, equivalently, such that $a[y] = 0$. When R is a field, α, as shown in [Wyman and Sain, 1981], is actually an isomorphism. To investigate the ring case, it is useful to consider the following notion, first introduced in [Conte and Perdon, 1982].

Definition 3 *Let $\mathcal{M} \subseteq \mathcal{N}$ be R-modules. The closure of $\mathcal{M}$ in $\mathcal{N}$, denoted by $CL_N(\mathcal{M})$ or simply $CL(\mathcal{M})$ if no confusion arises, is the R-module defined by*

$$CL_N(\mathcal{M}) = \{x \in \mathcal{N}, \text{ such that } ax \in \mathcal{M} \\ \text{for some } a \neq 0, \ a \in R\}. \tag{3}$$

If $\mathcal{M} = CL_N(\mathcal{M})$, $\mathcal{M}$ is said to be closed in $\mathcal{N}$.

A key property of closed submodules over a P.I.D. R is the following one.

Proposition 2 *[Conte and Perdon, 1982] Let R be a P.I.D.. Then, a submodule $\mathcal{M} \subseteq \mathcal{N}$ is closed in $\mathcal{M} \subseteq R^n$ if and only if it is a direct summand of $\mathcal{M}$, i.e. there exists a submodule $\mathcal{W} \subseteq \mathcal{N}$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{W}$, or, equivalently, any basis of $\mathcal{N}$ can be completed to a basis of $\mathcal{M}$.*

The notion of torsion submodules is clearly related to the notion of closure and, for what we are concerned, this allows to get the following results.

Proposition 3 *The module $Tor(\Omega\mathcal{Y}/N\Omega\mathcal{U})$ is finitely generated as an R-module if and only if $N(U(z))$ is $R(z)$-closed in $Y(z)$.*

Proposition 4 *Assume that $N(U(z))$ is $R(z)$-closed in $Y(z)$ and let $\alpha : \mathcal{Z} \to Tor(\Omega\mathcal{Y}/N\Omega\mathcal{U})$ be the $R[z]$-homomorphism defined in Proposition 1, then*

- i) $\alpha(\mathcal{Z}) = Tor((D\Omega\mathcal{Y} + N\Omega\mathcal{U})/N(\Omega\mathcal{U})) \subseteq Tor(\Omega\mathcal{Y}/N\Omega\mathcal{U})$
- ii) *α is an $R[z]$-isomorphism if and only if $N(\Omega\mathcal{U}) \cap \Omega\mathcal{Y} \subseteq D\Omega\mathcal{Y} + N\Omega\mathcal{U}$.*

It is useful to remark, as in [Conte and Perdon, 1984], that, besides the case in which R is a field, α is an $R[z]$-isomorphism if $G(Z) = D^{-1}(z)N(Z)$ is a Bezout factorization and G(z) is left or right invertible.

III. Zeros and Zero Dynamics for Time-delay Systems

Let us consider, now, a linear, time-invariant, time-delay system Σ_d with non commensurable delays $h_1, \ldots, h_k$, $h_i \in \mathbb{R}^+$, for $i = 1, \ldots, k$, described by equations of the the form

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{k}\sum_{j=0}^{a} A_{ij}x(t - jh_i) \\ & \quad + \sum_{i=1}^{k}\sum_{j=0}^{b} B_{ij}u(t - jh_i) \\ y(t) &= \sum_{i=1}^{k}\sum_{j=0}^{c} C_i x(t - jh_i) \end{cases} \tag{4}$$

where A_{ij}, B_{ij}, and C_{ij} are matrices of suitable dimensions with entries in the field of real number $\mathbb{R}$.

In the last years, a great research effort has been devoted to the development of analysis and synthesis techniques for this kind of systems, mainly extending tools and methods from the framework of classical linear systems (see e.g. the Proceedings of the IFAC Workshops on Linear Time Delay Systems from 1998 to 2007). Many of the difficulties in dealing with systems of the form (4) is due to the fact that their state space has infinite dimension. In order to circumvent this, it is useful to associate to a time-delay system a suitable system with coefficients in a ring of polynomials with real coefficients (compare with Remark 1), as described in the following.

For any delay h_j, let us introduce the delay operator δ_j defined, for any time function $f(t)$, by $\delta_j f(t) = f(t - h_j)$. Accordingly, we can re-write the system (4) as

$$\Sigma_d = \begin{cases} \dot{x}(t) &= \sum_{i=1}^{a}\sum_{j=0}^{k} A_{ij}\delta_j^i x(t) \\ & \quad + \sum_{i=1}^{b}\sum_{j=0}^{k} B_{ij}\delta_j^i u(t) \\ y(t) &= \sum_{i=1}^{c}\sum_{j=0}^{k} C_{ij}\delta_j^i x(t) \end{cases}$$

Now, by formally replacing the delay operators δ_j by the algebraic unknowns Δ_j, it is possible to associate to Σ_d the discrete-time system Σ over the ring $R = \mathbb{R}[\Delta_1, ..., \Delta_k]$ defined by equations of the form (1) where the matrices A, B, C are given by $A = \sum_{i=1}^{a}\sum_{j=0}^{k} A_{ij}\Delta_j^i$, $B = \sum_{i=1}^{b}\sum_{j=0}^{k} B_{ij}\Delta_j^i$, $C = \sum_{i=1}^{c}\sum_{j=0}^{k} C_{ij}\Delta_j^i$. Actually, the time-delay system Σ_d and the associated system Σ over $R = \mathbb{R}[\Delta_1, ..., \Delta_k]$ are quite different objects from a dynamical point of view, but they share a number of structural properties that depend on the defining matrices. In particular, control problems concerning the input/output behavior of Σ_d can be naturally formulated in terms of the input/output behavior of Σ, transferring them from the time-delay framework to the ring framework. Since systems with coefficients in a ring have finite dimensional state modules, algebraic methods, similar to those of linear algebra, as well as geometric methods apply. Then, solutions to specific problems found in the ring framework can be interpreted in the time-delay framework for solving the original problem (see [Conte and Perdon, 2000a], [Conte and Perdon, 2005] and the references therein).

It should be noted that in this approach the use of systems over rings, beside being instrumental in clarifying the problem and in helping intuition, provides a sound basis for the use of ring algebra. Here, we use the correspondence between time-delay systems and systems over rings to derive a notion of Zero Modules and Zero Dynamics for the first ones and for investigating related control problems. More precisely, we state the following Definition.

Definition 4 *Given a time-delay system Σ_d of the form (4), the Zero Module of Σ_d is the Zero Module $\mathcal{Z}$ of the associated system Σ over the ring $R = \mathbb{R}[\Delta_1, ..., \Delta_k]$.*

In case $\mathcal{Z}$ is a free R-module and the Zero Dynamics of Σ is defined, we can consider that notion also for Σ_d.

Definition 5 *Given a time-delay system Σ_d of the form (4), the Zero Dynamics of Σ_d is the Zero Dynamics of the associated system Σ over the ring $R = \mathbb{R}[\Delta_1, ..., \Delta_k]$, if the latter is defined.*

Remark that if $\mathcal{Z}$ is represented as the pair (R^m, Z), the dynamics described in the ring framework by the equation $\zeta(t+1) = Z\zeta(t)$ gives rise to an associated dynamics $\dot{\zeta}(t) = Z\zeta(t)$ in the time-delay framework by substituting, according to the correspondence illustrated above, the indeterminate Δ by the delay operator $delta$ in the matrix Z. Then, the Zero Dynamics of a time-delay system can be viewed, in general, as a time-delay dynamics.

IV. Invertibility and Phase Minimality

One of the most interesting aspects in the notions of zeros and of Zeros Dynamics is the relation with inversion problems. Roughly, one can expect that the zeros of $G(z)$ appear, in some sense, as poles of any inverse transfer function, if some exists, and, henceforth, that they characterize (part of) the dynamics of any inverse systems. The situation, in the ring framework, has been investigated in [Conte and Perdon, 1984](compare also with [Wyman and Sain, 1981]).

Let us recall that right invertibility of a system Σ with coefficients in the ring R can be characterized by the fact that its transfer function G is surjective, while left invertibility of Σ can be characterized by the fact that its transfer function G is injective and its image ImG is a direct summand of $\mathcal{Y}(z) = \mathcal{Y} \otimes R(z)$. In the following, we will denote by G_{inv} any inverse of a given transfer function G and by $\Sigma_{inv} = (A_{inv},\ B_{inv},\ C_{inv},\ X_{inv})$ its canonical realization. Then, we can recall the following results of [Conte and Perdon, 1984].

Proposition 5 *Given a left (respectively, right) invertible system $\Sigma = (A, B, C, X)$ with coefficients in the ring R and transfer function G, let G_{inv} denote a left (respectively, right) inverse of G and let $\Sigma_{inv} = (A_{inv},\ B_{inv},\ C_{inv},\ X_{inv})$ be its canonical realization. Then, the relation $G_{inv}\ G = Identity$ induces an injective $R[z]$-homomorphism $\psi : \mathcal{Z} \to X_{inv}$ (respectively, the relation $G\ G_{inv} = Identity$ induces a surjective $R[z]$-homomorphism $\varphi : X_{inv} \to \mathcal{Z}$) between the Zero Module of Σ and the state module of the canonical realization of G_{inv}.*

The above Proposition says that the Zero Module $\mathcal{Z}$ is contained, in a suitable algebraic sense, into the canonical state module X_{inv} of any inverse. On this basis, it is interesting to investigate the situations in which, possibly, $\mathcal{Z}$ coincides with X_{inv}, namely the situations in which the $R[z]$-homomorphism ψ (respectively, φ) is an isomorphism. In such case, the inverse can be viewed, in a suitable sense, as a minimal or reduced inverse. In case the ring R is a P.I.D. (this holds, in particular, when dealing with systems associated to time-delay systems with commensurate delays) and K denotes its field of fractions, using the notation of Proposition 5, we can give the following Definition.

Definition 6 *([Conte and Perdon, 1984] Definition 3.4) Given a transfer function G, a left (respectively, right) inverse G_{inv} of G is said to be an essential inverse if $X_{inv}/\psi(\mathcal{Z}) \otimes K = 0$ (respectively, if $Ker\varphi \otimes K = 0$).*

The above Definition reduces to that given in [Wyman and Sain, 1981] when R is itself a field. Moreover, it implies that the Zero Module of Σ and the state module X_{inv} of an essential inverse have the same rank and, in case of right inverse, they are isomorphic (see [Conte and Perdon, 1984], Section 3). A more practical result is the following.

Proposition 6 *Let $G(z) = D^{-1}(z)N(z)$ be a Bezout factorization (respectively, when R is a P.I.D., a coprime factorization). Then,*
$i)$ in case of right invertibility, there exists an essential right inverse G_{ess} of G if $N(z) = Q(z)N'(z)$, where the matrix $Q(z)$ is invertible over $R(z)$ and the matrix $N'(z)$ is right invertible over $R[z]$; moreover, $G_{ess}(z) = M(z)Q^{-1}(z)D(z)$, where the matrix $M(z)$ is a right inverse over $R[z]$ of $N'(z)$, is a factorization of $G_{ess}(z)$ where $M(z), Q(z)$ are right coprime and $Q^{-1}(z)D(z)$ is a Bezout (respectively, coprime) factorization;
$ii)$ in case of left invertibility, there exists an essential left inverse G_{ess} of G if $N(z) = N'(z)Q(z)$, where the matrix $Q(z)$ is invertible over $R(z)$ and the matrix $N'(z)$ is left invertible over $R[z]$; moreover, $G_{ess}(z) = Q^{-1}(z)P(z)D(z)$, where $P(z)$ is a left inverse over $R[z]$ of $N'(z)$, is a Bezout (respectively, coprime) factorization of $G_{ess}(z)$.

The explicit factorizations of essential inverses provided by the above Proposition and the content of Proposition 1 explain the relation between inverses and Zero Modules in terms of the numerator matrix $N(Z)$ of a Bezout factorization of $G(z)$. Further results concerning the case in which R is a P.I.D. are stated in [Conte and Perdon, 1984].
Using the corrspondence between time-delay systems and systems with coefficients in a ring described in Section III, the notion of essential inverse, as well as the results of the above Propositions, can be directly extended to the time delay-framework.
The main interest in the above results is related to the construction of stable inverses. Since a ring cannot, in general, be endowed with a natural metric structure, the notion of stability for systems with coefficients in a ring R can only be given in a formal way. However, in the correspondence between the ring framework and the time-delay one, formal stability can be related to the classical notion. In order to proceed, let us introduce the formal concept of Hurwitz set and the related notion of formal stability (see also [Habets, 1994]).

Definition 7 *Given a ring R, a subset $\mathcal{H} \subseteq R[z]$ of polynomials with coefficients in R in the indeterminate z is said to be an Hurwitz set if*

i) *$\mathcal{H}$ is multiplicatively closed,*
ii) *$\mathcal{H}$ contains at least an element of the form $z - \alpha$, with $\alpha \in R$,*
iii) *$\mathcal{H}$ contains all monic factors of all its elements.*

Given a Hurwitz set $\mathcal{H}$, a system Σ of the form (1) with coefficients in R is said to be $\mathcal{H}$-stable if $det(zI-A)$ belongs to $\mathcal{H}$.
In order to interpret the formal notion in a practical sense, let us consider the ring $R = \mathbb{R}[\Delta_1, ..., \Delta_k]$, which comes into the picture when studying time-delay systems of the form (4). In that case, letting the Hurwitz set $\mathcal{H}$ be defined as

$$\begin{array}{c} \mathcal{H} = \{p(z, \Delta_1, ..., \Delta_k) \in R[z], \\ \text{such that } \ p\left(\gamma, e^{-\gamma h_1}, ..., e^{-\gamma h_k}\right) \neq 0 \\ \text{for all complex number } \ \gamma \text{ with } \mathrm{Re}\gamma \geq 0\}, \end{array} \tag{5}$$

we have that stability of a system Σ_d in the time-delay framework corresponds to $\mathcal{H}$-stability of the associated system Σ in the ring framework (see [Datta and Hautus, 1984]).
Now, it is possible to introduce an abstract notion of phase minimality in the ring framework.

Definition 8 *Given a Hurwitz set $\mathcal{H}$, a system Σ of the form (1) with coefficients in R for which the Zero Dynamics is defined is said to be $\mathcal{H}$-minimum phase if its Zero Dynamics is $\mathcal{H}$-stable.*

Letting $\mathcal{H}$ be chosen as in (5), the notion of $\mathcal{H}$-minimum phase can be used to characterized minimum phase time-delay systems. Then, on the basis of Proposition 5, we can state that non-minimum phase, time-delay systems cannot have stable inverses. For a given time-delay system Σ_d, stable inverses can be constructed by means of essential inverses of the associated system Σ with coefficients in R, if the hypothesis of Proposition 6 hold.

V. Geometric Tools and Analysis of the Zero dynamics

A different point of view for looking at Zeros is that based on the so-called geometric approach (see [Basile and Marro, 1992], [Wohnam, 1985]). Extensions to the framework of systems over rings of geometric methods and tools have been considered by many authors (see [Conte and Perdon, 2000a] and [Perdon and Anderlucci, 2006] for an account of the geometric approach to systems with coefficients in a ring). The basic notion of the geometric approach we will need in the following are briefly recalled below.

Definition 9 *([Hautus, 1982]) Given a system Σ, defined over a ring R by equations of the form (1), a submodule $\mathcal{V}$ of its state module $\mathcal{X}$ is said to be*

i) *(A, B)-invariant, or controlled invariant, if and only if $A\mathcal{V} \subseteq \mathcal{V} + ImB$;*
ii) *(A, B)-invariant of feedback type if and only if there exists an R-linear map $F : \mathcal{X} \to \mathcal{U}$ such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.*

Any feedback F as in ii) above is called a friend of $\mathcal{V}$.

While (A, B)-invariance is a purely geometric property, controlled invariance is a notion related to system dynamics which is equivalent to invariance with respect to a closed loop dynamics. For systems with coefficients in a ring, an (A, B)-invariant submodule $\mathcal{V}$ is not necessarily of feedback type and therefore it cannot always be made invariant with respect to a closed loop dynamics, as it happens in the special case of systems with coefficients in the field of real numbers $\mathbb{R}$. Equivalence between the (generally weaker) geometric notion of (A, B)-invariance and the (generally stronger) dynamic notion of feedback type invariance holds if $\mathcal{V}$ is a direct summand of $\mathcal{X}$, that is $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$ for some submodule $\mathcal{W}$ (see [Conte and Perdon, 1998]).
Given a submodule $\mathcal{K} \subseteq \mathcal{X}$, there exists a maximum (A, B)-invariant submodule of $\mathcal{X}$ contained in $\mathcal{K}$, denoted by $\mathcal{V}^*(\mathcal{K})$, but there may not be a maximum (A, B)-invariant submodule of feedback type contained in $\mathcal{K}$.
The computation of $\mathcal{V}^*(\mathcal{K})$ is not difficult for systems with coefficients in the field of real numbers $\mathbb{R}$, since $\mathcal{V}^*(\mathcal{K})$ coincides with the limit of the sequence $\{\mathcal{V}_k\}$ defined by

$$\begin{array}{rcl} \mathcal{V}_0 & = & \mathcal{K} \\ \mathcal{V}_{k+1} & = & \mathcal{K} \cap A^{-1}(\mathcal{V}_k + ImB) \end{array} \tag{6}$$

and the limit itself is reached in a number of steps lesser than or equal to the dimension of the state space. For systems with coefficients in a ring, the sequence (6), which is non-increasing, may not converge in a finite number of steps and, in such case, an algorithm for computing $\mathcal{V}^*(\mathcal{K})$ is in general not available. In case R is a P.I.D., however, using a different

characterization, the problem of computing $\mathcal{V}^*(\mathcal{K})$ has been satisfactorily solved (see [Assan et al., 1999b]).
Together with the notion of controlled invariance, it is useful to consider the following one.

Definition 10 *Given a system Σ, defined over a ring R by equations of the form (1), a submodule $\mathcal{S}$ of its state module $\mathcal{X}$ is said to be*

i) *(A, C)-invariant, or conditioned invariant, if and only if $A(\mathcal{S} \cap KerC) \subseteq \mathcal{S}$;*
ii) *injection invariant if and only if there exists an R-linear map $H : \mathcal{Y} \to \mathcal{X}$ such that $(A + CH)\mathcal{S} \subseteq \mathcal{S}$.*

Any output injection H as in ii) above is called a friend of $\mathcal{S}$.

In the ring framework, (A, C)-invariance is a weaker property than injection invariance. Given a submodule $\mathcal{K} \subseteq \mathcal{X}$, there exists a minimum (A, C)-invariant submodule of $\mathcal{X}$ containing $\mathcal{K}$, usually denoted by $\mathcal{S}^*(\mathcal{K})$, but there may not be a minimum injection invariant submodule containing $\mathcal{K}$.
As in the field case, it is not difficult to show that, denoting simply by $\mathcal{V}^*$ the (A, B)-invariant submodule $\mathcal{V}^*(KerC)$ and by $\mathcal{S}^*$ the (A, C)-invariant submodule $\mathcal{S}^*(ImB)$, the submodule $\mathcal{R}^*$ defined by

$$\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^* \tag{7}$$

is the smallest (A, B)-invariant submodule of $\mathcal{V}^*$ containing $\mathcal{V}^* \cap ImB$. Moreover, if $\mathcal{V}^*$ is of feedback type with a friend F, also $\mathcal{R}^*$ is of feedback type and it has the same friends (see [Basile and Marro, 1992], [Wohnam, 1985]). The main result relating the Zero Module with the geometric objects we have introduced can now be stated.

Proposition 7 *([Conte and Perdon, 1984], Proposition 4.2, 4.3) Given a system Σ, defined over a ring R by equations of the form (1), with zero module $\mathcal{Z}$, assume that Σ is reachable and observable (respectively, when R is a P.I.D., that it is a minimal realization of its transfer function G) and that $G(\mathcal{U}(z))$ is closed in $\mathcal{Y}(z)$.*
Then, $\mathcal{V}^*/\mathcal{R}^*$ is R-isomorphic to $\mathcal{Z}$.
If, in addition, $\mathcal{V}^*$ is of feedback type and F is one of its friends, $\mathcal{V}^*/\mathcal{R}^*$ endowed with the $R[z]$-module structure induced by the R-morphism $(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*} : \mathcal{V}^*/\mathcal{R}^* \to \mathcal{V}^*/\mathcal{R}^*$ is $R[z]$-isomorphic to $\mathcal{Z}$.

Beside establishing a connection between the algebraic notion of Zero Module and the geometric notion of controlled invariant submodule, the above Proposition provides a practical way to study the Zero Dynamics of Σ. The submodules $\mathcal{V}^*$ and $\mathcal{R}^*$, as well as a friend F, if any exists, can in fact be computed by means of suitable algorithms in several interesting situations (see [Assan et al., 1999a], [Assan et al., 1999b]). Then, the Zero Dynamics turns out to be defined only if $\mathcal{V}^*/\mathcal{R}^*$ is a free R-module and, in that case, it is $\mathcal{H}$-stable only if $det(zI - Z)$, where Z is a matrix that describes the R-morphism $(A+BF)|_{\mathcal{V}^*/\mathcal{R}^*} : \mathcal{V}^*/\mathcal{R}^* \to \mathcal{V}^*/\mathcal{R}^*$, belongs to the Hurwitz set $\mathcal{H}$.

However, it may happen that $\mathcal{V}^*/\mathcal{R}^*$ is a free R-module, but no friends exist, as $\mathcal{V}^*$ is not of feedback type. In this case, the Zero Dynamics is defined, but its analysis on the basis of the definition may be complicated. In order to handle situations of this kind, we can proceed as follows.
Letting $dim\mathcal{V}^* = s$, we construct the extended system $\Sigma_e = (A_e, B_e, C_e, \mathcal{X}_e)$, with state module $\mathcal{X}_e = \mathcal{X} \oplus R^s$, for which

$$A_e = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}; B_e = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}; C_e = \begin{bmatrix} C & 0 \end{bmatrix} \tag{8}$$

where I and 0 denote, respectively, the identity matrix and null matrices of suitable dimensions. Denoting by V a matrix whose columns span $\mathcal{V}^*$ in $\mathcal{X}$, the submodule $\mathcal{V}_e$ spanned in $\mathcal{X} \oplus R^s$ by the columns of the matrix $[V^T I]^T$ is easily seen to be (A_e, B_e)-invariant and direct summand of $\mathcal{X} \oplus R^s$. Hence, $\mathcal{V}_e$ is of feedback type and, since the canonical projection $\pi : \mathcal{X} \oplus R^s \to \mathcal{X}$ is such that $\pi(\mathcal{V}_e) = \mathcal{V}^*$, it can be viewed as an extension of $\mathcal{V}^*$. In addition, we have that $\mathcal{V}_e$ is contained in the kernel $KerC_e$ of the output map of Σ_e. Moreover, denoting by $CL(\mathcal{R}^*)$ the closure of $\mathcal{R}^*$ in $\mathcal{V}^*$, we have the following results.

Proposition 8 *In the above situation and with the above notations, let $\mathcal{R}_e$ be defined as $\mathcal{R}_e = \pi^{-1}(CL(\mathcal{R}^*))$. Then:*

i) *$\mathcal{R}_e$ is an (A_e, B_e)-invariant submodule of $\mathcal{X}_e$;*
ii) *$\mathcal{R}_e$ is the minimum closed submodule of $\mathcal{V}_e$ that contains $\mathcal{V}_e \cap ImB_e$;*
iii) *the canonical projection π induces an R-isomorphism between the quotient module $\mathcal{V}_e/\mathcal{R}_e$ and the quotient module $\mathcal{V}^*/(CL(\mathcal{R}^*))$.*

It follows from the above Proposition, that the system extension produces an R-module $\mathcal{V}_e/\mathcal{R}_e$ and an R-morphism $(A_e + B_eF_e)|_{\mathcal{V}_e/\mathcal{R}_e}$, where F_e is a friend of $\mathcal{V}_e$, that form a pair akin to the pair $\mathcal{V}^*/\mathcal{R}^*$ and $(A + BF)|_{\mathcal{V}^*/\mathcal{R}^*}$ considered in the last statement of Proposition 7. However, the R-morphism $(A_e + B_eF_e)|_{\mathcal{V}_e/\mathcal{R}_e}$ can be constructed also when $\mathcal{V}^*$ is not of feedback type and, henceforth, no friend F is available to define the R-morphism $(A + BF)$. In addition, if the Zero Dynamics is defined, $\mathcal{R}^*$ is closed in $\mathcal{V}^*$ and, therefore, the quotient module $\mathcal{V}_e/\mathcal{R}_e$ and the quotient module $\mathcal{V}^*/\mathcal{R}^*$ are isomorphic. This allows us to state the following result, which is of help in analyzing the Zero Dynamics in many situations.

Proposition 9 *In the above context and with the above notations, assuming that the Zero Dynamics of Σ is defined, let $\mathcal{V}_e/\mathcal{R}_e$ be a free R-module of dimension m and let Z be a matrix representing the R-morphism , $(A_e+B_eF_e)|_{\mathcal{V}_e/\mathcal{R}_e} : \mathcal{V}_e/\mathcal{R}_e \to \mathcal{V}_e/\mathcal{R}_e$, where F_e is a friend of $\mathcal{V}_e$. Then, the Zero Dynamics of Σ can be represented by the pair (R^m, Z).*

Example 1 *This example, that was already discussed in [Conte and Perdon, 2008], can be revisited here in the light of Definition 2. Let us consider the time-delay system Σ_d*

described by the equations

$$\Sigma_d = \begin{cases} \dot{x}_1(t) &= x_3(t-h) + u(t-h) \\ \dot{x}_2(t) &= x_1(t) + x_3(t-h) + u(t-h) \\ \dot{x}_3(t) &= x_2(t) + x_3(t-h) \\ y(t) &= x_3(t) \end{cases}$$

and the associated system $\Sigma = (A, B, C, X)$ with coefficients in $R = \mathbb{R}[\Delta]$ and matrices

$$A = \begin{bmatrix} 0 & 0 & \Delta \\ 1 & 0 & \Delta \\ 0 & 1 & \Delta \end{bmatrix}; B = \begin{bmatrix} \Delta \\ \Delta \\ 0 \end{bmatrix}; C = [0\ 0\ 1].$$

Computations in the ring framework show that $\mathcal{V}^*$ is the submodule of R^3 spanned by the vector $V = (\Delta\ 0\ 0)^T$ and that $\mathcal{R}^* = \{0\}$. The controlled invariant $\mathcal{V}^*$ is not of feedback type (because it is not closed), but it is free. Therefore, the Zero Dynamics of Σ is defined and it can be represented as a suitable pair (R^m, Z) in order to check, for instance, phase minimality.

To analyze the Zero Dynamics, in accordance with the above construction, we consider the extension $\Sigma_e = (A_e, B_e, C_e, X_e)$, where

$$A_e = \begin{bmatrix} 0 & 0 & \Delta & 0 \\ 1 & 0 & \Delta & 0 \\ 0 & 1 & \Delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; B_e = \begin{bmatrix} \Delta & 0 \\ \Delta & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; C_e = [0\ 0\ 1\ 0]$$

We have, now, that $\mathcal{V}_e$ is the submodule of R^4 spanned by the vector $[V^T\ 1]^T = (\Delta\ 0\ 0\ 1)^T$ and $\mathcal{R}_e = \{0\}$. The Zero Dynamics of Σ, as well as that of Σ_d, is defined and can be explicitly evaluated. A friend F_e of $\mathcal{V}_e$ is given, for instance, by $F_e = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Therefore, the dynamic matrix $A_c = (A_e + B_e F)$ of the compensated system is $A_c = \begin{bmatrix} 0 & 0 & \Delta & -\Delta \\ 1 & 0 & \Delta & -\Delta \\ 0 & 1 & \Delta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Since $A_c[V^T\ 1]^T = -[V^T\ 1]^T$, the Zero Dynamics turns out to be given by $(R, [-1])$ or, in other terms by

$$\begin{gathered} \xi(t+1) = -\xi(t) \\ (\dot{\xi}(t) = -\xi(t) \text{ in the time-delay framework}) \end{gathered} \tag{9}$$

with $\xi \in R$ (respectively $\xi \in \mathbb{R}$).

Alternatively, we could analyze directly the Zero Dynamics, starting from its definition. The transfer function matrix of Σ is

$$G(z) = \frac{\Delta z + \Delta}{z^3 - \Delta z^2 - \Delta z - \Delta}$$

and then, by Proposition 1, the zero module $\mathcal{Z}$ of Σ can be viewed, through the injective $R[z]$-morphism α, as a submodule of $Tor(\Omega\mathcal{Y}/(\Delta z + \Delta)\Omega\mathcal{U})$. We have, in our case, $Tor(\Omega\mathcal{Y}/(\Delta z + \Delta)\Omega\mathcal{U}) = \mathbb{R}[\Delta, z]/(\Delta z + \Delta)\mathbb{R}[\Delta, z]$ and, since any element $p(\Delta, z) \in \mathbb{R}[\Delta, z]$ can be written in a unique way as $p(\Delta, z) = \Delta p'(\Delta, z) + p''(z) = \Delta[(z+1)q(\Delta, z) + q'(\Delta)] + p''(z)$, we can say that, denoting equivalence classes by brackets, any element $[p(\Delta, z)] \in Tor(\Omega\mathcal{Y}/(\Delta z + \Delta)\Omega\mathcal{U})$ can be written in a unique way as $[p(\Delta, z)] = [\Delta q'(\Delta)] + [p''(z)]$, for suitable polynomials $q'(\Delta) \in \mathbb{R}[\Delta]$ and $p''(z) \in \mathbb{R}[z]$. It turns out that $\alpha(\mathcal{Z})$ coincides with the submodule of $\mathbb{R}[\Delta, z]/(\Delta z + \Delta)\mathbb{R}[\Delta, z]$ consisting of all elements of the form $[\Delta q'(\Delta)]$ with $q'(\Delta) \in \mathbb{R}[\Delta]$. Inspection shows that, for any element of that kind, $z[\Delta q'(\Delta)] = [z\Delta q'(\Delta)] = [-\Delta q'(\Delta)]$ and therefore we can conclude that the $R[z]$-module $\mathcal{Z}$ can be viewed as defined by the pair $(R, [-1])$, in accordance with what expressed by equation (9).

To conclude, we can say that, taking the Hurwitz set $\mathcal{H}$ as in (5), Σ is $\mathcal{H}$-minimum phase and, respectively, Σ_d is minimum phase.

Example 2 *Let us consider the time-delay system Σ_d described by the equations*

$$\Sigma_d = \begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_1(t) + u_1(t-h_1) + u_2(t-h_2) \\ y(t) &= x_2(t) \end{cases}$$

and the associated system $\Sigma = (A, B, C, X)$ with coefficients in $R = \mathbb{R}[\Delta_1, \Delta_2]$ and matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ \Delta_1 & \Delta_2 \end{bmatrix}; C = [0\ \ 1].$$

Computations in the ring framework show that $\mathcal{V}^*$ is the submodule of R^2 given by $span\{(\Delta_1\ 0)^T, (\Delta_2\ 0)^T\}$ and that $\mathcal{R}^* = 0$. In this case, the Zero Module $\mathcal{Z} = \mathcal{V}^*/\mathcal{R}^* = \mathcal{V}^*$ is not free, since it has a minimal set of generators which are not linearly independent. Hence, the Zero Dynamics of Σ is not defined.

VI. Applications

The notion of Zero Dynamics we have defined can be employed in dealing with control problems that imply inversion or geometric decomposition of the state space. As explained in Section III, problems formulated in the time-delay framework can be interpreted in the ring framework, using the correspondence between time-delay systems and systems over rings. If a solution is found, it can generally be interpreted in the time-delay framework. However, one of the most effective use of this techniques is in pointing out the obstruction that prevents a problem to be solvable.

A. Decoupling problems

Zero Dynamics can be used to characterizes the fixed dynamics with respect to feedbacks which make a system maximally unobservable. To see this, recall that $\mathcal{V}^*$ represents the largest submodule of the state module $\mathcal{X}$ that can be made unobservable by means of a feedback, either a static one, in case $\mathcal{V}^*$ is of feedback type, or a dynamic one, in case it is not. This property is fundamental in dealing with the problem of decoupling the output of the system from a disturbance input (see [Conte and Perdon, 1995]).

Given a system Σ, with coefficient in R, of the form (1), let us consider the submodule $\mathcal{V}^*$ of its state module. In order to deal with all possible situations at the same time, let us consider a system $\Sigma_e = (A_e, B_e, C_e, X_e)$ that, in case $\mathcal{V}^*$

is not of feedback type, is an extension of Σ constructed as in Section V and that, in case $\mathcal{V}^*$ is of feedback type, coincides with Σ. In both cases we have the submodules $\mathcal{V}_e$ and $\mathcal{R}_e$, which, in particular, when Σ_e coincides with Σ, coincide with $\mathcal{V}^*$ and $CL(\mathcal{R}^*)$. Now, let us assume that the Zero Dynamics of Σ is defined and that $\mathcal{V}_e$ is a direct summand of $\mathcal{X}_e$ (as we have seen in Section V, this is true by construction if Σ_e is actually an extension of Σ, but it must be assumed explicitly in the other situation). Remark that the existence of the Zero Dynamics implies, in particular, that $\mathcal{R}^*$ is closed, that is $\mathcal{R}^* = CL(\mathcal{R}^*)$. In both situations, then, we can write $X_e = X \oplus R^r = \mathcal{R}_e \oplus W_1 \oplus W_2$ for some submodules W_1 and W_2, such that $\mathcal{V}_e = \mathcal{R}_e \oplus W_1$ and $X_e = \mathcal{V}_e \oplus W_2$. Writing A_e and B_e in that basis and partitioning accordingly, we get

$$A_e = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}; B_e = \begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix}. \tag{10}$$

Compensating the system by means of a state feedback F_e that is a friend of $\mathcal{V}_e$, we can force the motions originating in $\mathcal{V}_e$ to remain in $KerC_e$, making the system maximally unobservable and, actually, decoupling the output from possible inputs whose image is in $\mathcal{V}_e$. The dynamic matrix $A_c = (A_e + B_e F)$ of the compensated system, for any friend $F_e = [\ F_1 \quad F_2 \quad F_3\ ]$ of $\mathcal{V}_e$, takes, in such case, the form

$$A_c = \begin{bmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 & A_{13} + B_1F_3 \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} + B_3F_3 \end{bmatrix} \tag{11}$$

showing that the dynamics of the block A_{22} remains fixed for any choice of F_e and that, being described by $(A_e + B_eF_e)|_{\mathcal{V}_e/\mathcal{R}_e} = A_{22}$, it coincides with Zero Dynamics of Σ. As a consequence, stability of the decoupled system cannot be achieved if Σ is not minimum phase.

B. *Inversion of time delay systems*

Inversion of time delay-system has been studied, using methods and results from the theory of systems with coefficients in a ring, in [Conte and Perdon, 2000b] and [Conte, Perdon, Iachini, 2001]. Let us consider, for sake of illustration, only the case of Single Input/Single Output, or SISO, systems, the extension to more general situations being worked out in the above mentioned papers. Given a SISO time-delay system Σ_d described by the set of equations

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i x(t-ih) + \sum_{i=0}^{b} b_i u(t-ih) \\ y(t) = \sum_{i=0}^{c} c_i x(t-ih) \end{cases} \tag{12}$$

and the corresponding SISO system Σ, defined, over the principal ideal domain $R = \mathbb{R}[\Delta]$, by the set of equations

$$\Sigma = \begin{cases} x(t+1) = A(\Delta)x(t) + b(\Delta)u(t) \\ y(t) = c(\Delta)x(t) \end{cases} \tag{13}$$

one can apply the Silverman Inversion Algorithm in the ring framework by evaluating recursively y(t+1), y(t+2),...... Since the general formula (where Δ has been omitted for simplicity) yields

$$y(t+k) = cA^k x(t) + \sum_{i==}^{k-1} cA^{k-i-1}\ b\ u(t+i),$$

either $cA^{k-1}\ b = 0$ for all $k \geq 1$, or there exists k_0, necessarily lesser than or equal to $dim\ \mathcal{X}$, such that for all $k < k_0\ cA^{k-1}\ b = 0$ and $cA^{k_0-1}\ b \neq 0$. In such case, one gets

$$\begin{array}{l} y(t+1) = cAx(t) \\ y(t+2) = cA^2x(t) \\ \vdots \\ y(t+k_0) = cA^{k_0}x(t) + cA^{k_0-1}bu(t) \end{array} \tag{14}$$

It is clear, then, that Σ is invertible if and only if $cA^{k_0-1} = c(\Delta)cA(\Delta)^{k_0-1}b(\Delta)$ is an invertible element of $R = \mathbb{R}[\Delta]$. If this is the case, an inverse in the ring framework is given by the system

$$\Sigma_{inv} = \begin{cases} z(t+1) &= (A - b(cA^{k_0-1}b)^{-1}cA^{k_0})z(t) + \\ & \quad b(cA^{k_0-1}b)^{-1}y(t+k_0) \\ u(t) &= -(cA^{k_0-1}b)^{-1}cA^{k_0}z(t) + \\ & \quad (cA^{k_0-1}b)^{-1}y(t+k_0) \end{cases} \tag{15}$$

From the expression of the inverse in the ring framework it is easy to derive that of an inverse in the original time-delay framework. The defining matrices of this will simply be obtained by substituting the algebraic indeterminate Δ with the delay operator δ. Now, following [Conte and Perdon, 2000b], we introduce the following Definition.

Definition 11 *Given the SISO system $\Sigma = (A, b, c, \mathcal{X})$ with coefficients in the ring R, assume that there exists k_0, such that $cA^{k-1}\ b = 0$ for all $k < k_0$ and $cA^{k_0-1}\ b \neq 0$. Then, we say that Σ has finite relative degree equal to k_0. If, in addition, $cA^{k_0-1}\ b$ is an invertible element of R, we say that the relative degree is pure. Alternatively, if $cA^{k-1}\ b = 0$ for all $k \geq 1$, we say that Σ has no finite relative degree.*

Invertibility of Σ is therefore characterized by the fact that the system has pure, finite relative degree. If we write the transfer function matrix $G(z)$ of Σ as

$$G(z) = c(zI - A)^{-1}b = d(z)^{-1}n(z)$$

where $d(z)$, $n(z)$ are polynomials in $R[z]$, that is they are polynomials in the indeterminate z with coefficients in R, and $d(z)$ is monic, it is not difficult to see that the leading coefficient of $n(z)$ is just $cA^{k-1}\ b$. If Σ has pure relative degree k_0, than the zero module $\mathcal{Z}$ of Σ is isomorphic, by Proposition 1, to the torsion $R[z]$-module $R[z]/n(z)R[z]$. In such case, the notion of $\mathcal{H}$-minimum phase system discussed at the end of Section IV can be easily characterized in terms of the numerator polynomial $n(z)$.
Essential inverses of $\mathcal{H}$-minimum phase SISO, time-delay systems are $\mathcal{H}$-stable and, on the other hand, non minimum phase systems do not have stable inverses.

Extension of the Silverman Algorithm and of the above discussion to the multi input-multi output case is possible due to the abstract, algebraic nature of our arguments (see [Conte and Perdon, 2000b] and [Conte, Perdon, Iachini, 2001]. The inverse system Σ_{inv} constructed by means of the Silverman Inversion Algorithm, when it exists, is not essential. In facts, its dimension is the same of that of the system Σ, which is equal to the degree of the denominator polynomial $d(z)$, while the dimension of $\mathcal{Z}$, being equal to the degree of the numerator polynomial $n(z)$, is smaller than that. Reducing Σ_{inv} to an essential inverse, in general, may be complicated, so, to construct an essential inverse, it is preferable to make use of the explicit decomposition given in Proposition 6.

Example 3 *Assuming that the transfer function of a given n-dimensional, SISO system Σ, with coefficients in the ring $\mathbb{R}[\Delta]$, is $G = d(z)^{-1}n(z)$ with $n(z) = c_0 + c_1 z + ... + c_{n-1}z^{n-1}$ and $c_{n-1} \neq 0$, the relative degree of Σ is 1 and $cA^{k_0-1}\ b = cb = c_{n-1}$. If c_{n-1} is invertible, it is not difficult to see, using a suitable realization of Σ, that the characteristic polynomial of the dynamical matrix of Σ_{inv} is $p(z) = (c_{n-1})^{-1}(c_0 + c_1 z + ... + c_{n-1}z^{n-1})z$.*

From the above results and discussion, it follows that the use of inversion as a synthesis procedure in the framework of time-delay systems can be dealt with, in connection with the issue of stability of inverses, by studying Zero Modules and Zero Dynamics.

C. Tracking problems for time delay systems

Given a SISO time-delay system Σ_d described by the set of equations (4) and the corresponding system Σ, defined, over the principal ideal domai $R = \mathbb{R}[\Delta]$, by the set of equations (1), let us consider the problem of designing a compensator which forces Σ_d to track a reference signal $r(t)$ (see [Conte, Perdon and Moog, 2007]). Working in the ring framework, we consider the extended system

$$\Sigma_E = \begin{cases} x(t+1) = Ax(t) + bu(t) \\ e(t) = cx(t) - r(t) \end{cases} \tag{16}$$

whose output is the tracking error and, assuming that Σ has pure relative degree k_0, we apply the Silverman Inversion Algorithm. This gives the following relation

$$e(t + k_0) = cA^{k_0}x(t) + cA^{k_0-1}bu(t) - r(t + k_0)$$

Then, choosing a real polynomial $p(z) = z^{k_0} + \sum_{i=o}^{k_0-1} a_i z^j$ in such a way that it is in the Hurwitz set $\mathcal{H}$, we can construct the compensator

$$\Sigma_C = \begin{cases} z(t+1) &= Az(t) + bu(t) \\ u(t) &= -(cA^{k_0-1}b)^{-1}(cA^{k_0}z(t) \\ &\quad -r(t+k_0)+ \\ &\quad -(cA^{k_0-1}b)^{-1}\sum_{i=0}^{k_0-1} a_i e(t+i) \end{cases} \tag{17}$$

whose action on Σ causes the error to evolve according to the following equation

$$e(t + k_0) = \sum_{i=o}^{k_0-1} a_i e(t+i) = cA^{k_0}(x(t) - z(t)) \tag{18}$$

$\mathcal{H}$-stability of the compensator is of course a key issue and, since its construction is based on inversion, it can be dealt with as for inverses. If Σ, and hence Σ_E, are $\mathcal{H}$-minimum phase (that is: their Zero Dynamics is $\mathcal{H}$–stable); an $\mathcal{H}$-stable reduced compensator can be obtained by means of an essential inverse.
Then, as in the case of inversion, it is easy to derive a compensator in the time-delay framework from Σ_C. Its defining matrices are obtained from those of Σ_C by substituting the algebraic indeterminate Δ with the delay operator δ. Thank to (18) and to the corresponding relation in the time-delay framework, the tracking error behaves in a desired way. More precisely, in case the initial conditions for Σ are known and the compensator can be initialized accordingly, then, the compensated system tracks asymptotically the reference signal.
In case the correct initialization is not possible, if Σ is globally asymptotically stable, the tracking error can be made arbitrarily small for t sufficiently large. Extension to the MIMO case is possible (see [Conte, Perdon and Moog, 2007]), although the situation complicates and results are relatively weaker.

REFERENCES

[Assan et al., 1999a] J. Assan, J.F. Lafay and A.M. Perdon. Computing the maximum pre-controllability submodules over a Noetherian ring. *Systems and Control Letters*, 37, 1999.

[Assan et al., 1999b] J. Assan, J.F. Lafay, A.M. Perdon and J.J. Loiseau. Effective computation of maximal controlled invariant submodules over a Principal Ring. *Proceedings 38th IEEE-CDC*, Phoenix, AZ, 1999, pp. 4216–4221.

[Basile and Marro, 1992] G. Basile and G. Marro. Controlled and Conditioned Invariants in Linear System Theory. *Prentice Hall*, Englewood Cliffs, New Jersey, 1992.

[Brewer et al., 1986] J. Brewer, J. Bunce and F. Van Vleck.Linear systems over commutative rings. *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, vol. 104, 1986.

[Conte and Perdon, 1982] G. Conte and A. M. Perdon. Systems over a principal ideal domain. A polynomial approach. *SIAM Journal on Control and Optimization*, vol. 20,1982.

[Conte and Perdon, 1984] G. Conte and A. M. Perdon. An algebraic notion of zeros for systems over rings. *Lectures Notes in Control and Information Science*, Springer Verlag, vol. 58, 1984.

[Conte and Perdon, 1995] G. Conte and A. M. Perdon. The disturbance decoupling problem for systems over a ring. *SIAM Journal on Control and Optimization*, 33, 1995.

[Conte and Perdon, 1998] G. Conte and A. M. Perdon. The Block Decoupling Problem for Systems over a Ring. *IEEE Trans. Automatic Control*, 43, 1998.

[Conte and Perdon, 2000a] G. Conte and A. M. Perdon. Systems over Rings: Geometric Theory and Applications. *Annual Review in Control*, 24, 2000.

[Conte and Perdon, 2000b] G. Conte and A. M. Perdon. Invertibility and Inversion for Systems over Rings and Applications to Delay-differential Systems. *Proc. 39th IEEE Conference on Decision and Control*, Sydney, Australia, 2000.

[Conte, Perdon, Iachini, 2001] G. Conte, A.M. Perdon andR. Iachini Inversion Problems for Time-delay Systems via Systems over Rings. Proc. IFAC Symposium on System Structure and Control, Prague, Czech Republic, [2001] [Conte and Perdon, 2002] G. Conte and A. M. Perdon. Functional controllability and right invertibility for systems over rings. *IMA Journal Mathematical Control and Information*, Vol. 19, 1, 2002.

[Conte and Perdon, 2005] G. Conte and A. M. Perdon. Modeling Time-delay Systems by means of Systems with Coefficients in a Ring. *Proc. Workshop on Modelling and Control of Complex Systems*, Ayia Napa, Cyprus, 2005.

[Conte and Perdon, 2007] G. Conte and A. M. Perdon. Zeros in linear time-delay systems. *Proc. Workshop on Advances in Statistical Control, System Theory and Engineering Education*, Notre Dame, IN, 2007.

[Conte and Perdon, 2008] G. Conte and A. M. Perdon. A Notion of Zero Dynamics for linear time-delay systems. *Proc. IFAC World Conference 2008*, Seoul, South Korea, 2007.

[Conte, Perdon and Moog, 2007] G. Conte, A.M. Perdon and C.H. Moog. Inversion and Tracking Problems for Time Delay Linear Systems. *Applications of Time-Delay Systems" , LNCIS 352*, J. Chiasson and J.J. Loiseau Eds., Springer.Verlag [2007] pp 267-284

[Datta and Hautus, 1984] K.B. Datta and M.L.J. Hautus. Decoupling of multivariable control systems over unique factorization domains. *SIAM J. on Control and Optimization*, 22, 1, 1984.

[Habets, 1994] L. Habets. Algebraic and computational aspects of time-delay systems. *PhD Dissertation*, Technical University of Eindhoven, The Netherlands, 1994.

[Hautus, 1982] M.L.J. Hautus. Controlled Invariance in Systems over Rings. *Springer Lecture Notes in Control and Information Sciences*, Springer verlag, Berlin, vol. 39, 1982.

[Kamen, 1991] E. W. Kamen. Linear system over rings: from R. E. Kalman to the present. *Mathematical System Theory - The Influence of R. E. Kalman*, A. C. Antoulas Ed., Springer Verlag, Berlin, 1991.

[Lam, 1978] T. Y. Lam. Serre Conjecture. *Lecture Notes in Matematics*, Springer Verlag, NY, vol. 635, 1978.

[Lang, 1984] S. Lang. Algebra.*Advanced Book Program*, Addison-Wesley, Menlo Park, California, 1984.

[Perdon and Anderlucci, 2006] A M. Perdon and M. Anderlucci. Geometric Design of Observers for Linear Time-Delay Systems. *Proc. 6th IFAC Workshop on Time-Delay Systems*, l'Aquila, Italia (2006)

[Schrader and Sain, 1989] C. B. Schrader and M. K. Sain. Research on System Zeros: A Survey. *International Journal of Control*, 50, 1989.

[Sontag, 1976] E.D. Sontag. Linear systems over commutative rings: a survey. *Ricerche di Automatica*, 7, 1976

[Sontag,1981] E.D. Sontag. Linear systems over commutative rings: a [partial] updated survey. *Control science and technology for the progress of society*, Kyoto, Japan, 1981.

[Wohnam, 1985] M. Wohnam. Linear Multivariable Control: a Geometric Approach, 3rd Ed., Springer Verlag, New York, 1985.

[Wyman and Sain, 1981] B. Wyman and M. K. Sain. The Zero Module and Essential Inverse Systems. *IEEE Trans. Circuits and Systems* CAS-28, 1981, pp. 112-126