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***Lab*MACS**
Laboratory of Modeling, Analysis and Control of Dynamical System

Zeros and Zero Dynamics for Linear, Time-delay System

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Introduction

Time delay systems are interesting in connection with:

- industrial applications (where delays are unavoidable effects of the transportation of materials)

- tele-operated systems, networked systems, large Integrated Communication Control Systems or ICCS (where delays originates from dispatching information through slow or very long communication lines).

In the last years, a great research effort has been devoted to the development of analysis and synthesis techniques for time delay systems (Proc. IFAC Workshop on LTDS 2000, 2001, 2003, 2005).

Introduction

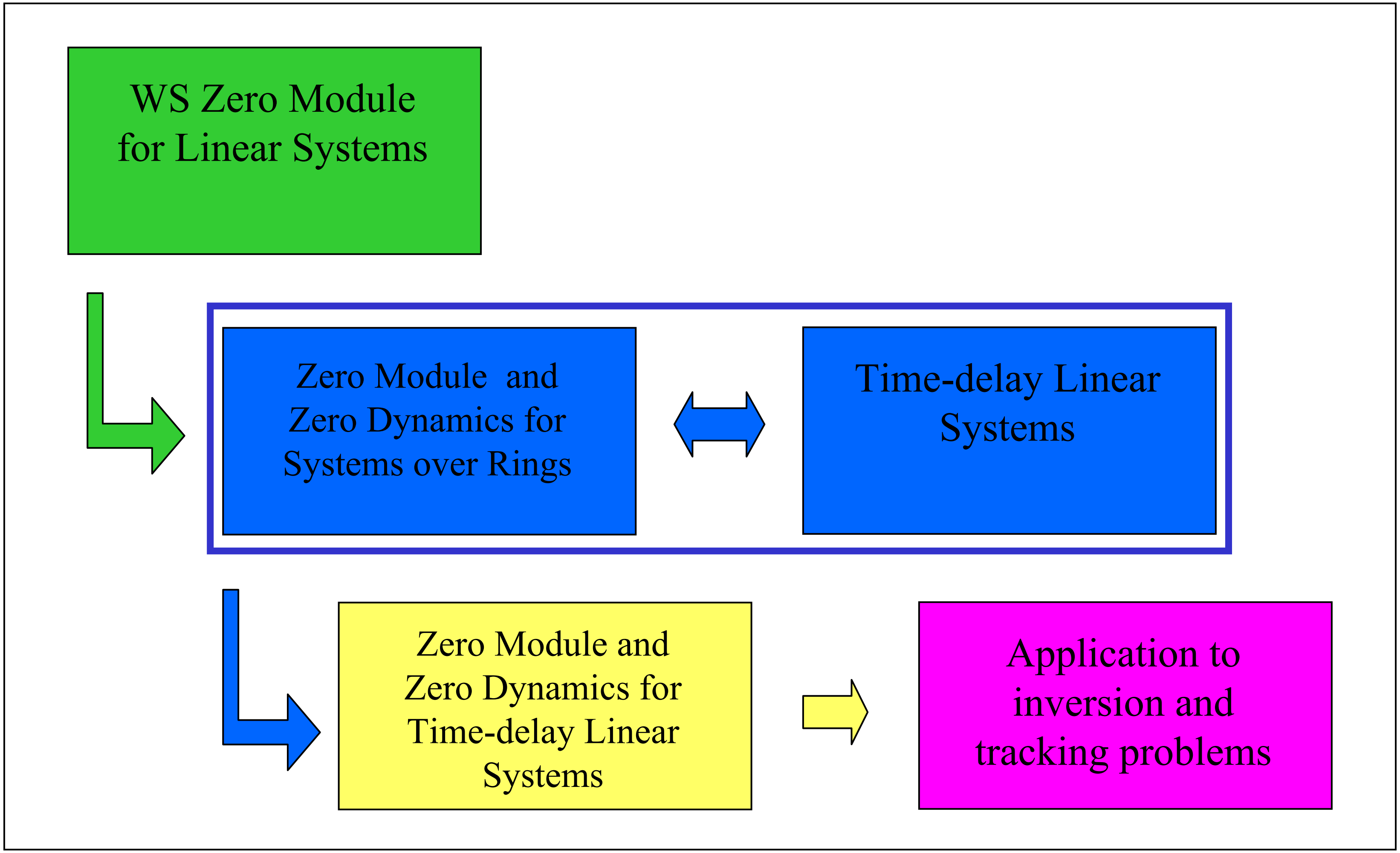
The notion of ZERO and of ZERO DYNAMICS play an important role in several control problems, especially when solutions require some sort of inversion.

For classical linear systems, ZERO and ZERO DYNAMICS can be characterized in abstract algebraic terms by the notion of ZERO MODULE (Wyman,Sain-1981).

The notion of ZERO MODULE can be generalized to other classes of dynamical systems, notably to that of systems with coefficients in a ring.

By exploiting the relations between systems with coefficients in a ring and time-delay systems, suitable notions of ZERO and ZERO DYNAMICS can be defined these latter.

Introduction



Time delay systems

Time delay system with uncommensurable delays:

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=1}^{k} \sum_{j=0}^{a} A_{ij} x(t - jh_i) + \sum_{i=1}^{k} \sum_{j=0}^{b} B_{ij} u(t - jh_i) + \\ y(t) = \sum_{i=1}^{k} \sum_{j=0}^{c} C_{ij} x(t - jh_i) \end{cases}$$

$t \in \Re$, continuous time axis

$x \in X = \Re^n$, state value space

{states=functions $x: [T-ah_i, T) \to \Re^n$} (∞-dim. $\Re$-vector space)

$u \in U = \Re^m$, input value space (m-dim. $\Re$-vector space)

$y \in Y = \Re^p$, output value space (p-dim. $\Re$-vector space)

A_{ij}, B_{ij}, C_{ij} real matrices of suitable dimensions

h_i for $i=1,\ldots,k$ are fixed time delays

Systems with coefficients in a ring

Given a ring R, a system Σ with coefficients in R is a quadruple (A,B,C,X) where A, B, C are matrices of dimensions $n\times n$, $n\times m$, $p\times n$ with entries in R and $X= R^n$.

Dynamical interpretation:

$$\boxed{\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}}$$

$t \in Z$, ordered set of integer numbers (discrete time)

$x \in X = R^n$, state module (n-dim. free R-module)

$u \in U = R^m$, input module (m-dim. free R-module)

$y \in Y = R^p$, output module (p-dim. free R-module)

Time delay $\Leftrightarrow$ Coefficients in a ring

Time delay system

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i x(t-ih) + \sum_{i=0}^{b} B_i u(t-ih) \\ y(t) = \sum_{i=0}^{c} C_i x(t-ih) \end{cases}$$

Delay operator δ, defined, for any time function f(t), by $\delta f(t) = f(t-h)$

$$A = \sum_{i=0}^{a} A_i \delta^i$$

$$B = \sum_{i=0}^{b} B_i \delta^i$$

$$C = \sum_{i=0}^{c} C_i \delta^i$$

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i \delta^i x(t) + \sum_{i=0}^{b} B_i \delta^i u(t) \\ y(t) = \sum_{i=0}^{c} C_i \delta^i x(t) \end{cases}$$

Substitute δ with the indeterminate Δ

System w. c. i. the ring $R = \Re[\Delta]$

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Time delay $\Leftrightarrow$ Coefficients in a ring

Time delay system

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i x(t-ih) + \sum_{i=0}^{b} B_i u(t-ih) \\ y(t) = \sum_{i=0}^{c} C_i x(t-ih) \end{cases}$$

t, x, u, and y have different meanings in the two frameworks: e.g. $x(t) \in \Re^n$ in the time delay fremework, $x(t) \in R^n$ in the ring framework; notations are kept equal by abuse.

$\Updownarrow$

System w. c. in the ring $R = \Re[\Delta]$

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Zero module for systems w.c.in a ring

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

R[z] ring of polynomials in the indeterminate z with coefficients in R

$R(z) = S^{-1}R[z]$ localization at the multiplicative set S of all monic polynomials

Transfer Function Matrix $G_\Sigma = C(zI-A)^{-1}B$ (entries in R(z))

R(z)-morphism $C(zI-A)^{-1}B : U\otimes R(z) \to Y\otimes R(z)$

$u(z) \in U\otimes R(z)$, $u(z) = \sum_{t_0}^{\infty} u_t z^{-t}$, $u_t \in U$ (input sequence)

$y(z) \in Y\otimes R(z)$, $y(z) = \sum_{t_0}^{\infty} y_t z^{-t}$, $y_t \in Y$ (output sequence)

Zero module for systems w.c.in a ring

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

$$G_\Sigma = C(zI-A)^{-1}B : U\otimes R(z) \to Y\otimes R(z)$$

R[z]-modules

$\Omega U = U\otimes R[z] \subseteq U\otimes R(z)$

$\Omega Y = Y\otimes R[z] \subseteq Y\otimes R(z)$

Definition (CP-1983, after WS-1981) Given the system $\Sigma = (A,B,C,X)$ with coefficients in the ring R and transfer function matrix G_Σ, the Zero Module of Σ is the R[z]-module Z_Σ defined by

$$Z_\Sigma = (G_\Sigma^{-1}(\Omega Y) + \Omega U)/(\mathrm{Ker}G + \Omega U)$$

Zero module for systems w.c.i. a ring

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

$G_\Sigma = C(zI-A)^{-1}B : U \otimes R(z) \rightarrow Y \otimes R(z)$

$Z_\Sigma = (G^{-1}(\Omega Y) + \Omega U)/(\mathrm{Ker}G + \Omega U)$

Proposition (CP-1983) Let $G_\Sigma = D(z)^{-1}N(Z)$ be a coprime factorization. The canonical projection p_N: $\Omega Y \rightarrow \Omega Y/N\Omega U$ induces an injective R[z]-homomorphism α: $Z_\Sigma \rightarrow \mathrm{Tor}(\Omega Y/N\Omega U)$.

As a consequence of the above Proposition we have the following foundamental result

Z_Σ is a finitely generated, torsion R[z]-module

Zero module for time-delay systems

Time delay system

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i x(t-ih) + \sum_{i=0}^{b} B_i u(t-ih) \\ y(t) = \sum_{i=0}^{c} C_i x(t-ih) \end{cases}$$

Delay operator δ, defined, for any time function f(t), by $\delta f(t) = f(t-h)$

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Substitute δ with the indeterminate Δ

System w. c. in the ring $R = \Re[\Delta]$

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Zero module for time-delay systems

Time delay system

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$$Z_\Sigma = (G^{-1}(\Omega Y) + \Omega U)/(\mathrm{Ker}G + \Omega U) = Z_{\Sigma_d}$$

ZERO MODULE

System w. c. in the ring $R = \Re[\Delta]$

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Zero module for time-delay systems

Time delay system

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i x(t-ih) + \sum_{i=0}^{b} B_i u(t-ih) \\ y(t) = \sum_{i=0}^{c} C_i x(t-ih) \end{cases}$$

Definition Given a time-delay system Σ_d, let Σ be the associated system with coefficients the ring R. The Zero Module Z_{Σ_d} of Σ_d is the Zero Module Z_Σ of Σ.

System w. c. in the ring $R = \Re[\Delta]$

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Zero dynamics

If the zero module Z_Σ of Σ is free, over the ring R, then it can be represented as $Z_\Sigma = (R^m, D)$, where $D: R^m \rightarrow R^m$ is an R-homomorphism. Then, we can consider the following notion.

Definition Given a system Σ with coefficients in the ring R, whose zero module Z_Σ can be represented as the pair (R^m, D), the Zero Dynamics of Σ is the dynamics induced on R^m by D, that is by the dynamic equation $z(t+1) = Dz(t)$, for $z \in R^m$.

Remark that in case Z_Σ cannot be represented as a pair (R^m, D), the Zero Dynamics is not defined.

Zero module and geometric structure

$\Sigma = (A,B,C,X)$ with coefficients in R:

a controlled invariant submodule (c.i.s.) of X is a submodule $V \subseteq X$ such that $A(V) \subseteq V + \mathrm{Im}B$

feedback property: there exists an R-morphism $F: X \rightarrow U$ such that $(A+BF)V \subseteq V$ (F is called a *friend* of V).

V^* = maximum c.i.s. contained in KerC

R^* = minimum c.i.s. containing $\mathrm{Im}B \cap V^*$

Proposition Given $\Sigma = (A,B,C,X)$, w.c.in R and $C(zI-A)^{-1}B = D^{-1}N$ coprime, let $N(U \otimes R(z))$ be a direct summand of $Y \otimes R(z)$ and let V^* have the feedback property with a friend F. Then, V^*/R^* endowed with the R[z] structure induced by (A+BF) is isomorphic to Z_Σ.

Zero module and geometric structure

If V^* has the feedback property with a friend F and V^*/R^* is a free R-module, say $V^*/R^* = R^m$, letting D be a matrix that represents $(A + BF)_{V^*/R^*}$ with respect to the canonical basis of R^m, it is be possible to represent the Zero Dynamics of Σ as the dynamics induced on R^m by D: $z(t+1) = Dz(t)$, for $z \in R^m$.

☞ The above characterization of Zero Dynamics allows us to analyse it in a simple, practical way, avoiding the necessity of working with $R[z]$-modules and of involved computations. Unfortunately, it holds only if V^* has the feedback property (strong requirement).

Zero module and geometric structure

Proposition Given $\Sigma = (A,B,C,X)$, w.c.in R, it is possible to construct, in a canonic way, a dynamical extension Σ_e of Σ such that V_e^* has the feedback property with a friend F_e. Then, if V_e^*/ R_e^* is free, it is isomorphic to the largest free submodule of V^*/ R^*.

Proposition In the above context and with the above notations, assuming that the Zero Dynamics of Σ is defined, let V_e^*/ R_e^* be a free R-module of dimension m and let D be a matrix representing the R-morphism $(A_e + B_eF_e)_{Ve^*/ Re^*}$ with respect to the canonical basis of R^m. Then, the ZeroDynamics of Σ is the dynamics induced on R^m by D

$$z(t+1) = Dz(t), \qquad \text{for } z \in R^m .$$

Zero dynamics for time-delay systems

Definition Given a time-delay system Σ_d, let Σ be the associated system with coefficients the ring R. The Zero Dynamics of Σ_d is that of Σ, if the latter is defined.

Proposition Given the time-delay system Σ_d, with commensurable delays, let Σ be the associated system with coefficients the ring R. Then, if Σ is left invertible and V* is free, the Zero Dynamics of Σ is defined and so is that of Σ_d.

Zeros and phase minimality

Hurwitz set: a set H of monic polynomials in R[z]

- H contains at least one linear monomial $z + a$ with $a \in R$;
- H is multiplicatively closed;
- any factor of an element in H belongs to H.

Definition A system $\Sigma = (A,B,C,X)$ w.c.i. R is said to be

- H-stable if $\det(zI-A)$ belongs to H
- H-minimum phase if its zero dynamics is defined and H-stable.

For systems associated to time-delay ones, $R = \Re[\Delta]$

$$H = \{p(z,\Delta) \in \Re[z, \Delta], \text{ such that } p(s,e^{-hs}) \neq 0 \text{ for all } s \in C \text{ with } Re(s) \geq 0\}$$

H-stability (phase min.) in the ring framework ⟺ Asymptotic stability (phase min.) in the time-delay framework

Inversion problems

Proposition Given a left (respectively, right) invertible system $\Sigma = (A,B,C,X)$ with coefficients in the ring R and transfer function G, let G_{inv} denote a left (respectively, right) inverse of G and let $\Sigma_{inv} = (A_{inv}, B_{inv}, C_{inv}, X_{inv})$ be its canonical realization. Then, the relation $G_{inv}\, G$ = Identity induces an injective R[z]-homomorphism $\psi : Z_\Sigma \to X_{inv}$ (respectively, the relation $G\, G_{inv}$ = Identity induces a surjective R[z]-homomorphism $\varphi: X_{inv} \to Z_\Sigma$) between the Zero Module of Σ and the state module X_{inv} of the canonicalrealization of G_{inv}.

In case the Zero Dynamics of Σ is defined, but not minimum phase, the above Proposition allows us to say that Σ has no H-stable inverses.

Tracking problems

Problem Given a SISO time-delay system Σ_d and the corresponding system Σ w.c.in $R = \Re[\Delta]$, consider the problem of designing a compensator which forces Σ_d to track a reference signal r(t).

Consider the extended system

$$\Sigma_E = \begin{cases} \dot{x}(t+1) = Ax(t) + bu(t) \\ e(t) = cx(t) - r(t) \end{cases}$$

whose output is the tracking error and apply the Silverman Inversion Algorithm.

Tracking problems

This gives the relation

$$e(t+k_0) = cA^{k_0}x(t) + cA^{k_0-1}bu(t) - r(t+k_0)$$

Then, choosing a real polynomial $p(z) = z^{k_0} + \sum_{i=0}^{k_0-1} a_i z^i$ in such a way that it is in the Hurwitz set H, we can construct the compensator

$$\Sigma_C = \begin{cases} z(t+1) = Az(t) + bu(t) \\ u(t) = -(cA^{k_0-1}b)^{-1}(cA^{k_0}z(t) - r(t+k_0)) + \\ \qquad -(cA^{k_0-1}b)^{-1}\sum_{i=0}^{k_0-1} a_i e(t+i) \end{cases}$$

whose action on Σ causes the error to evolve according to the equation

$$e(t+k_0) + \sum_{i=0}^{k_0-1} a_i e(t+i) = cA^{k_0}(x(t) - z(t))$$

Tracking problems

The compensator

$$\Sigma_C = \begin{cases} z(t+1) = Az(t) + bu(t) \\ \quad u(t) = -(cA^{k_0-1}b)^{-1}(cA^{k_0}z(t) - r(t+k_0)) + \\ \qquad -(cA^{k_0-1}b)^{-1}\sum_{i=0}^{k_0-1} a_i e(t+i) \end{cases}$$

solves the tracking problem.

H-stability of the compensator is a key issue and, since its construction is based on inversion, it can be dealt with by using phase minimality.

If Σ_d and Σ, and hence Σ_E, are H-minimum phase (that is: their zero dynamics is H-stabe), an H-stabe compensator is obtained.

Example

Example.

Consider the time-delay system

$$\Sigma_d = \begin{cases} \dot{x}_1(t) &= x_3(t-h) + u(t-h) \\ \dot{x}_2(t) &= x_1(t) + x_3(t-h) + u(t-h) \\ \dot{x}_3(t) &= x_2(t) + x_3(t-h) \\ y(t) &= x_3(t) \end{cases}$$

and the associated system $\Sigma = (A,B,C,X)$ with coefficients in $R = R[\Delta]$ and matrices

$$A = \begin{bmatrix} 0 & 0 & \Delta \\ 1 & 0 & \Delta \\ 0 & 1 & \Delta \end{bmatrix}; B = \begin{bmatrix} \Delta \\ \Delta \\ 0 \end{bmatrix}; C = [0 \ 0 \ 1]$$

$V^* = \text{span}\,(\Delta\ 0\ 0)^T \subseteq R^3$, V^* is not of feedback type (because it is not closed), but it is free. $R^* = \{0\}$.

The Zero Dynamics is defined and it can be represented as a suitable pair (R^m, Z) in order to check, for instance, phase minimality.

Example

Example (continued).

To analyze the Zero Dynamics, we consider the extension $\Sigma_e =$ (A_e, B_e, C_e, X_e), wi

$$A_e = \begin{bmatrix} 0 & 0 & \Delta & 0 \\ 1 & 0 & \Delta & 0 \\ 0 & 1 & \Delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; B_e = \begin{bmatrix} \Delta & 0 \\ \Delta & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; C_e = [0\ 0\ 1\ 0]$$

V_e *= span $(\Delta\ 0\ 0\ 1)^T \subseteq R^4$ is of feedback type , $F_e = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

The dynamic matrix $A_c = (A_e + B_e F)$ of the compensated system is

$$A_c = \begin{bmatrix} 0 & 0 & \Delta & -\Delta \\ 1 & 0 & \Delta & -\Delta \\ 0 & 1 & \Delta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Example

Example (continued).

The Zero Dynamics turns out to be given by ($\mathbb{R}$, $[-1]$) or, in other terms, by the dynamic equation

$$\xi(t+1) = -\xi(t)$$

$\xi(t) = -\xi(t)$ in the time-delay framework.

We can conclude that the system is minimum phase.

Conclusion

The notions of Zero Module and of Zero Dynamics have been introduced in the time-delay framework, by exploiting the correspondence between system with coefficients in a ring and time-delay systems and the algebraic characterization of Zeros.

Stability of the Zero Dynamics and Phase Minimality can then be defined and used for the construction of stable inverses and of stable solutions to tracking problems in the time delay framework.