Sensor classification for control and diagnosis problems, a structural approach

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Abstract— **In this paper we consider the sensor network design problem. Given a system with its sensor network and a property which is satisfied by this system with the existing sensors, we will classify the sensors with respect to their importance relatively to the preservation of this property in case of failure. More precisely we will characterize the sensors which are critical, i.e. which failure leads to property loss and those which are useless for the property. We will also quantify the relative importance of the sensors which are neither useless nor critical. The properties which will be studied here are observability and Fault Detection and Isolation (FDI), we will then provide the sensor classification for these properties in case of sensor failure. The proposed graph approach is visual, easy to handle and close to the physical structure of the system.**

Keywords- Linear structured systems, Fault Detection and Isolation, Sensor failure, Sensor classification

I. Introduction

Control and diagnosis of dynamical systems require measures or estimation of system variables via sensors. In this paper we consider the sensor network design problem. This problem amounts to find sets of variables to be measured by sensors for observation or control purposes as observability, disturbance rejection or diagnosis for example. In this paper we address the sensor classification problem in the following sense. Given a system with its sensor network and a property P which is satisfied by this system with the existing sensors, we will classify the sensors with respect to their importance relatively to the preservation of the property P in case of failure. More precisely we will characterize the sensors which are critical, i.e. which failure leads to property loss and those which are useless for the property. We will also quantify the relative importance of the sensors which are neither useless nor critical. We consider here linear structured system models [1], [2], they represent a large class of parameter dependent linear systems. This allows us to use graph theory in order to easily exploit the structure of the process irrespective of the parameter values. The properties which will be studied here are observability and Fault Detection and Isolation (FDI), we will then provide the sensor classification for these properties in case of sensor failure. The Fault Detection and Isolation (FDI) problem has received considerable attention in the past ten years [3], [4]. It consists of building residuals from the available data and isolating, whenever possible, the faults using the residuals. In [5], the authors provide a qualitative classification of sensors with respect to their importance concerning observability. They determine the critical sensors, called essential, which failure leads to observability loss, as well as the useless ones, which may fail without impacting system observability. The complexity of the classification algorithms is polynomial with respect to the dimension of the system. The contribution of this approach is to provide with a unified framework allowing, with only a structural knowledge on the system, to determine which sensors are compulsory to use or useless to preserve a given property. Furthermore we propose a quantification of the respective importance of the useful sensors. For this purpose we define for each sensor a criticity degree related with its importance for the FDI solvability problem in case of sensor failure. It turns out that this criticity degree is zero for useless sensors, one for essential ones and from zero to one for sensors which are neither useless nor essential. The larger the criticity degree is, the worst for FDI solvability in case of failure of this sensor. The proposed graph approach is visual, easy to handle and close to the physical structure of the system. The underlying ideas are general and can be applied to other classes of models and properties.

The paper is structured as follows. The problem is formulated in Section 2. Structured systems are presented in Section 3. In Section 4, we provide with a characterization of useless and essential sensors for the observer-based FDI problem. A quantitative classification of sensors is given in Section 5. Concluding remarks end the paper.

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II. Problem formulation

A. Observer-based FDI problem

Let us consider the following linear time-invariant system:

$$\Sigma \begin{cases} \dot{x}(t) = Ax(t) + Lf(t) + Bu(t) \\ y(t) = Cx(t) + Mf(t) \end{cases} . \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f(t) \in \mathbb{R}^r$ the fault vector, $u(t) \in \mathbb{R}^m$ is the control input vector and $y(t) \in \mathbb{R}^p$ the measured output vector. Each output variable is provided by a specific sensor. For the sake of simplicity, in the following we will denote by y_i either the ith sensor or the ith output variable. A, C, B, L and M are matrices of appropriate dimensions.

A dedicated residual set is designed using a bank of r observers for system (1), according to the dedicated observer scheme [3].

The ith observer of this bank of r observers is designed for a system of type (1) as follows:

$$\dot{\hat{x}}^i(t) = A\hat{x}^i(t) + K^i(y(t) - C\hat{x}^i(t)) + Bu(t), \tag{2}$$

where $\hat{x}^i(t) \in \mathbb{R}^n$ is the state of the ith observer, K^i is the observer gain to be designed such that $\hat{x}^i(t)$ asymptotically converges to $x(t)$, when $f(t) = 0$.
The residuals are defined as :

$$r_i(t) = Q^i(y(t) - C\hat{x}^i(t)), \text{ for } i = 1, \ldots, r, \quad (3)$$

where Q^i is a $1 \times p$ matrix.

Definition 1: Let Σ be a linear system as in Equation (1) with the set of sensors $Y = \{y_1, \ldots, y_p\}$. The bank of observer-based FDI problem associated with Y consists has a solution if the system is observable and there exist matrices K^i and Q^i, such that, for $i = 1, 2, \ldots, r$, the fault to residual transfer matrix is non zero, proper and diagonal. It is clear that the solvability of this problem relies on the available sensors, a sufficient amount of information is needed for observability and to detect and isolate the faults via the dedicated observer scheme.

B. Sensor classification for FDI

In this paper, we tackle the problem of property preservation under sensor failure. We define a *failing sensor* as a sensor which is down *i.e.* whose measure is no more available. This output will then no more appear in the model. We will study the preservation of properties (e.g. observability) of dynamic systems under sensor failure. We point out different classes of sensors mainly the essential ones which are compulsory to preserve the property and the useless ones which do not play any role for the problem under consideration. A property $\mathcal{P}$ is a function mapping from $Y = \{y_1, y_2, \ldots, y_p\}$ into $\{0, 1\}$ and the property $\mathcal{P}$ is true when $\mathcal{P}(Y) = 1$.
For a given property $\mathcal{P}$ we define now different categories of sensors.

Definition 2: Let Σ be the linear system defined by 1 with a property $\mathcal{P}$ such that $\mathcal{P}(Y) = 1$. We call *admissible sensor set for the property* $\mathcal{P}$ a set of sensors $V \subseteq Y$ such that $\mathcal{P}(V) = 1$.

1) A sensor y^* is called a *useless sensor* if for any admissible sensor set V containing y^*, $V \backslash \{y^*\}$ is still an admissible sensor set for $\mathcal{P}$ where $V \backslash \{y^*\}$ is the set V minus the sensor y^*. A sensor which is not useless is called a *useful sensor*.
2) A sensor y^* is called an *essential sensor* if y^* belongs to any admissible sensor set V. The set of essential sensors is a subset of the set of useful sensors.

In the sequel, following [5] we will apply these notions to classify the sensors for the FDI problem in the context of structured systems. We recall the determination of the sets of useless and essential sensors. Then we provide with a classification of the sensors with respect to their relative importance for this property. This is the main contribution. Besides this qualitative classification we will quantify the criticity degree of each sensor in case of sensor failure.

III. LINEAR STRUCTURED SYSTEMS

A. Definitions and basic properties

We will consider models based on the available physical knowledge on the system. These models capture the relations between internal variables but without fixing the precise value of the parameters. In this paper we will consider linear structured systems as in [1]. We consider linear systems as described in (1), but with parameterized entries and denoted by Σ_Λ

$$\Sigma_\Lambda \left\{ \begin{array}{c} \dot{x}(t) = Ax(t) + Lf(t) + Bu(t) \\ y(t) = Cx(t) + Mf(t) \end{array} \right. . \quad (4)$$

This system is called a linear structured system if the entries of the composite matrix $J = \begin{bmatrix} A & L & B \\ C & M & 0 \end{bmatrix}$ are either fixed zeros or independent parameters (not related by algebraic equations). $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ denotes the set of independent parameters of the composite matrix J. More details can be found in [2].
For such systems one can study generic properties i.e. properties which are true for almost all values of the parameters collected in Λ [6].
A directed graph $G(\Sigma_\Lambda) = (W, Z)$ can be easily associated with the structured system Σ_Λ of type (4) where the matrix $\begin{bmatrix} A & L & B \\ C & M & 0 \end{bmatrix}$ is structured:
• the vertex set is $W = U \cup F \cup X \cup Y$ where U, F, X and Y are the control input, fault, state and output sets given by $\{u_1, u_2, \ldots, u_m\}$, $\{f_1, f_2, \ldots, f_r\}$, $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_p\}$ respectively,
• the arc set is $Z = \{(u_i, x_j) | B_{ji} \neq 0\} \cup \{(f_i, x_j) | L_{ji} \neq 0\} \cup \{(x_i, x_j) | A_{ji} \neq 0\} \cup \{(x_i, y_j) | C_{ji} \neq 0\} \cup \{(f_i, y_j) | M_{ji} \neq 0\}$, where A_{ji} (resp. B_{ji} C_{ji},L_{ji},M_{ji}) denotes the entry (j, i) of the matrix A (resp. B, C,L,M).
Recall that a directed path P in $G(\Sigma_\Lambda)$ from a vertex i_0 to a vertex i_q is a sequence of arcs $(i_0, i_1), (i_1, i_2), \ldots, (i_{q-1}, i_q)$ such that $i_t \in V$ for $t = 0, 1, \ldots, q$ and $(i_{t-1}, i_t) \in W$ for $t = 1, 2, \ldots, q$. If $i_0 \in F$ and, $i_q \in Y$, P is called a fault-output path. If $i_0 \in V_1$ and $i_q \in V_2$, where V_1 and V_2 are two subsets of V, P is called a V_1-V_2 path. Moreover, if i_0 is the only vertex of P which belongs to V_1 and i_l is the only vertex of P which belongs to V_2, P is called a *direct* V_1-V_2 path.
A set of paths with no common vertex is said to be vertex disjoint. A V_1-V_2 linking of size k is a set of k vertex disjoint V_1-V_2 paths. A linking is maximal when k is maximal.

B. Observability of linear structured systems

The structural controllability or the dual notion of observability has been studied in several papers following the pioneering work of [1].
Introduce now the concept of contraction which is the dual notion of the dilation defined by Lin for the study of controllability.

Definition 3: Let Σ_Λ be the linear structured system defined by (4) with associated graph $G(\Sigma_\Lambda)$, vertex set Z and arc set W. Consider a set $S \subseteq X$. Denote $E(S)$ the set of vertices:

$$E(S) = \{z_j \in Z \mid \exists \quad x_i \in S \quad \text{such that} \quad (x_i, z_j) \in W\} \tag{5}$$

S is said to be a contraction if $card(S) > card(E(S))$ where $card(.)$ denotes the cardinality of a set.

Recall the graph characterization of the structural observability, which will be useful later [1], [6].

Theorem 1: Let Σ_Λ be the linear structured system defined by (4) with associated graph $G(\Sigma_\Lambda)$. The system is structurally observable if and only if:

1) The system Σ_Λ is output-connected,
2) $G(\Sigma_\Lambda)$ contains no contraction.

C. Fault Detection and Isolation

Give now the result concerning the diagonal FDI problem by using a bank of observers, which was stated first in [7].

Theorem 2: Consider the structured system with r faults Σ_Λ as defined in (4) and its associated graph $G(\Sigma_\Lambda)$. The bank of observer-based diagonal FDI problem of Definition 1, is generically solvable if and only if:

- Σ_Λ is structurally observable
- $k = r$ where k is the size of a maximal fault-output linking in $G(\Sigma_\Lambda)$.

The second condition of Theorem will be referred to as the rank condition for FDI in the sequel.

IV. Sensor classification for Fault Detection and Isolation

This section gives a summary of the main results of [5] and [8]. We classify the sensors for three properties, the output connection, the contraction avoidance and the rank condition $k = r$. Then we combine these three classifications to obtain the sensor classification for the FDI problem.

A. Sensor classification for output connection

In this sub-section we classify the sensors with respect to the output connection property. We start with an observable structured system and we wonder if output connection is preserved under sensor failure. For this purpose we define irreducible separators as minimal subsets of output vertices which loss output disconnect some state vertices, see [5]. The sensors corresponding to irreducible separators of dimension 1 are shown to be essential for output connection. The sensors belonging to no irreducible separator are shown to be useless for output connection. We will illustrate this classification on the following example.

Example 1: Consider the linear structured system with 3 control inputs, 6 faults and 10 sensors illustrated in Figure 1

From Figure 1, $G(\Sigma_\Lambda)$ is output connected. One can observe that removing y_1 output disconnects x_9 and removing $\{y_6, y_7, y_8\}$ output disconnects x_{14}, these two sets are indeed irreducible separators. It can be shown that we have 8 irreducible separators included in Y. They are $S_1 = \{y_1\}$; $S_2 = \{y_2\}$; $S_3 = \{y_3, y_5\}$; $S_4 = \{y_3, y_4\}$;

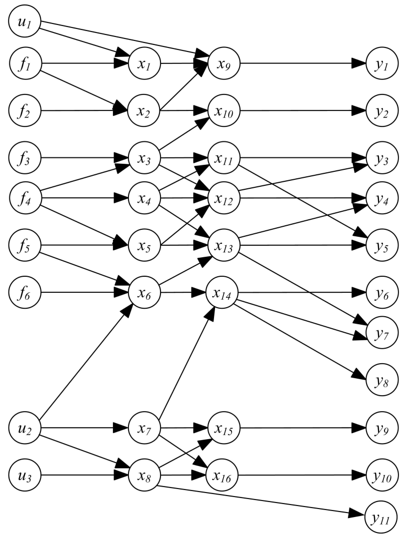

Fig. 1. Graph $G(\Sigma_\Lambda)$ of Example 1

$S_5 = \{y_4, y_5, y_7\}$; $S_6 = \{y_6, y_7, y_8\}$; $S_7 = \{y_9\}$ and $S_8 = \{y_{10}\}$. It turns out that, for the output connection, y_{11} is a useless sensor because it does not belong to any irreducible separator. All the others sensors are useful. The following sensors are essential for the output connection y_1, y_2, y_9, y_{10} because they belong to the irreducible separators of cardinality one, S_1, S_2, S_7, S_8 respectively.

B. Sensor classification for contraction avoidance

In this sub-section we classify the sensors with respect to the contraction avoidance property. This will be performed using a bipartite graph associated with the system Σ_Λ. The bipartite graph is well suited for generic matrix rank computations. The bipartite graph of this system is $B(\Sigma_\Lambda) = (B^+, B^-; W')$ where the sets B^+ and B^- are two disjoint vertex sets and W' is the arc set.

A *matching* in a bipartite graph $B(\Sigma_\Lambda) = (B^+, B^-; W')$ is an arc set $M \subseteq W'$ such that the arcs in M have no common vertex. The cardinality of a matching, *i.e.* the number of arcs it consists of, is also called its size. A matching M is called maximum if its cardinality is maximum. In general, a maximum matching is not unique. The maximum matching problem is the problem of finding such a matching of maximal cardinality. This problem can be solved using very efficient algorithms based on alternate augmenting chains or ideas of maximum flow theory [9]. It is know that there is no contraction on Σ_Λ if and only if the maximum matching on $B(\Sigma_\Lambda)$ is of cardinality n where n is number of state vertices.

Using the classical Dulmage-Mendelsohn decomposition [6] of $B(\Sigma_\Lambda)$, we can then characterize the contractions of the graph $G(\Sigma_\Lambda)$. The essential sensors for contraction avoidance correspond to the output vertices which failure induces a decrease in the dimension of the maximal matching. The useless sensors for contraction avoidance correspond to the output vertices which are of no use for building a maximal matching. The DM-Decomposition $B(\Sigma_\Lambda)$ associated with the system Σ_Λ of Example 1 is given in Fig. 2.

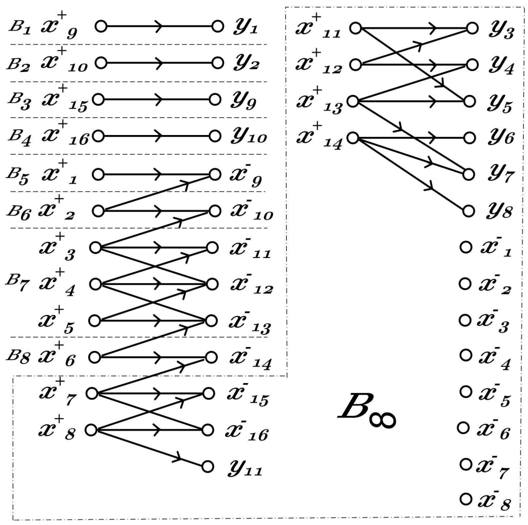

Fig. 2. Graph $B(\Sigma_\Lambda)$ and the DM-decomposition of Example 1

we have 4 essential sensors $\{y_1, y_2, y_9, y_{10}\}$ for the contraction avoidance because removing one of them induces a decrease in the dimension of the maximal matching. Any maximum matching with minimal number of outputs covers 4 output vertices of $\{y_3, y_4, y_5, y_6, y_7, y_8\}$ but none of them covers y_{11}. Then, y_{11} is a useless sensor and $\{y_1, \dots y_{10}\}$ are useful sensors.

C. Sensor classification for the rank condition of FDI

The useless sensors of Definition 2 for the FDI can then be characterized as follows.

Theorem 3: Consider the linear structured system Σ_Λ with sensor set Y and its associated graph $G(\Sigma_\Lambda)$. A sensor $y_i \in Y$ is useless if and only if there is no F-$\{y_i\}$ path in $G(\Sigma_\Lambda)$, where F is the fault set.

Define now an important set of vertices.

Definition 4: The set of *essential* vertices V_{ess} of $G(\Sigma_\Lambda)$ is the set of vertices which belong to any maximal size fault-output linking.

From [8], one can state the following.

Theorem 4: Consider the linear structured system Σ_Λ with its associated graph $G(\Sigma_\Lambda)$. The set of essential sensors for the rank condition of FDI, is given by Y_e=$Y \cap V_{ess}$, where Y is the sensor set and V_{ess} is the set of essential vertices of $G(\Sigma_\Lambda)$.

The corresponding set of essential vertices is $V_{ess} = \{y_1, y_2, f_1, \dots, f_6, x_1, \dots, x_6, x_9, \dots, x_{14}\}$, see Figure 3.

- The sensors y_1 and y_2 are essential sensors because they belong to $Y \cap V_{ess}$.
- The sensors y_9, y_{10} and y_{11} are useless sensors because they belong to no F-Y path.
- The sensors $y_1, \dots, y_8$, are useful since they are not useless.

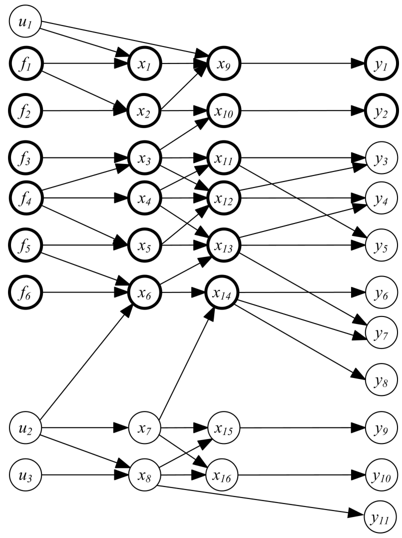

Fig. 3. The set of essential vertices of Example 1

D. Sensor classification for Fault Detection and Isolation

To conserve the Fault Detection and Isolation property in case of sensor failure, the system with the remaining sensors must be output connected, have no contraction and satisfy the rank condition. It is clear that a sensor which is essential for output connection, contraction avoidance or for the rank condition is essential for FDI property preservation. A sensor which is useless for output connection, contraction avoidance and for the rank condition is useless for FDI property preservation. Denote E(Output connection) (resp. L(Output connection), F(Output connection)) the set of essential sensors (resp. useless and useful sensors) for output connection. Denote E(contraction avoidance) (resp. L(contraction avoidance), F(contraction avoidance)) the set of essential sensors (resp. useless and useful sensors) for contraction avoidance. Denote E(rank condition) (resp. L(rank condition), F(rank condition)) the set of essential sensors (resp. useless and useful sensors) for rank condition. We have then the following result.

Theorem 5: Let Σ_Λ be the linear structured system defined by (4). For the problem of FDI preservation, using the previous notations we have:

- The set of essential sensors is given by E(FDI)= E(Output connection) $\cup$ E(contraction avoidance) $\cup$ E(rank condition).
- The set of useless sensors is given by L(FDI)= L(Output connection) $\cap$ L(contraction avoidance) $\cap$ L(rank condition).

From the previous subsections it follows that E(FDI)=$\{y_1, y_2, y_9, y_{10}\}$ and L(FDI)= $\{y_{11}\}$.

From the results of [5] and [8] it follows that:

Proposition 1: Let Σ_Λ be the linear structured system defined by (4) and for the problem of FDI preservation, using the previous notations. The determination of the two classes E and L of sensors for the FDI preservation problem can be done in a polynomial time.

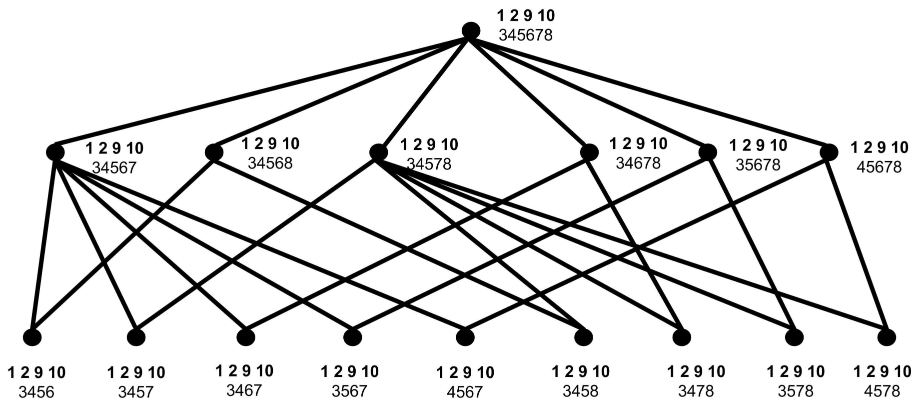

Fig. 4. Sensor network for Example 1

V. Quantitative classification of sensors

In this section we quantify the relative importance of the sensors relatively to their importance for a property $\mathcal{P}$. Introduce first the criticity degree.

A. The Criticity Degree

Definition 5: (**Criticity Degree**) Let K be the set of admissible sensor sets containing no useless sensor. Let K_{y^*} be the set of admissible sensor sets of K containing y^*. The *criticity degree* of a sensor y^* for the property $\mathcal{P}$ is defined as the ratio between the cardinality of K_{y^*} and the the cardinality of K. The criticity degree of y^* is denoted $\mathcal{D}(y^*)$.

The following Proposition can be proved.

Proposition 2: Let Σ be the linear system defined by 1 with a property $\mathcal{P}$ such that $\mathcal{P}(Y) = 1$.

- The criticity degree of a useless sensor is zero
- The criticity degree of an essential sensor is one
- The criticity degree of a useful sensor is greater than zero and lower than or equal to one.

Proof: By Definition 5, the criticity degree is greater than or equal to zero and lower than or equal to one. From Definition 2, when y^* is useless, the cardinality of K_{y^*} is zero, therefore the criticity degree is zero. From Definition 2, when y^* is essential it belongs to any admissible sensor set, then the sets K and K_{y^*} have the same cardinality, therefore the criticity degree is one. A useful sensor belongs to at least one admissible sensor set without useless sensor, therefore the criticity degree is greater than zero.

For two sensors $y_i \neq y_j$, if $\mathcal{D}(y_i) > \mathcal{D}(y_j)$ it turns out that the sensor y_i is more critical than y_j because the number of remaining admissible sensor sets when y_i fails is lower than in the case where y_j fails. In effect, $1 - \mathcal{D}(y_i) < 1 - \mathcal{D}(y_j)$, which implies

$$1 - Card(K_{y_i})/Card(K) < 1 - Card(K_{y_j})/Card(K)$$

or $Card(K) - Card(K_{y_i}) < Card(K) - Card(K_{y_j})$ where $Card(K) - Card(K_{y_i})$ is the number of remaining admissible sets when the sensor y_i fails.

B. The network of admissible sensor sets

In this subsection we use a network to represent all the possible situations in case of sensor failure. The nodes are all the sets of available sensors and an arc corresponds to the failure of a particular sensor. The top node of the network is the set of all sensors, the bottom node is an empty set of sensors. The network is decomposed in $(p + 1)$ levels, the nodes of a given level having the same cardinality. An arc starts from a node a of level l and ends in a node b of level $l - 1$ and represents the loss of a single sensor. In this case, node b is called a *successor* of node a and node a is called a *predecessor* of node b. The loss of several sensors is represented by a path in the sensor network.

For each node on the network, we can check whether the associated combination of sensors is an admissible sensor set or not. It should be noted that if a node is not an admissible sensor set, then none of its successors is an admissible sensor set. We will now define a reduced sensor network as follows: we start with the previous sensor network and remove the nodes corresponding to non admissible sensors sets or sensor sets containing useless sensors. This reduced sensor network contains all the relevant information to compute the criticity degrees. On Example 1 and for the solvability of the bank of observer based diagonal FDI problem, we obtain the sensor network of Figure 4.

From Definition 5, the criticity degrees of the sensors are as follows:

$\mathcal{D}(y_1) = 1$ $\quad\mathcal{D}(y_2) = 1$ $\quad\mathcal{D}(y_3) = 13/16$
$\mathcal{D}(y_4) = 13/16$ $\quad\mathcal{D}(y_5) = 13/16$ $\quad\mathcal{D}(y_6) = 10/16$
$\mathcal{D}(y_7) = 13/16$ $\quad\mathcal{D}(y_8) = 10/16$ $\quad\mathcal{D}(y_9) = 1$
$\mathcal{D}(y_{10}) = 1$ $\quad\mathcal{D}(y_{11}) = 0$

This illustrates the fact that y_1, y_2, y_9, y_{10} are essential, that y_{11} is useless and that y_3, y_4, y_5, y_7 are more important for FDI than y_6 and y_8 in case of sensor failure.

VI. Concluding remarks

In this paper we have classified the sensors with respect to their importance for the solvability of the observer based FDI problem. The proposed analysis is mainly based on the system structure and is parameter independent. This analysis quantifies the degree of criticity of sensors which can be useful in order to perform a robust FDI in case of possible sensor failure. The proposed graph approach uses standard algorithms. This analysis is very general and can be applied to other problems of control and diagnosis.

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