Simple Polynomial Approach to Nonlinear Control

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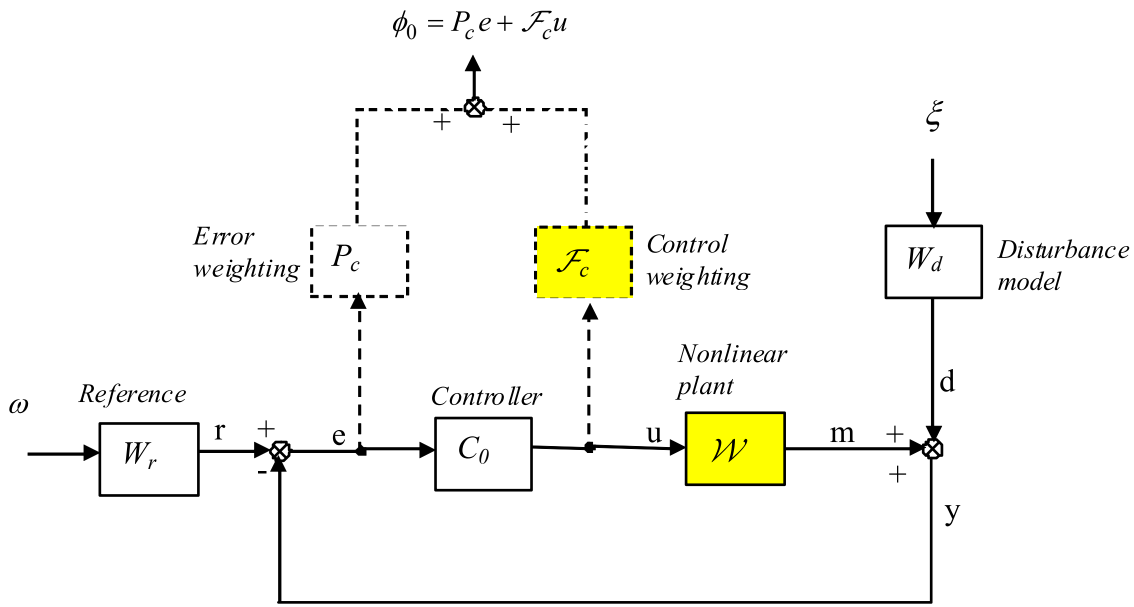
Nonlinear Multivariable CL System



Diagonal matrix of delays

Nonlinear plant model: $(\mathcal{W}u)(t) = z^{-\Lambda}(\mathcal{W}_k u)(t)$

Linear disturbance model: $W_d = A_f^{-1} C_d$

Linear reference model: $W_r = A_f^{-1} E_r$

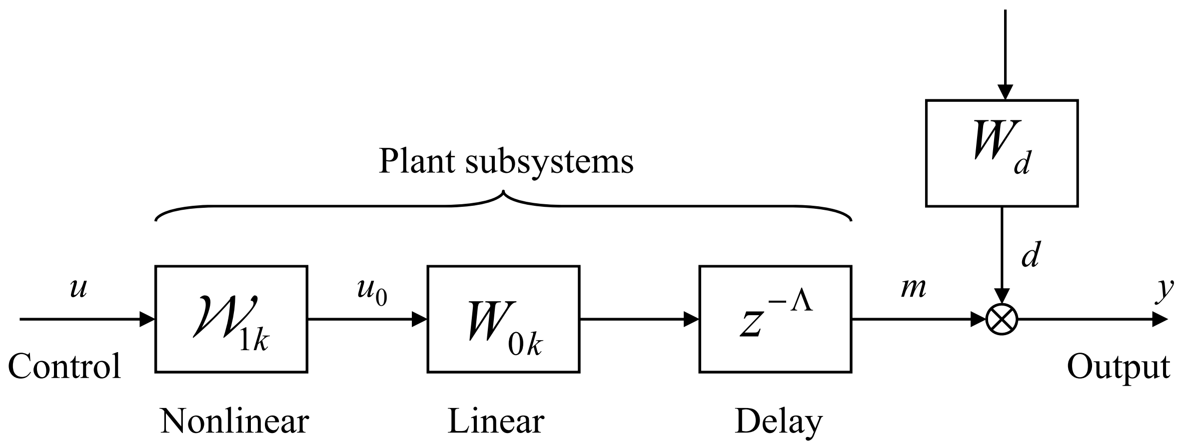
- Advantage of NGMV is only knowledge of NL plant model required is *ability to compute an output* for a given control input sequence.

Nonlinear Plant Model

- Plant model may be given in a very general form, e.g.:
  - state-space formulation
  - neural network / neuro-fuzzy model
  - look-up table
  - Fortran/C code

$$f(u, y) = 0$$

- Only need to compute the output to given input signal
- Can include linear/NL components, e.g. Hammerstein model with static input NL's

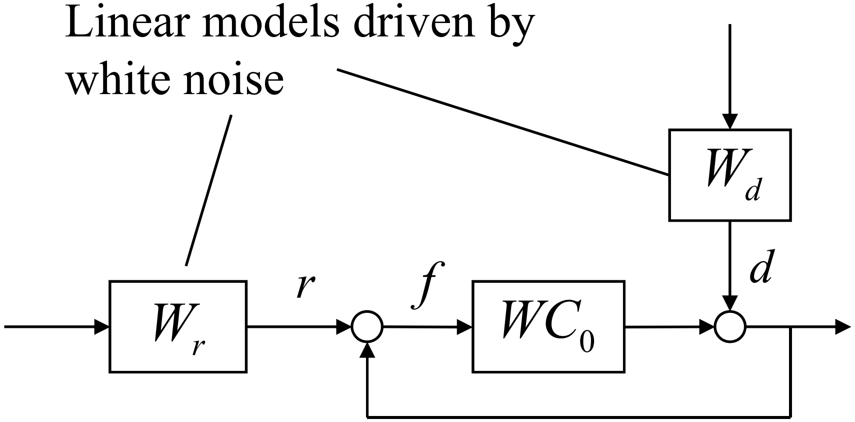


- Only knowledge of NL plant model required is *ability to compute an output* for a given control sequence.

Equivalent Model

- GOAL: Combine all stochastic inputs into one noise signal

Linear models driven by white noise



$$f(t) = Y_f(z^{-1})\varepsilon(t)$$

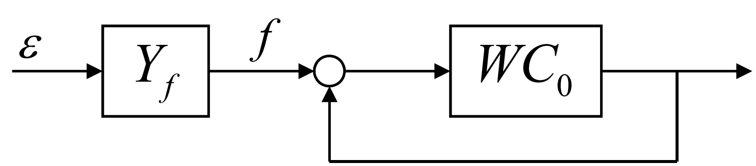
$\varepsilon(t)$ - zero mean white noise of unit variance

$$f(t) = r(t) - d(t)$$

$$\Phi_{ff} = \Phi_{rr} + \Phi_{dd} = W_r W_r^* + W_d W_d^*$$

$$Y_f Y_f^* = \Phi_{ff}$$

Schur spectral factor



Reminder: ***Minimum Variance Control***

- First look at the simple *MV* problem: $\min_u E[y^2(t)]$

$$y(t) = W_k u(t-k) + Y_f \varepsilon(t)$$

- Plant model:

$$Y_f = F + z^{-k} R$$

~ Diophantine equation

$$y(t+k) = W_k u(t) + Y_f \varepsilon(t+k)$$

$$= F\varepsilon(t+k) + W_k u(t) + R\varepsilon(t)$$

statistically independent terms

$$\varepsilon(t) = Y_f(z^{-1})^{-1} f(t)$$

- Optimal control: $u^{MV}(t) = -\dfrac{R}{W_k}\varepsilon(t) = -\dfrac{R}{W_k Y_f} f(t)$

- MV control assumptions:
  - The plant W_k has stable inverse (minimum-phase)
  - Reference and disturbance models are representative of the actual signals acting on the system.

Nonlinear Generalised Minimum Variance Control

NGMV Problem Formulation

- General NGMV cost function to be minimized:

$$J_{NGMV} = E[\phi_0^2(t)]$$

where $$\phi_0(t) = P_c e(t) + (\mathcal{F}_c u)(t)$$

$P_c = P_{cn} P_{cd}^{-1}$ - linear error weighting (matrix fraction)

$(\mathcal{F}_c u)(t) = z^{-\Lambda} (\mathcal{F}_{c\,k} u)(t)$ - control weighting (possibly nonlinear)

- Control weighting assumed invertible and potentially nonlinear to compensate for plant nonlinearities in appropriate cases
- Weighting selection is restricted by closed-loop stability needs

Nonlinear GMV Problem Solution

Split the output into two statistically non-overlapping terms:

$$\phi_0(t+\Lambda) = (\mathcal{F}_{ck} - P_c \mathcal{W}_k) u(t) + P_c Y_f \varepsilon(t+\Lambda)$$

$$\phi_0(t+\Lambda) = \boxed{F_0 \varepsilon(t+\Lambda)} + \boxed{\left((\mathcal{F}_{ck} - P_c \mathcal{W}_k) u(t) + R\varepsilon(t)\right)}$$

statistically independent

$$P_c Y_f = F_0 + z^{-\Lambda} R$$

~ Diophantine equation

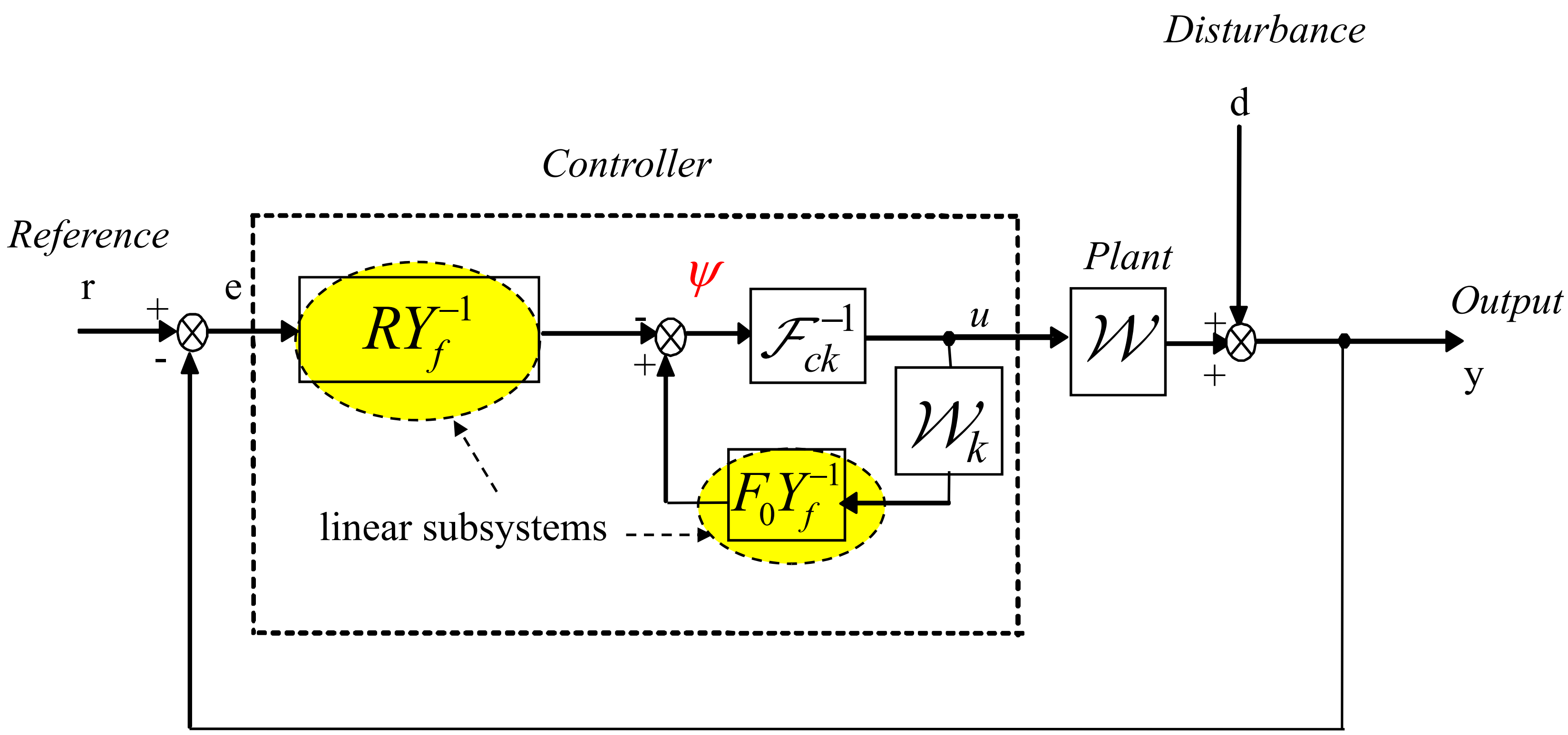
$$\varepsilon(t) = Y_f(z^{-1})^{-1} f(t)$$

NGMV control: $u^{NGMV}(t) = (\mathcal{F}_{ck} - P_c \mathcal{W}_k)^{-1}(-R\varepsilon(t))$

Need stable causal nonlinear operator inverse

Implementation of the NGMV Controller

$$u^{NGMV}(t) = -[(\mathcal{F}_{ck} - F_0 Y_f^{-1} \mathcal{W}_k)^{-1} R Y_f^{-1} e](t)$$



- *The controller is nonlinear but fixed!*

Selection of the Dynamic Cost Weightings

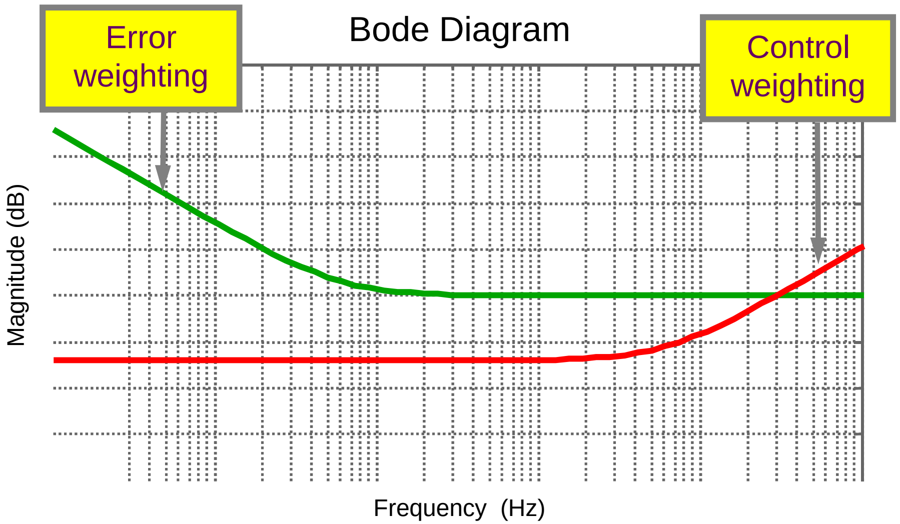
- Restriction on choice of weightings: need invertible nonlinear operator

$$\left(P_c \mathcal{W}_k - \mathcal{F}_{ck} \right)$$

- Find a non-zero control weighting is necessary for non-invertible plants
- Admissible and meaningful choice of weightings important.

Typically

- P_c large at low frequencies to guarantee integral action
- $\mathcal{F}_{ck}$ large at high frequencies to provide sufficient controller roll-off



Stable NL Operator Inverse

and Starting Point for Weighting Selection

- **Necessary condition for optimality:** Operator $\left(P_c\mathcal{W}_k - \mathcal{F}_{ck}\right)$ must have a stable inverse and for linear systems be minimum-phase.
- To show this is satisfied for a wide class of systems consider case where $\mathcal{F}_{ck}$ is linear and $\mathcal{F}_{ck} = -F_k$. Then:

$$\left(P_c\mathcal{W}_k + F_k\right)u = F_k\left(F_k^{-1}P_c\mathcal{W}_k + I\right)u$$

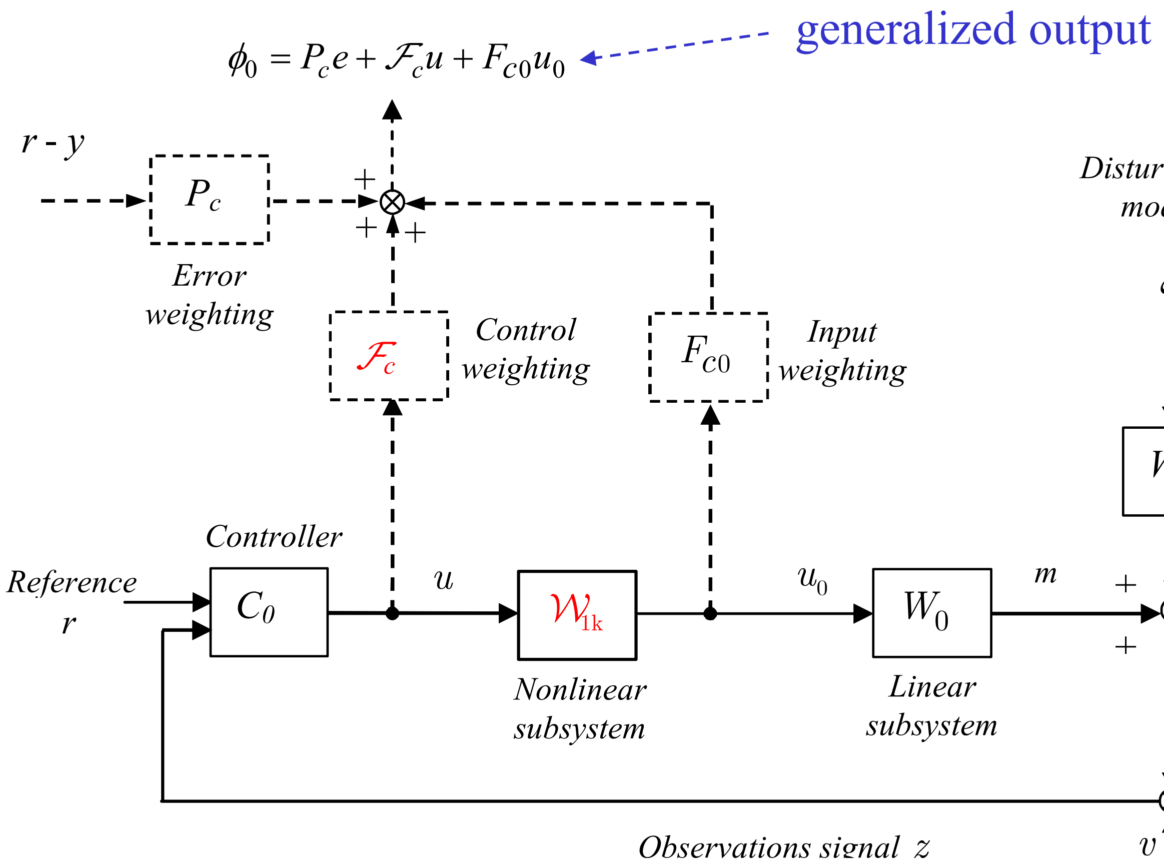
Like return-difference for a feedback system with a delay-free plant and controller: $K_c = F_k^{-1}P_c$.

- Consider delay-free plant W_k and assume PID controller K_{PID} exists to stabilize the closed-loop. Then a starting point for weighting choice that will ensure operator $\left(P_c\mathcal{W}_k + F_k\right)$ is stably invertible is $P_c = K_{PID}, \quad F_k = 1$

Provides weightings that lead to a stable inverse for the NL operator.

Predictive Controller For Nonlinear Processes:

System Model



GPC Criterion

- **Typical GPC cost function:**

$$J = E\left\{ \sum_{j=0}^{N} \mathrm{e}_p(t+j+k)^T \mathrm{e}_p(t+j+k) + \lambda_j^2 u_0(t+j)^T u_0(t+j)) \middle| t \right\}$$

  - Error signal $e_p = r_p - y_p$ may be dynamically weighted
  - Prediction and control horizons equal
  - Time delay included in the cost

Using vector notation:

$$J = E\left\{ (R_{t+k,N} - Y_{t+k,N})^T (R_{t+k,N} - Y_{t+k,N}) + U_{t,N}^{0T} \Lambda_N^2 U_{t,N}^0 \mid t \right\}$$

"sum of squares" criterion

Linear Subsystem Polynomial Matrix Models

The polynomial matrix system models, for the *(r x m)* multivariable system W_0 may now be introduced.

Controlled Auto-Regressive Moving Average (CARMA) model, representing the linear subsystem of the plant model in *GP*C design, defined as:

$$A(z^{-1})y(t) = B_{0k}(z^{-1})u_0(t-k) + C_d(z^{-1})\xi(t)$$

$$[W_{0k}(z^{-1}) \;\; W_d(z^{-1})] = A(z^{-1})^{-1}[B_{0k}(z^{-1}) \;\; C_d(z^{-1})]$$

$$z(t) = A^{-1}(z^{-1})B_{0k}(z^{-1})u_0(t-k) + Y_f(z^{-1})\varepsilon(t)$$

Define the right coprime model for the *weighted spectral factor:*

$$P_c(z^{-1})Y_f(z^{-1}) = D_{fp}(z^{-1})A_f^{-1}(z^{-1})$$

Then the weighted output $y_p(t) = P_c y(t)$ and the observations signal:

$$z_p(t) = P_c(z^{-1})W_{0k}(z^{-1})u_0(t-k) + D_{fp}(z^{-1})A_f^{-1}(z^{-1})\varepsilon(t)$$

Diophantine Equations

First Diophantine: $E_j(z^{-1})A_f(z^{-1}) + z^{-j-k}H_j(z^{-1}) = D_{fp}(z^{-1})$

This equation may be written in the transfer operator form:

$$E_j(z^{-1}) + z^{-j-k}H_j(z^{-1})A_f^{-1}(z^{-1}) = D_{fp}(z^{-1})A_f^{-1}(z^{-1})$$

Prediction equation: Substituting the expression for the weighted observations:

$$z_p(t) = P_c(z^{-1})W_{0k}(z^{-1})u_0(t-k) + D_{fp}(z^{-1})A_f^{-1}(z^{-1})\varepsilon(t)$$
$$= P_c(z^{-1})W_{0k}(z^{-1})u_0(t-k) + (E_j(z^{-1}) + z^{-j-k}H_j(z^{-1})A_f^{-1}(z^{-1}))\varepsilon(t)$$

Substituting from the innovations: $\varepsilon(t) = Y_f^{-1}z(t) - D_f^{-1}B_{0k}u_0(t-k)$ obtain:

$$z_p(t) = P_c(z^{-1})W_{0k}(z^{-1})u_0(t-k) + E_j(z^{-1})\varepsilon(t)$$
$$+ z^{-j-k}H_j(z^{-1})A_f^{-1}(z^{-1})\left(Y_f^{-1}(z^{-1})z(t) - D_f^{-1}(z^{-1})B_{0k}(z^{-1})u_0(t-k)\right)$$

The optimal predictor to minimise the estimation error variance follows as:

$$\hat{y}_p(t+j+k\,|\,t) = \left[H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) + E_j(z^{-1})B_{1k}(z^{-1})u_f(t+j)\right]$$

where $u_f(t) = D_{f1}^{-1}(z^{-1})u_0(t)$.

Predicted Weighted Output

A second Diophantine equation is required to break up the term: $E_j(z^{-1})B_{1k}(z^{-1})$ into a part with a *j+1* step delay and a part depending on $D_{f1}(z^{-1})$. For j ≥ 0, introduce the following equation, which has the solution (G_j, S_j), of *smallest degree* for G_j :

Second Diophantine: $G_j(z^{-1})D_{f1}(z^{-1}) + z^{-j-1}S_j(z^{-1}) = E_j(z^{-1})B_{1k}(z^{-1})$

where $\deg(G_j(z^{-1})) = j$. The prediction equation may now be obtained (for j ≥ 0) as:

$$\hat{y}_p(t+j+k \mid t) = H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) + (G_j(z^{-1})D_{f1}(z^{-1}) + z^{-j-1}S_j(z^{-1}))u_f(t+j)$$
$$= H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) + G_j(z^{-1})u_0(t+j) + S_j(z^{-1})u_f(t-1)$$

Define the signal: $f_j(t)$ in terms of past outputs and inputs, as:

$$f_j(t) = H_j(z^{-1})D_{fp}^{-1}(z^{-1})z_p(t) + S_j(z^{-1})u_f(t-1)$$

Thus, the *predicted weighted output* may be written, for j ≥ 0, as:

$$\hat{y}_p(t+j+k \mid t) = G_j(z^{-1})u_0(t+j) + f_j(t)$$

Matrix Representation of the Prediction Equations

The future weighted outputs are to be predicted in the following section for inputs computed in the interval: $\tau \in [t, t+N]$. The equation may therefore be used to obtain the following vector equation for the weighted output at future times:

$$\begin{bmatrix} \hat{y}_p(t+k \mid t) \\ \hat{y}_p(t+1+k \mid t) \\ \vdots \\ \\ \hat{y}_p(t+N+k \mid t) \end{bmatrix} = \begin{bmatrix} g_0 & 0 & \cdots & 0 & 0 \\ g_1 & g_0 & 0 & \cdots & 0 \\ \vdots & g_1 & g_0 & & \vdots \\ & \vdots & & \ddots & \\ g_N & g_{N-1} & \cdots & & g_0 \end{bmatrix} \begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ \\ u_0(t+N) \end{bmatrix} + \begin{bmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ \\ f_N(t) \end{bmatrix}$$

Vector Form of Prediction Equations

Introducing an obvious definition of terms for the matrices in the above equation the vector form of the predicted weighted outputs may be written as:

$$\hat{Y}_{t+k,N} = G_N U^0_{t,N} + F_{t,N}$$

The vector of free response predictions $F_{t,N}$ may also be written as:

$$F_{t,N} = \begin{bmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ f_N(t) \end{bmatrix} = \begin{bmatrix} H_0(z^{-1}) \\ H_1(z^{-1}) \\ \vdots \\ H_N(z^{-1}) \end{bmatrix} D_{fp}^{-1}(z^{-1}) z_p(t) + \begin{bmatrix} S_0(z^{-1}) \\ S_1(z^{-1}) \\ \vdots \\ S_N(z^{-1}) \end{bmatrix} u_f(t-1)$$

$$= H_{NZ}(z^{-1}) z_p(t) + S_{NZ}(z^{-1}) u_f(t-1)$$

Vector Forms of Future Signals

Future set point knowledge: It is reasonable to assume in many applications that the future variations of the set-point or reference signal $\{r(t)\}$ are predetermined, at least over a fixed future horizon of N steps. The weighted reference is assumed to include the stable weighting: $r_p(t) = P_c(z^{-1})\,r(t)$. The vectors of *future weighted* reference, output and input signals may also be defined as:

$$R_{t,N} = \begin{bmatrix} r_p(t) \\ r_p(t+1) \\ \vdots \\ r_p(t+N) \end{bmatrix} \qquad R_{t,N} = \begin{bmatrix} r_p(t) \\ r_p(t+1) \\ \vdots \\ r_p(t+N) \end{bmatrix} \qquad U_{t,N}^{0} = \begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ u_0(t+N) \end{bmatrix}$$

Theorem : Equivalent Minimum Variance Problem

Consider the minimisation of the *GPC* cost index for the system:

$$J = E\{\sum_{j=0}^{N} \mathrm{e}_p(t+j+k)^T \mathrm{e}_p(t+j+k) + \lambda_j^2 u_0(t+j)^T u_0(t+j)) \Big| \, t\}$$

where the nonlinear subsystem: $\mathcal{W}_{1k} = I$ and the vector of optimal *GPC* controls is

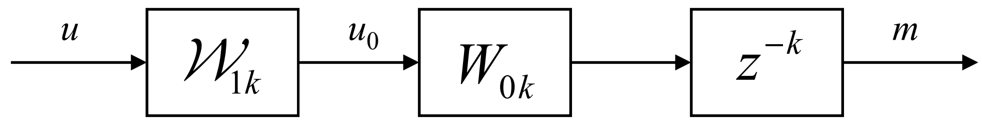
given by: $U_{t,N}^0 = \left(G_N^T G_N + \Lambda_N^2\right)^{-1} G_N^T \left(R_{t+k,N} - F_{t,N}\right)$. If the cost index is

redefined to have a multi-step variance form: $\tilde{J}(t) = E\{\Phi_{t+k,N}^T \Phi_{t+k,N} \mid t\}$,

where $\Phi_{t+k,N} = P_{CN}(R_{t+k,N} - Y_{t+k,N}) + F_{CN}^0 U_{t,N}^0$

and the cost weightings: $P_{CN} = Y^{-T} G_N^T$ and $F_{CN}^0 = -Y^{-T} \Lambda_N^2$. ***Then the vector of future optimal controls is identical to the vector of GPC controls.*** ■

Nonlinear Predictive GMV Problem



- Actual input to the system is the control signal $u(t)$ rather than input to the linear subsystem $u_0(t)$
- Cost function for the nonlinear problem therefore includes an additional control signal costing term

$$\Phi^0_{t+k,N} = P_{CN} E_{t+k,N} + F^0_{CN} U^0_{t,N} + \boxed{(\mathcal{F}_{c\,k,N} U_{t,N})}$$

Nonlinear costing

- When $N = 0$, the problem simplifies to the single step non-predictive NGMV control
- Control design involves specifying the dynamic weightings P_c, $\mathcal{F}_{ck}$, and the constant Λ weighting for the original GPC cost

Theorem: NL Predictive GMV Control

Let error weighting $P_c(z^{-1})$ and the input weightings $\{\lambda_0, \ldots, \lambda_N\}$ be specified and assume the *control signal weighting*: $(\mathcal{F}_c u)(t) = (\mathcal{F}_{ck} u)(t-k)$ where $\mathcal{F}_{ck}$ is full rank and invertible. The *multi-step* cost-function: $J_p = E\{\Phi^{0T}_{t+k,N} \Phi^0_{t+k,N} \mid t\}$

The signal $\Phi^0_{t+k,N}$ includes the vector of future error, input and control signal costing terms: $\Phi^0_{t+k,N} = P_{CN} E_{t+k,N} + F^0_{CN} U^0_{t,N} + (\mathcal{F}_{c\,k,N} U_{t,N})$ where the effective weightings :

$P_{CN} = Y^{-T} G_N^T$, $F^0_{CN} = -Y^{-T} \Lambda_N^2$ and $\mathcal{F}_{c\,k,N}$ may be a diagonal control weighting.

Define the constant matrix factor Y to satisfy $Y^T Y = G_N^T G_N + \Lambda_N^2$ then using the *receding horizon philosophy the* control law:

$$U_{t,N} = -(\mathcal{F}_{c\,k,N} - Y \mathcal{W}_{1\mathrm{k,N}})^{-1} P_{CN} (R_{t+k,N} - F_{t,N})$$

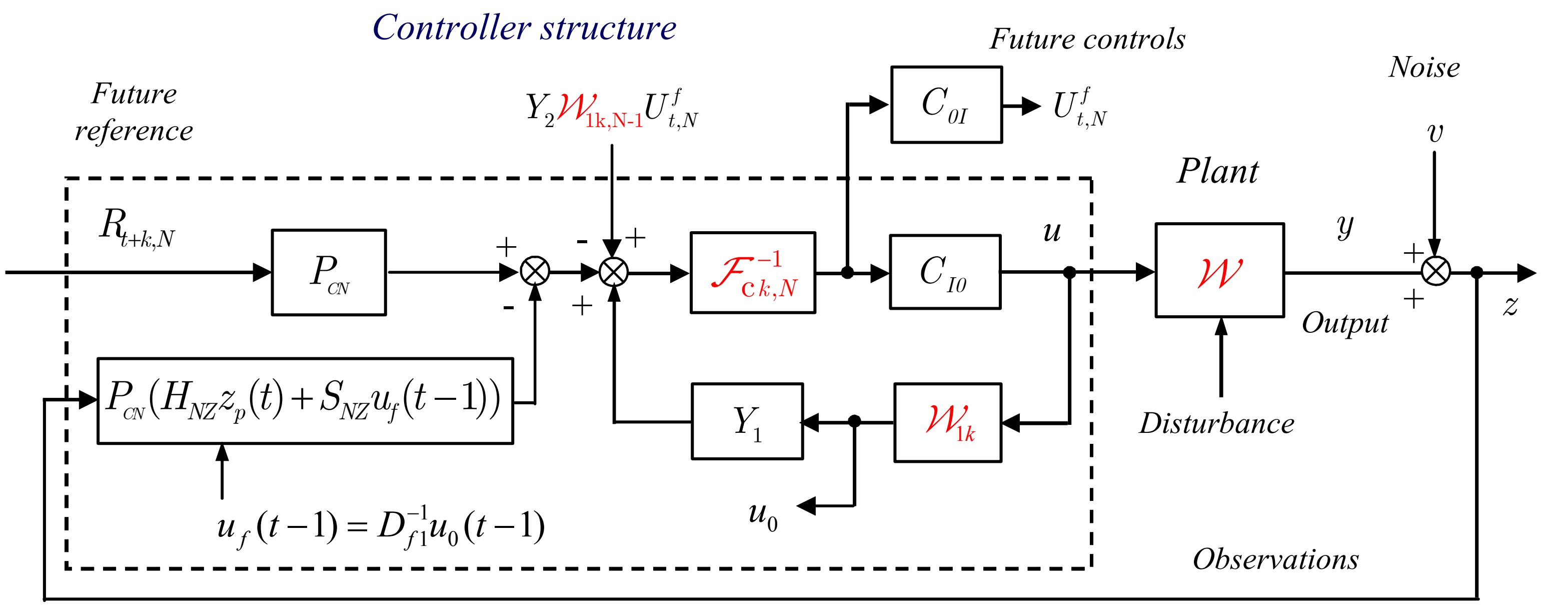
or equivalently:

$$U_{t,N} = -\mathcal{F}^{-1}_{c\,k,N} \left(P_{CN} (R_{t+k,N} - F_{t,N}) - Y \mathcal{W}_{1\mathrm{k,N}} U_{t,N} \right)$$

where the signals: $F_{t,N} = H_{NZ}(z^{-1}) z(t) + S_{NZ}(z^{-1}) u_f(t-1)$ and $u_f(t) = D_{f1}^{-1}(z^{-1}) u_0(t)$.

NPGMV Controller – Polynomial Form

$$u(t) = -\mathcal{F}_{ck}^{-1} C_{I0} \left(P_{CN} (R_{t+k,N} - F_{t,N}) - Y \mathcal{W}_{1k,N} U_{t,N} \right)$$

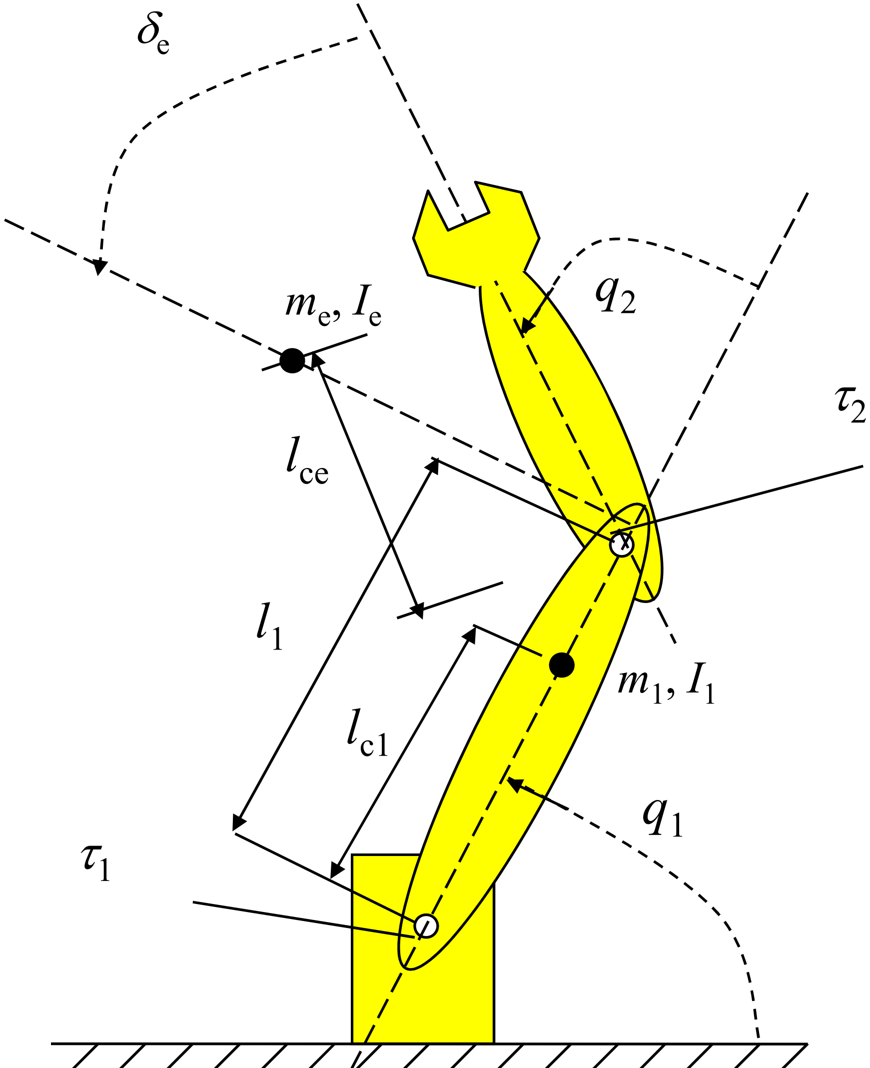


- The solution involves solving two sets of polynomial Diophantine equations
- Equivalent to the state-space version

Robotics Application of Nonlinear Predictive Control

Robotics Application

Two-link robotic manipulator

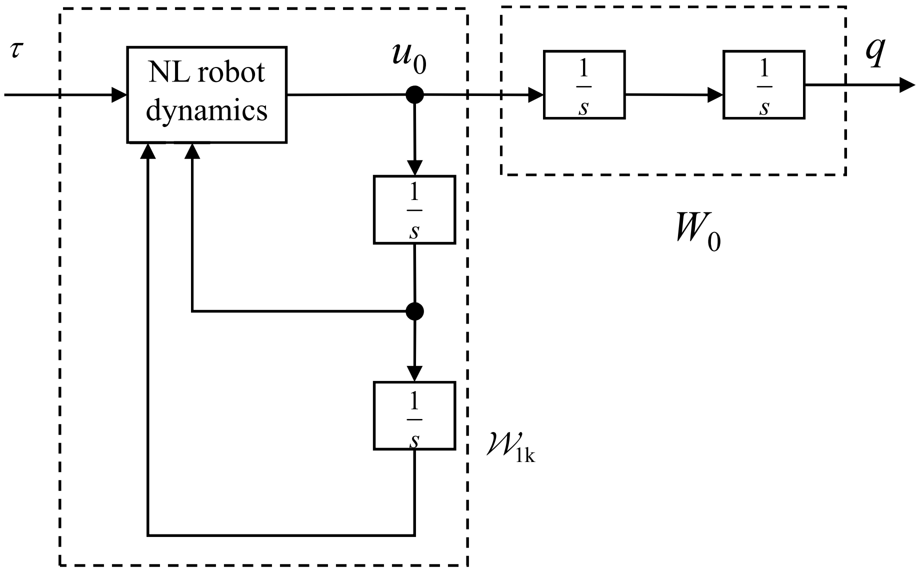


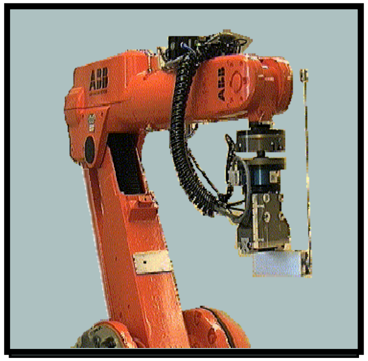
After "Applied Nonlinear Control" by Slotine and Li, 1991.

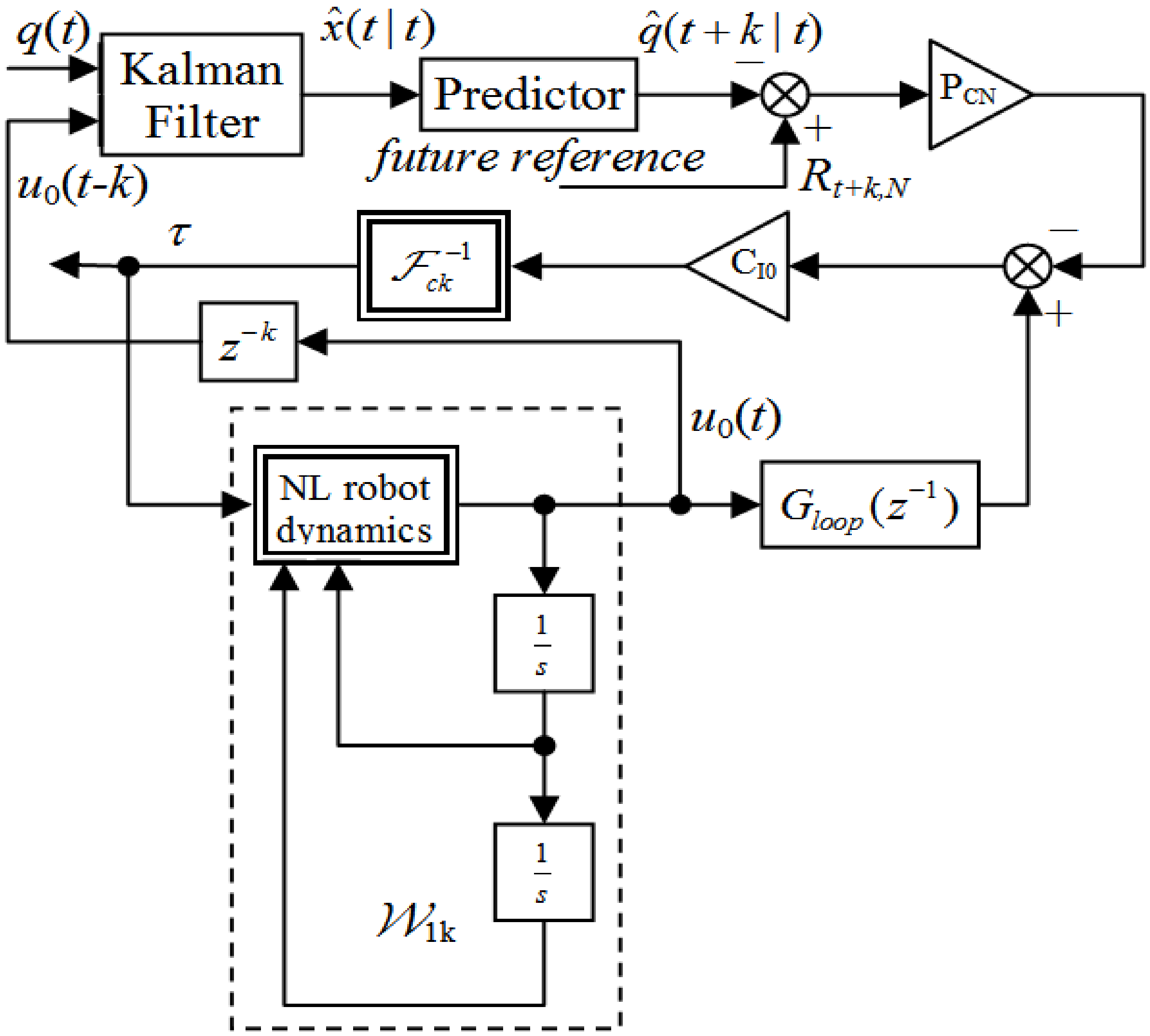
Nonlinear model:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h(\dot{q}_1 + \dot{q}_2) \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

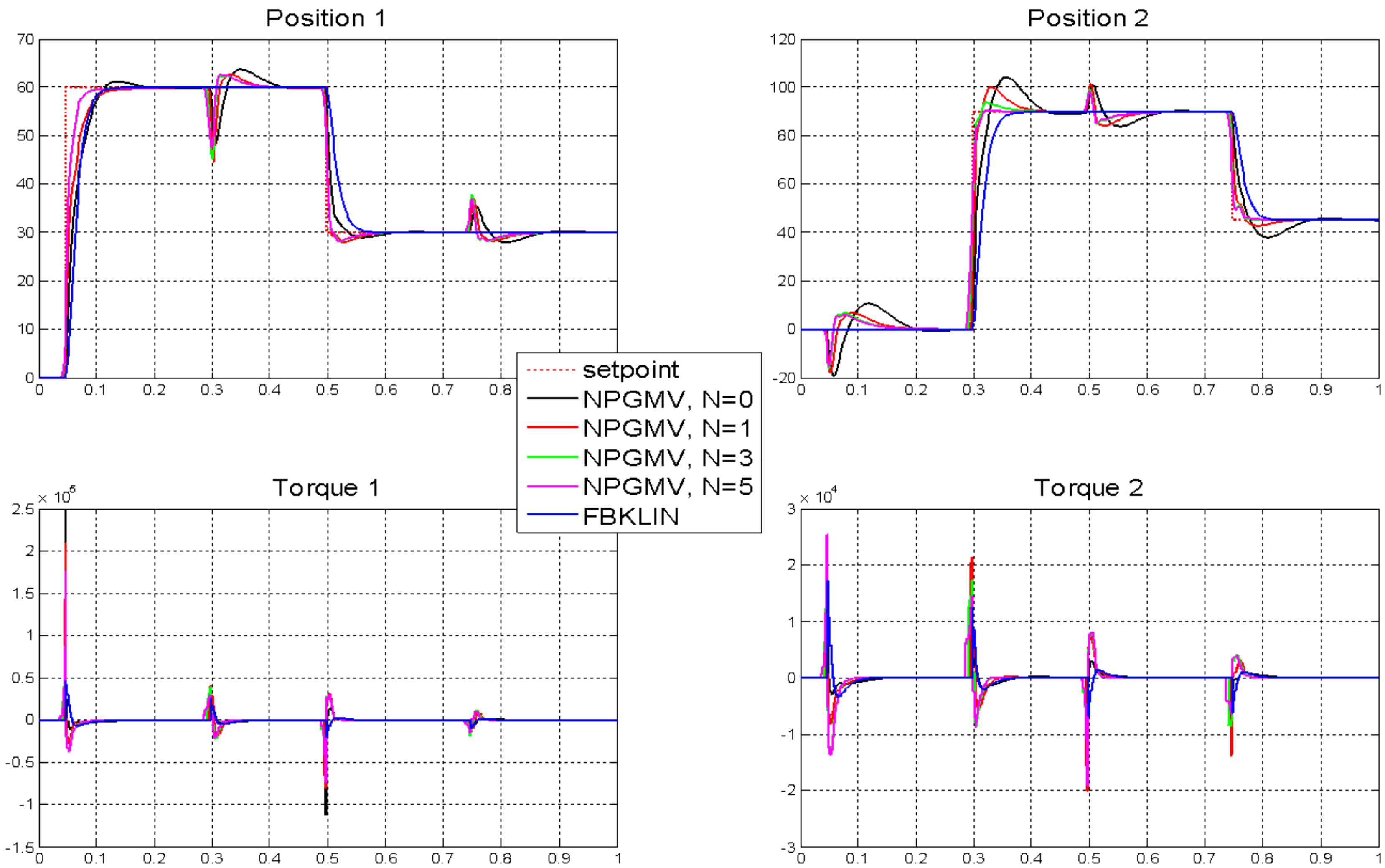
Separation of the model into the nonlinear and linear subsystems



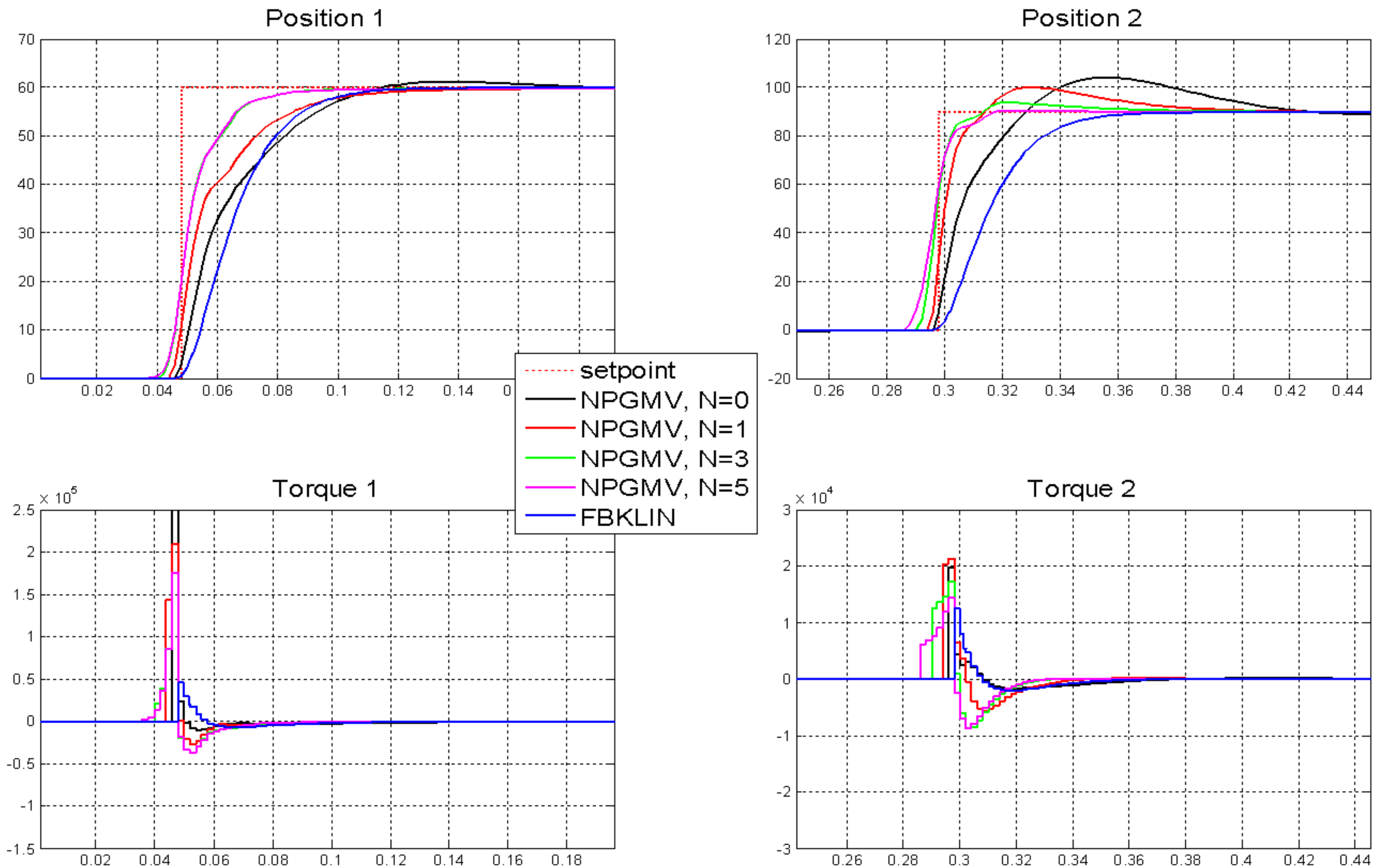




NPGMV controller structure for robot control



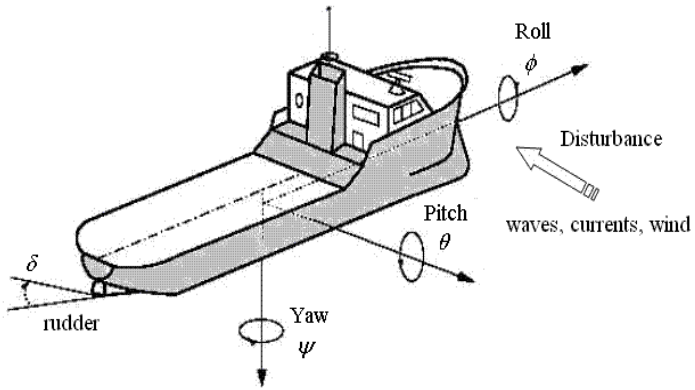
Position Control of the Two-link manipulator. NPGMV control with $N = 0, 1, 3, 5$ and Feedback linearization



Position Control of the Two-link manipulator. NPGMV control with $N = 0, 1, 3, 5$ and Feedback linearization (close-up views)

Marine Systems
Roll Stabilization Example

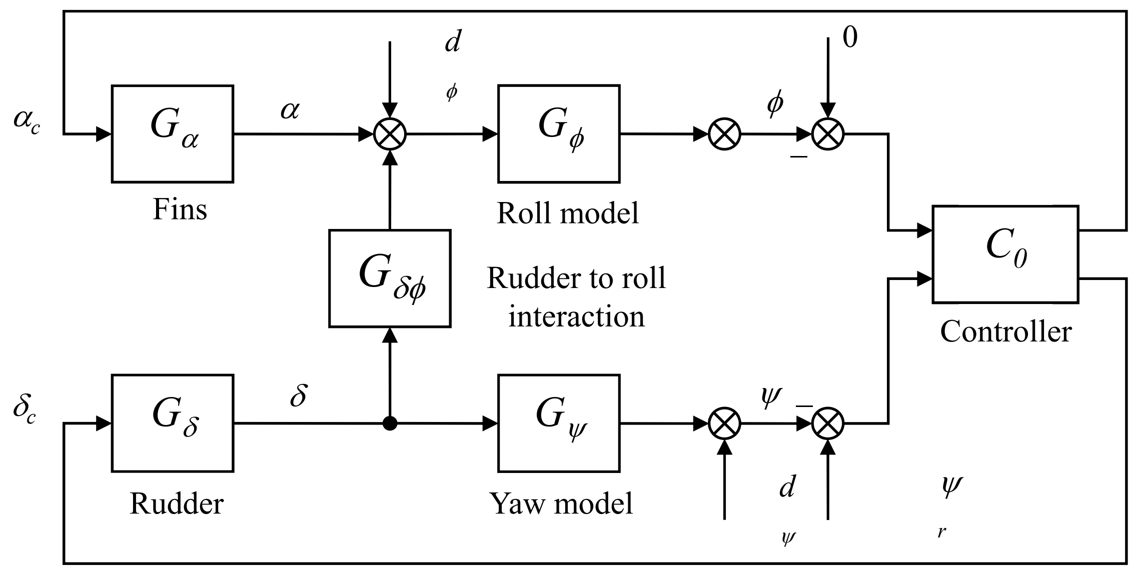
Roll and Yaw Control Using Fins and Rudder



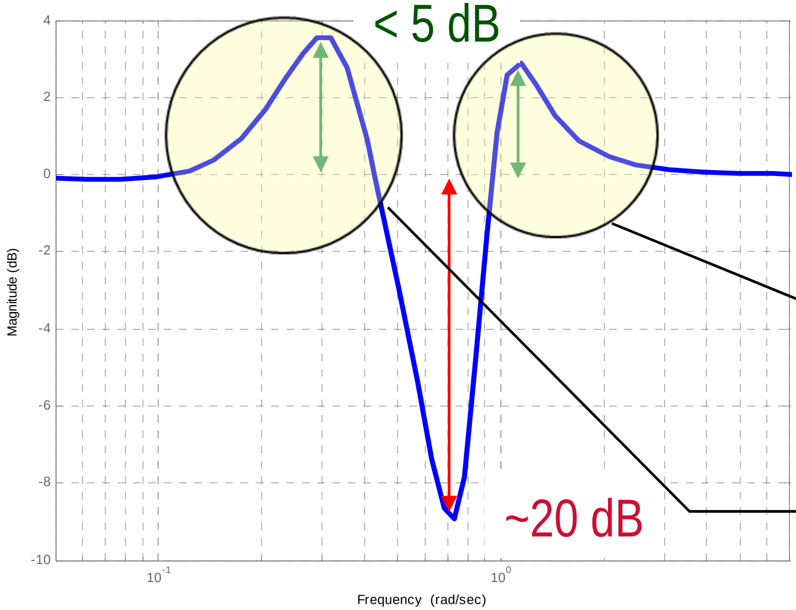
- Ship heading controlled by rudder
- Roll motion reduced by both fin and rudder action
- Difficulty: rudder to roll interaction is non-minimum phase!

Control objective:

Roll reduction and yaw trajectory tracking subject to angle and rate limits on rudder and fins.



Ship Roll Stabilisation Problem



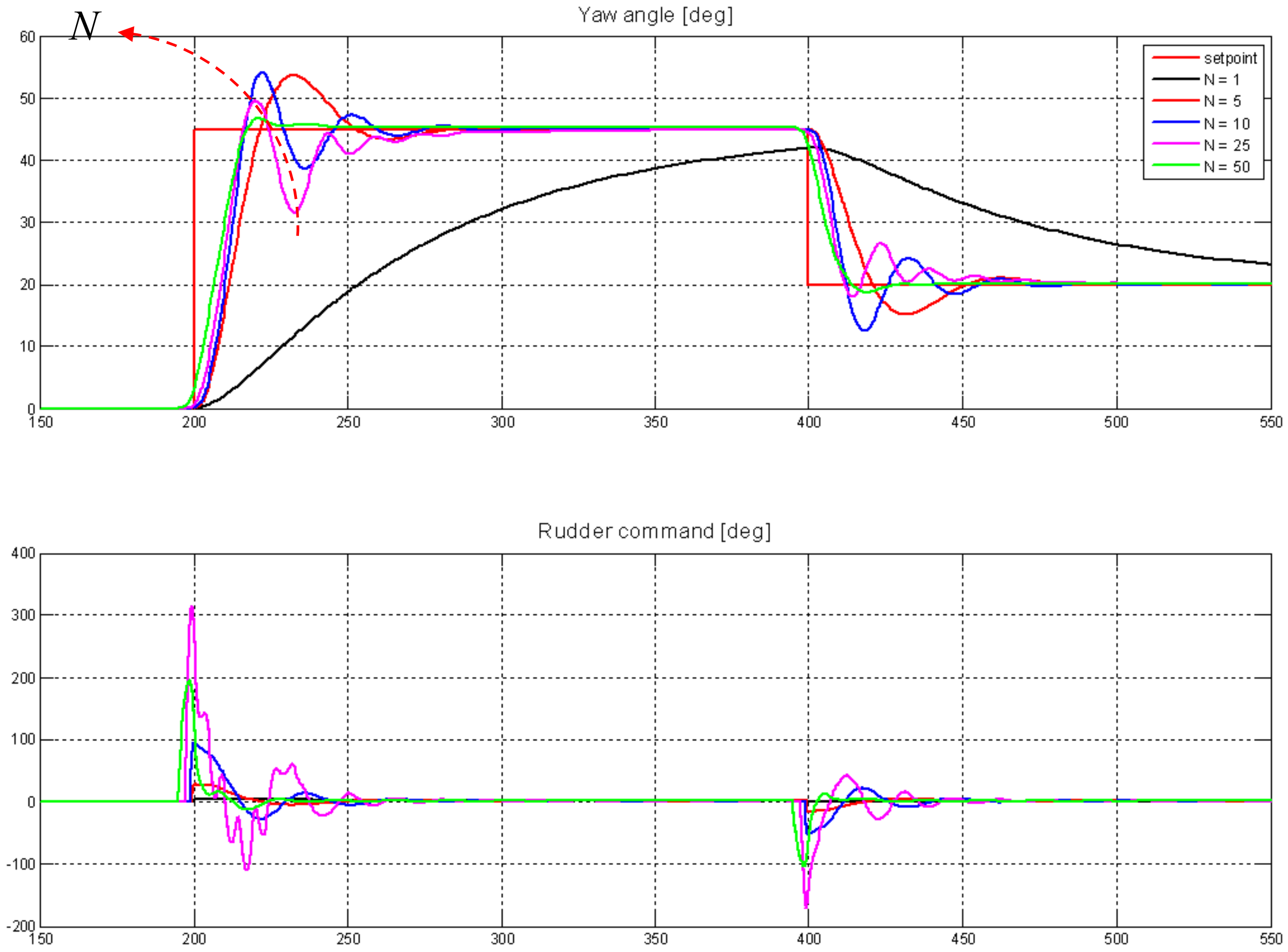
(1) Achieve good roll stabilization
(2) Do not hit rudder constraints
(3) Keep the vessel on course

HF rejection may cause rudder slew rate saturation

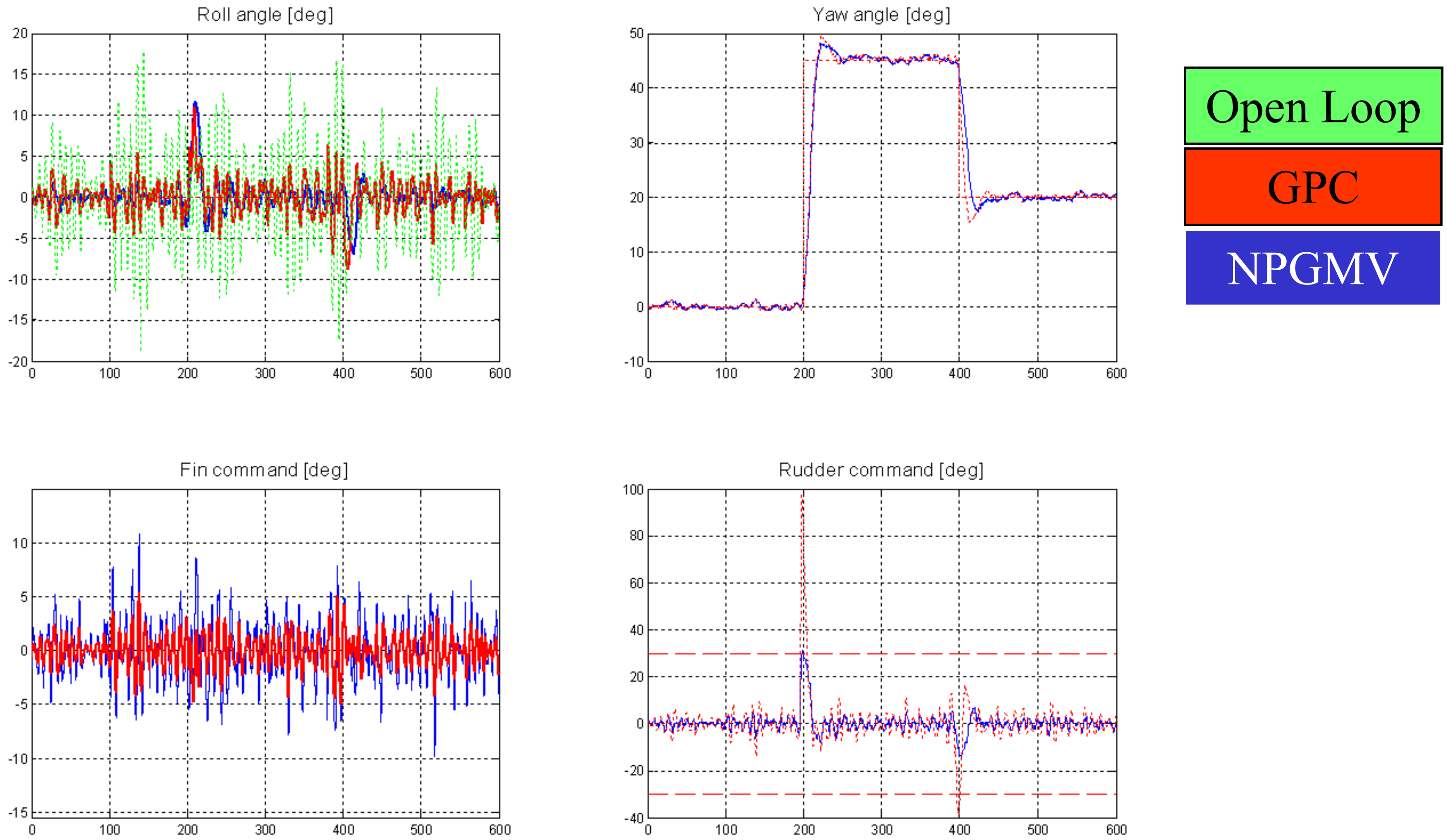
LF rejection may cause rudder angle saturation

➡ Compensate roll motion in a well-defined frequency band (0.3-1.2 rad/sec)

Ship GPC Control Results for Varying *N*



Example: GPC and NPGMV Results



Concluding Remarks

- *A practical NL controller must be simple but we need some mathematical basis to understand behavior.*
- *NGMV is a candidate and the patriarch for a family of more complicated and specialist solutions.*
- *The ability to handle black box models is important industrially.*
- *Nonlinear predictive is a model based fixed controller without uncertainty of linearization around a trajectory - so interesting.*
- *Extendable further to hybrid and/or complex systems.*
- *LabVIEW toolbox including new tools next !*
- *Dual Estimation problems equally interesting.*

Nonlinear Book

For new book on nonlinear control, to be published next year:

M. J. Grimble and P. Majecki, ***Nonlinear Industrial Control,***

Springer, Heidelberg, Germany 2009

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