Deadbeat Response is l_2 0ptimal

Vladimír Kučera
Fellow IEEE, Fellow IFAC
Director, Masaryk Institute of Advanced Studies
Professor, Faculty of Electrical Engineering
Czech Technical University in Prague
Czech Republic
http://www.muvs.cvut.cz
kucera@muvs.cvut.cz

***Abstract*—A typical linear control strategy in discrete-time systems, deadbeat control produces transients that vanish in finite time. On the other hand, the linear-quadratic control stabilizes the system and minimizes the l_2 norm of its transient response. Quite surprisingly, it is shown that deadbeat systems are l_2 optimal, at least for reachable systems.**

The proof makes use of polynomial matrix fractions and structure theorem for linear time-invariant multivariable systems, the notions introduced by W.A. Wolovich in the early seventies.

The result demonstrates the flexibility offered by the linear-quadratic regulator design and is an exercise in inverse optimality. The linear-quadratic regulator gain is unique, whereas the deadbeat feedback gains are not. Only one deadbeat gain is linear-quadratic optimal. An alternative construction of such a gain, based on solving an algebraic Riccati equation, is thus available.

Keywords - linear systems; finite impulse response; l_2 norm minimization

I. DEADBEAT REGULATOR

We consider a linear system (A, B) described by the state equation

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, ... \tag{1}$$

where $u_k \in I\!R^m$ and $x_k \in I\!R^n$. The objective of *deadbeat regulation* is to determine a linear state feedback of the form

$$u_k = -Lx_k \tag{2}$$

that drives each initial state x_0 to the origin in a least number of steps.

We define the *reachability subspaces* of system (1) by

$$R_0 = 0,$$
$$R_k = \text{range}[B \;\; AB \;\; ... \;\; A^{k-1}B], \quad k = 1, 2, ...$$

Hence R_k is the set of states of (1) that can be reached from the origin in k steps by applying an input sequence u_0, u_1, ..., u_{k-1}. When $R_n = I\!R^n$, the system (A, B) of (1) is said to be *reachable*.

Define the integers

$$q_k = \text{dimension } R_k - \text{dimension } R_{k-1}$$

and for $k = 1, 2, ..., m$ let

$$r_i = \text{cardinality } \{q_k : q_k \geq i\}.$$

The integers $r_1 \geq r_2 \geq ... \geq r_m$ are the *reachability indices* of system (1).

We further define the controllability subspaces for (1) by

$$C_0 = 0,$$
$$C_k = \{x \in I\!R^n : F^k x \in R_k\}, \quad k = 1, 2, ...$$

Thus C_k is the set of all states of (1) that can be steered to the origin in k steps by an appropriate control sequence u_0, u_1, ..., u_{k-1}. When $C_n = I\!R^n$, the system (A, B) of (1) is said to be *controllable*. It follows from the definitions that reachability implies controllability and the converse is true whenever A is nonsingular.

The existence and construction of deadbeat control laws is described below. For each k = 1, 2, ... let S_1, S_2, ..., S_k be a sequence of $m \times q_1, m \times q_2, ..., m \times q_k$ matrices such that

$$\text{range}[BS_1 \;\; ABS_2 \;\; ... \;\; A^{k-1}BS_k] = \text{range } R_k.$$

Therefore S_1, S_2, ..., S_k serve to select a basis for R_k.

Theorem 1. [1] There exists a deadbeat control law (2) if and only if the system (A, B) of (1) is controllable. Let

$$L_0 = 0,$$
$$L_k = L_{k-1} + L'_k(A - BL_{k-1})^k, \quad k = 1, 2, ... \tag{3}$$

The original version of the paper was presented at the 3rd IEEE International Symposium on Communications, Control and Signal Processing, held in St Julians, Malta 2008.

where L'_k satisfies

$$L'_k[BS_1 \;\; ABS_2 \;\; \ldots \;\; A^{k-1}BS_k] = [0 \;\; \ldots \;\; 0 \;\; S_k].$$

Then $L = L_n$ is a deadbeat regulator gain. ⊲

The theorem identifies all deadbeat control laws via the recursive procedure (3). Actually the procedure can be terminated in q steps, where $q = \min\{k : C_{k+1} = C_k\}$. The resulting closed-loop system matrix is nilpotent with index q,

$$(A - BL)^q = 0. \tag{4}$$

If A is nonsingular, the recursive procedure (3) can be shortcut by setting $L_{q-1} = 0$. Then, the Jordan structure of $A - BL$ comprises m nilpotent blocks [3] of sizes $r_1, r_2, ..., r_m$ and the index of nilpotency equals $q = r_1$. In fact, this is the least size of Jordan blocks that can be achieved [4] in a reachable system (1) by applying state feedback (2).

II. Linear Quadratic Regulator

We consider a linear system described by the state equation (1),

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots$$

where $u_k \in I\!R^m$ and $x_k \in I\!R^n$. The objective of LQ regulation is to find a linear state feedback of the form (2),

$$u_k = -Lx_k$$

that stabilizes the closed-loop system

$$x_{k+1} = (A - BL)x_k$$

and, for every initial state x_0, minimizes the l_2 norm

$$\|y\|^2 = \sum_{k=0}^{\infty} y_k^{\mathrm{T}} y_k$$

of a specified output $y_k \in I\!R^p$ of the form

$$y_k = Cx_k + Du_k. \tag{5}$$

The existence and construction of an LQ control law is described below. We say that the system (A, B) of (1) is *stabilizable* if the system matrices can be transformed to the following form using an appropriate basis:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where the subsystem defined by the pair of matrices (A_{11}, B_1) is reachable and A_{22} is a stable matrix. We say that the system (A, B, C, D) defined by the state equation (1) and the output equation (5) is *left invertible* if its transfer function $C(zI_n - A)^{-1}B + D$ has full column normal rank. We further define the *system matrix* as the polynomial matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix}$$

and say that a complex number ζ is an *invariant zero* of the system (A, B, C, D) if the rank of $S(\zeta)$ is strictly less than the normal rank of $S(z)$.

Theorem 2. [2] Suppose that the system (A, B) of (1) is stabilizable. Suppose that the system (A, B, C, D) defined by (1) and (5) is left invertible and also has no invariant zeros on the unit circle $|z| = 1$. Then, there exists a unique LQ regulator gain given by

$$L = (D^T D + B^T XB)^{-1}(B^T XA + D^T C), \tag{6}$$

where X is the largest symmetric nonnegative definite solution of the algebraic Riccati equation

$$\begin{aligned} X &= A^{\mathrm{T}} XA + C^{\mathrm{T}} C \\ &\quad - (B^T XA + D^T C)^T (D^T D + B^T XB)^{-1} (B^T XA + D^T C). \end{aligned} \tag{7}$$ ⊲

The assumption of left invertibility for (A, B, C, D) is required in order for the matrix $D^{\mathrm{T}}D + B^{\mathrm{T}}XB$ to be nonsingular while the remaining assumptions are required for the existence of the requisite solution X of the algebraic Riccati equation.

III. The Deadbeat Regulator As an LQ Regulator

The aim of this section is to show that deadbeat control laws in *reachable* systems are LQ optimal. This will be done by constructing an output (5) of the system (1) so that the resulting LQ regulator gain is a deadbeat gain.

Let the system (A, B) of (1) be reachable with reachability indices $r_1, r_2, ..., r_m$. Then there exists a similarity transformation T that brings the matrices A and B to the standard reachability form [3],

$$A' = TAT^{-1}, \quad B' = TB \tag{8}$$

where A' is a top-companion matrix with nonzero entries in rows r_i, $i = 1, 2, ..., m$ and B' has nonzero entries only in rows r_i and columns $j \geq i$, $i = 1, 2, ..., m$.

Theorem 3. Suppose that the system (A, B) of (1) is reachable, with reachability indices $r_1 \geq r_2 \geq ... \geq r_m$ and with the matrix B having rank m. Let T be a similarity transformation that brings A and B to the standard reachability form. Then, the feedback gain L that is LQ optimal with respect to $C = T$ and $D = 0$ in (5) is a deadbeat gain.

Proof. Consider the transfer function of system (1) in polynomial matrix fraction form

$$(zI_n - A)^{-1}B = Q(z)P^{-1}(z) \qquad (9)$$

where P and Q are right coprime polynomial matrices in z of respective size $m \times m$ and $n \times m$, with P column reduced and column-degree ordered with column degrees $r_1 \geq r_2 \geq ... \geq r_m$. These integers are the reachability indices of (1).

The system (1) being reachable, the matrices $zI_n - A$ and B are left coprime. It follows from (9) that the denominator matrices $zI_n - A$ and $P(z)$ have the same determinant (in fact, the same invariant polynomials).

For any feedback (2) applied to system (1), one obtains

$$[zI_n - A \quad -B]\begin{bmatrix} I_n & 0 \\ -L & I_m \end{bmatrix} = [zI_n - (A - BL) \quad -B]$$

and

$$\begin{bmatrix} I_n & 0 \\ L & I_m \end{bmatrix}\begin{bmatrix} Q(z) \\ P(z) \end{bmatrix} = \begin{bmatrix} Q(z) \\ P(z) + LQ(z) \end{bmatrix}.$$

Then (9) implies that

$$[zI_n - (A - BL)]^{-1}B = Q(z)[P(z) + LQ(z)]^{-1}. \qquad (10)$$

Thus the closed-loop system transfer function matrices $zI_n - (A - BL)$ and B are left coprime while $P(z) + LQ(z)$ and $Q(z)$ are right coprime. It follows from (10) that the polynomial matrices $zI_n - (A - BL)$ and $P(z) + LQ(z)$ have the same determinant (in fact, the same invariant polynomials).

Now we show that an LQ regulator gain exists that is optimal with respect to $C = T$ and $D = 0$. Indeed, the pair (A, B) is reachable, hence stabilizable. The quadruple $(A, B, T, 0)$ corresponds to the transfer function $T(zI_n - A)^{-1}B$ whose column normal rank is m, hence the system is left invertible. The system matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ T & 0 \end{bmatrix}$$

has rank $n + m$ for all complex numbers z, which implies that $(A, B, T, 0)$ has no invariant zeros at all. Consequently, the assumptions of Theorem 2 are all satisfied, which shows the existence of an LQ optimal regulator gain (6).

Consider the associated algebraic Riccati equation (7). Add $z^{-1}(XA - XA) + (XA - XA)^{\mathrm{T}} z$ to the right-hand side of the equation in order to introduce polynomial matrix factorizations, then use (6) and (9) to get the following identity [3]

$$\begin{aligned} &[P(z^{-1}) + LQ(z^{-1})]^{\mathrm{T}}(D^T D + B^{\mathrm{T}} XB)[P(z) + LQ(z)] \\ &\quad = [CQ(z^{-1}) + DP(z^{-1})]^{\mathrm{T}}[CQ(z) + DP(z)]. \end{aligned} \qquad (11)$$

Define a polynomial $m \times m$ matrix F, which is column reduced and column-degree ordered with column degrees $r_1 \geq r_2 \geq ... \geq r_m$, by the equation

$$F^{\mathrm{T}}(z^{-1})F(z) = [CQ(z^{-1}) + DP(z^{-1})]^{\mathrm{T}}[CQ(z) + DP(z)] \qquad (12)$$

in such a way that its inverse F^{-1} is analytic in the domain $|z| \geq 1$. This matrix is referred to as the *spectral factor* and it is determined uniquely by (12) up to multiplication on the left by a constant orthogonal matrix.

The pair (A, B) being reachable, the matrices A and B can be brought to the reachability standard form (8) using the similarity transformation matrix T. The corresponding right coprime polynomial fraction matrices are related by

$$P'(z) = P(z), \quad Q'(z) = TQ(z)$$

and by the Structure Theorem of Wolovich [5], Q' has the block-diagonal form

$$Q'(z) = \text{block-diag}\,[\begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_1 - 1} \end{bmatrix}, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_2 - 1} \end{bmatrix}, ..., \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_m - 1} \end{bmatrix}].$$

The spectral factorization (12) reads

$$\begin{aligned} F^{\mathrm{T}}(z^{-1})F(z) &= Q^{\mathrm{T}}(z^{-1})T^{\mathrm{T}}TQ(z) \\ &= Q'^{T}(z^{-1})Q'(z) = \text{diag}\,[r_1, r_2, ..., r_m] \end{aligned}$$

so that

$$F(z) = \text{diag}\,[\sqrt{r_1}\, z^{r_1}, \sqrt{r_2}\, z^{r_2}, ..., \sqrt{r_m}\, z^{r_m}].$$

It follows from (11) that the LQ regulator that is optimal with respect to $C = T$ and $D = 0$ induces the closed-loop right denominator matrix $P(z) + LQ(z)$ with invariant factors $z^{r_1}, z^{r_2}, ...,, z^{r_m}$. The same factors are shared by the closed-loop left denominator matrix $zI_n - (A - BL)$. Therefore, $A - BL$ is nilpotent with Jordan structure comprising m nilpotent blocks of sizes $r_1, r_2, ..., r_m$. The nilpotency index of $A - BL$ is r_1, the largest reachability index. Q.E.D.

The restriction of Theorem 3 to reachable systems, while technically important, is actually a mild restriction as it covers the case of main practical interest. Controllable systems (1) that are not reachable possess a singular matrix A with nilpotent dynamics. Such systems are inherently discrete. In particular, the periodically sampled continuous time systems, considered in the discrete instants of time, have a nonsingular matrix A.

The other restriction applied in Theorem 3, namely B having full column rank m, is needed to guarantee the solvability of the LQ regulator. It represents no practical constraint, either. Indeed, if the rank of B is less than m, then the components of the control vector u are linearly dependent.

IV. Example

To illustrate, let us consider a system (1) described by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since

$$R_0 = 0, \; R_1 = \text{image}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; R_2 = R_3 = \text{image}\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] = I\!R^3$$

the system is reachable with reachability indices $r_1 = 2$, $r_2 = 1$. Since

$$C_0 = 0, \; C_1 = \text{image}\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C_2 = C_3 = \text{image}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I\!R^3$$

the system is controllable and $q = 2$.

Deadbeat gains can be calculated using Theorem 1. One can take

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

thus obtaining, recursively,

$$L_1 = \begin{bmatrix} \beta & 1+\beta & 0 \\ \alpha & \alpha & 1 \end{bmatrix}, \quad L_2 = L_3 = \begin{bmatrix} 1 & 2 & 0 \\ \alpha & \alpha & 1 \end{bmatrix}$$

for any real numbers α and β. Any and all deadbeat gains are given as

$$L = \begin{bmatrix} 1 & 2 & 0 \\ \alpha & \alpha & 1 \end{bmatrix}.$$

The closed-loop system (1), (2) is described by

$$A - BL = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ -\alpha & -\alpha & 0 \end{bmatrix},$$

which is a nilpotent matrix with index $q = 2$. Any initial state is driven to the controllability subspace C_1 in one step and then to the origin in the second step.

Let us now transform (1) to the reachability standard form. An appropriate similarity transformation T is found to be

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It allows calculating a deadbeat gain as the LQ regulator gain that is optimal with respect to $C = T$ and $D = 0$.

Since rank $B = 2$, the conditions of Theorem 2 are all satisfied. The algebraic Riccati equation (7) has a unique symmetric non-negative definite solution

$$X = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The resulting LQ regulator gain (6),

$$L = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

is indeed a deadbeat gain, corresponding to $\alpha = 0$. The other deadbeat gains, however, cannot be obtained using this approach.

V. Conclusion

Deadbeat control and LQ regulation in discrete-time systems, two control strategies that are so different in nature, are in fact related. It has been shown that a deadbeat control law can be obtained by solving a particular LQ regulator problem, at least for reachable systems. This demonstrates the flexibility offered by the LQ regulator design.

The LQ optimal regulator gain is unique, whereas the deadbeat feedback gains are not. Only one deadbeat gain is LQ optimal. An alternative construction of such a gain, based on solving an algebraic Riccati equation, is thus available.

References

[1] C. T. Mullis, "Time optimal discrete regulator gains," *IEEE Trans. Automat. Control*, vol. AC-17, pp. 265–266, 1972.

[2] A. Saberi, P. Sannuti, and B. M. Chen, *H_2 Optimal Control.* London: Prentice Hall, 1995.

[3] V. Kučera, "Deadbeat control, pole placement, and LQ regulation," *Kybernetika*, vol. 35, pp.681–692, 1999.

[4] V. Kučera, Analysis and Design of Discrete Linear Control Systems. London: Prentice Hall, 1991.

[5] W.A. Wolovich, *Linear Multivariable Systems*. NewYork: Springer, 1974.