Deadbeat Response

is $l_2$ Optimal

Vladimír Kučera
Czech Technical University in Prague

Bill Wolovich Celebratory Event
Cancun 2008, Mexico
Introduction

A typical linear control strategy in discrete-time systems, deadbeat control produces transients that vanish in finite time.

On the other hand, the linear-quadratic control stabilizes the system and minimizes the $l_2$ norm of its transient response.

Quite surprisingly, it is shown that deadbeat systems are $l_2$ optimal, at least for reachable systems.
Given a linear system \((A, B)\)

\[ x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots \]

where \(u_k \in \mathbb{S}^m\) and \(x_k \in \mathbb{S}^n\).

The objective of deadbeat regulation is to determine a linear state feedback of the form

\[ u_k = -Lx_k \]

that drives each initial state \(x_0\) to the origin in a least number of steps.
We define the \textit{reachability subspaces} by

\[ R_0 = 0, \]
\[ R_k = \text{range}[B \ AB \ ... \ A^{k-1}B], \quad k = 1, 2, \ldots. \]

When \( R_n = \mathbb{S}^n \), the system \((A, B)\) is said to be \textit{reachable}.

The system \((A, B)\) is said to be \textit{controllable} if there exists a basis in which

\[ A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \]

where \((A_1, B_1)\) is reachable and \(A_2\) is nilpotent.
Reachability Indices

For each $k = 1, 2, ...$
let $S_1, S_2, ..., S_k$ be a sequence of matrices such that

$$\text{range} [BS_1 \ ABS_2 \ ... \ A^{k-1}BS_k] = \text{range } R_k$$

Therefore $S_1, S_2, ..., S_k$ serve to select a basis for $R_k$.

The reachability indices $r_1, r_2, ..., r_m$ are defined by

$$r_i = \text{cardinality } \{ S_j, j = 1, 2, ... : \text{rank } S_j \geq i \}$$
**Theorem 1**

There exists a deadbeat control law if and only if the system \((A, B)\) is controllable.

Let

\[
L_0 = 0,
\]

\[
L_k = L_{k-1} + L_k'(A - BL_{k-1})^k, \quad k = 1, 2, ...
\]

where \(L_k'\) satisfies

\[
L_k'[BS_1 \quad ABS_2 \quad ... \quad A^{k-1}BS_k] = [0 \quad ... \quad 0 \quad S_k].
\]

Then \(L = L_n\) is a deadbeat regulator gain.

The closed-loop system matrix \(A - BL\) is nilpotent.
Given a linear system \((A, B)\)
\[ x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots \]
where \(u_k \in \mathbb{S}^m\) and \(x_k \in \mathbb{S}^n\).

The objective of LQ regulation is to find a linear state feedback of the form
\[ u_k = -Lx_k \]
that stabilizes the closed-loop system and, for every initial state \(x_0\), minimizes the \(l_2\) norm of a specified output \(y_k \in \mathbb{S}^p\) of the form
\[ y_k = Cx_k + Du_k \]
The system \((A, B)\) is said to be **stabilizable** if there exists a basis in which

\[
A = \begin{bmatrix}
A_1 & A_{12} \\
0 & A_2
\end{bmatrix}, \quad B = \begin{bmatrix}
B_1 \\
0
\end{bmatrix}
\]

where \((A_1, B_1)\) is reachable and \(A_2\) is stable.

The system \((A, B, C, D)\)

\[
x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k + Du_k, \quad k = 0, 1, ...
\]

is said to be **left invertible** if its transfer function has full column normal rank.
Invariant Zeros

We further define the system matrix as the polynomial matrix

\[ S(z) = \begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix} \]

and say that a complex number \( \zeta \) is an invariant zero of the system \( (A, B, C, D) \) if the rank of \( S(\zeta) \) is strictly less than the normal rank of \( S(z) \).
Theorem 2

Suppose that the system \((A, B)\) is stabilizable. Suppose that the system \((A, B, C, D)\) is left invertible and also has no invariant zeros on the unit circle \(|z| = 1\).

Then, there exists a unique LQ regulator gain given by

\[
L = (D^T D + B^T X B)^{-1} (B^T X A + D^T C),
\]

where \(X\) is the largest nonnegative definite solution of the algebraic Riccati equation

\[
X = A^T X A + C^T C
- (B^T X A + D^T C)^T (D^T D + B^T X B)^{-1} (B^T X A + D^T C)
\]
Let the system \((A, B)\) be reachable, with reachability indices \(r_1, r_2, \ldots, r_m\).

Then there exists a similarity transformation \(T\) that brings the matrices \(A\) and \(B\) to the *reachability standard form*,

\[
A' = T A T^{-1}, \quad B' = T B
\]

where \(A'\) is a top-companion matrix with nonzero entries in rows \(r_i, i = 1, 2, \ldots, m\) and \(B'\) has nonzero entries only in rows \(r_i\) and columns \(j \geq i, i = 1, 2, \ldots, m\).
Theorem 3

Suppose that the system \((A, B)\) is reachable, with reachability indices \(r_1 \geq r_2 \geq \ldots \geq r_m\) and with the matrix \(B\) having rank \(m\).

Let \(T\) be a similarity transformation that brings \(A\) and \(B\) to the reachability standard form.

Then, the feedback gain \(L\) that is LQ optimal with respect to \(C = T\) and \(D = 0\) is a deadbeat gain.
**Proof: Existence**

We first show that an LQ regulator gain exists that is optimal with respect to $C = T$ and $D = 0$. Indeed, the system $(A, B)$ is reachable hence stabilizable. The system $(A, B, T, 0)$ has a transfer function whose normal rank is $m$, so it is left invertible. The system matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ T & 0 \end{bmatrix}$$

has rank $n + m$ for all complex numbers $z$, hence $(A, B, T, 0)$ has no invariant zeros at all. The assumptions of Theorem 2 are all satisfied.
Proof: Polynomial Matrix Fractions

Write the transfer function of the system \((A, B)\) in the polynomial matrix fraction form

\[
(zI_n - A)^{-1} B = Q(z)P^{-1}(z)
\]

For any feedback applied to the system, one obtains

\[
[zI_n - (A - BL)]^{-1} B = Q(z)[P(z) + LQ(z)]^{-1}
\]

The system \((A, B)\) being reachable, these polynomial matrix fractions are coprime.

Thus the matrices \(zI_n - (A - BL)\) and \(P(z) + LQ(z)\) have the same invariant factors.
Proof: Matrix Identity

Using the polynomial fraction matrices $P$ and $Q$, the algebraic Riccati equation yields the identity

$$[P(z^{-1}) + LQ(z^{-1})]^T (D^T D + B^T XB)[P(z) + LQ(z)]$$

$$= [CQ(z^{-1}) + DP(z^{-1})]^T [CQ(z) + DP(z)]$$

Define a polynomial matrix $F$ by the equation

$$F^T(z^{-1})F(z) = [CQ(z^{-1}) + DP(z^{-1})]^T [CQ(z) + DP(z)]$$

in such a way that $F^{-1}$ is analytic in the domain $|z| \geq 1$. This matrix $F$ is referred to as the spectral factor.
Bring \((A, B)\) to the *reachability standard form* using the similarity transformation matrix \(T\). The corresponding polynomial fraction matrices are related by

\[
P'(z) = P(z), \quad Q'(z) = TQ(z)
\]

and by Structure Theorem, \(Q'\) has the block-diagonal form

\[
Q'(z) = \text{block-diag} \left[ \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_1-1} \end{bmatrix}, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_2-1} \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_m-1} \end{bmatrix} \right].
\]
Proof: Spectral Factorization

The spectral factorization reads
\[ F^T(z^{-1})F(z) = Q^T(z^{-1})T^TTQ(z) = Q'^T(z^{-1})Q'(z) = \text{diag} [r_1, r_2, \ldots, r_m] \]
so that
\[ F(z) = \text{diag} [\sqrt{r_1} z^{r_1}, \sqrt{r_2} z^{r_2}, \ldots, \sqrt{r_m} z^{r_m}] \]

The matrices \( P(z) + LQ(z) \) and \( zI_n - (A - BL) \) share the same invariant factors \( z^{r_1}, z^{r_2}, \ldots, z^{r_m} \).
Therefore, \( A - BL \) is nilpotent with Jordan structure comprising \( m \) nilpotent blocks of sizes \( r_1, r_2, \ldots, r_m \).
This proves that \( L \) is a deadbeat gain.
Conclusions

- Deadbeat control and LQ regulation, two strategies different in nature, are in fact related.
- A deadbeat control law can be obtained by solving a particular LQ regulator problem, at least for reachable systems.
- The LQ optimal regulator gain is unique, whereas the deadbeat feedback gains are not. Only one deadbeat gain is LQ optimal.
- An alternative construction of such a gain is thus available, solving the Riccati equation.


