Deadbeat Response is ℓ_2 Optimal

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Introduction

A typical linear control strategy
in discrete-time systems,
deadbeat control produces transients
that vanish in finite time.

On the other hand,
the *linear-quadratic control* stabilizes the system
and minimizes the l_2 norm of its transient response.

Quite surprisingly,
it is shown that *deadbeat systems are l_2 optimal*,
at least for reachable systems.

Deadbeat Control

Given a linear system (A, B)

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots$$

where $u_k \in \mathbb{R}^m$ and $x_k \in \mathbb{R}^n$.

The objective of *deadbeat regulation* is to determine a linear state feedback of the form

$$u_k = -Lx_k$$

that drives each initial state x_0 to the origin in a least number of steps.

We define the *reachability subspaces* by

$$R_0 = 0,$$
$$R_k = \text{range}[B \;\; AB \;\; \dots \;\; A^{k-1}B], \quad k = 1, 2, \dots .$$

When $R_n = S^n$, the system (A, B) is said to be *reachable*.

The system (A, B) is said to be *controllable* if there exists a basis in which

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where (A_1, B_1) is reachable and A_2 is nilpotent.

Reachability Indices

For each $k = 1, 2, ...$

let $S_1, S_2, ..., S_k$ be a sequence of matrices

such that

$$\text{range}[BS_1 \;\; ABS_2 \;\; \dots \;\; A^{k-1}BS_k] = \text{range}\, R_k$$

Therefore $S_1, S_2, ..., S_k$ serve to *select a basis* for R_k.

The *reachability indices* $r_1, r_2, ..., r_m$ are defined by

$$r_i = \text{cardinality}\,\{S_j,\; j = 1, 2, \dots : \text{rank}\; S_j \geq i\}$$

Theorem 1

There exists a deadbeat control law
if and only if the system (A, B) is controllable.

Let

$$L_0 = 0,$$
$$L_k = L_{k-1} + L'_k (A - BL_{k-1})^k, \quad k = 1, 2, \ldots$$

where L'_k satisfies

$$L'_k [BS_1 \;\; ABS_2 \;\; \ldots \;\; A^{k-1}BS_k] = [0 \;\; \ldots \;\; 0 \;\; S_k].$$

Then $L = L_n$ is a deadbeat regulator gain.

The closed-loop system matrix $A - BL$ is nilpotent.

Given a linear system (A, B)

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots$$

where $u_k \in S^{m}$ and $x_k \in S^{n}$.

The objective of LQ regulation

is to find a linear state feedback of the form

$$u_k = -Lx_k$$

that stabilizes the closed-loop system

and, for every initial state x_0, minimizes the l_2 norm

of a specified output $y_k \in S^{p}$ of the form

$$y_k = Cx_k + Du_k$$

The system (A, B) is said to be *stabilizable*
if there exists a basis in which

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where (A_1, B_1) is reachable and A_2 is stable.

The system (A, B, C, D)

$$x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k + Du_k, \quad k = 0, 1, \ldots$$

is said to be *left invertible*
if its transfer function has full column normal rank.

Invariant Zeros

We further define the *system matrix* as the polynomial matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix}$$

and say that a complex number ζ is an *invariant zero* of the system (A, B, C, D) if the rank of $S(\zeta)$ is strictly less than the normal rank of $S(z)$.

Theorem 2

Suppose that the system (A, B) is stabilizable.

Suppose that the system (A, B, C, D) is left invertible and also has no invariant zeros on the unit circle $|z| = 1$.

Then, there exists a *unique* LQ regulator gain given by

$$L = (D^T D + B^T XB)^{-1}(B^T XA + D^T C),$$

where X is the largest nonnegative definite solution of the algebraic Riccati equation

$$X = A^T XA + C^T C - (B^T XA + D^T C)^T (D^T D + B^T XB)^{-1}(B^T XA + D^T C)$$

Reachability Standard Form

Let the system (A, B) be reachable,

with reachability indices $r_1, r_2, \ldots, r_m$.

Then there exists a similarity transformation T

that brings the matrices A and B

to the *reachability standard form*,

$$A' = TAT^{-1}, \quad B' = TB$$

where A' is a top-companion matrix

with nonzero entries in rows r_i, $i = 1, 2, \ldots, m$

and B' has nonzero entries

only in rows r_i and columns $j \geq i$, $i = 1, 2, \ldots, m$.

Theorem 3

Suppose that the system (A, B) is reachable,
with reachability indices $r_1 \geq r_2 \geq \ldots \geq r_m$
and with the matrix B having rank m.

Let T be a similarity transformation
that brings A and B to the reachability standard form.

Then, the feedback gain L
that is LQ optimal with respect to $C = T$ and $D = 0$
is a deadbeat gain.

Proof: Existence

We first show that an LQ regulator gain exists that is optimal with respect to $C = T$ and $D = 0$.

Indeed, the system (A, B) is reachable hence stabilizable.

The system $(A, B, T, 0)$ has a transfer function whose normal rank is m, so it is left invertible.

The system matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ T & 0 \end{bmatrix}$$

has rank $n + m$ for all complex numbers z, hence $(A, B, T, 0)$ has no invariant zeros at all.

The assumptions of Theorem 2 *are all satisfied*.

Proof: Polynomial Matrix Fractions

Write the transfer function of the system (*A*, *B*)
in the polynomial matrix fraction form

$$(zI_n - A)^{-1}B = Q(z)P^{-1}(z)$$

For any feedback applied to the system, one obtains

$$[zI_n - (A - BL)]^{-1}B = Q(z)[P(z) + LQ(z)]^{-1}$$

The system (*A*, *B*) being reachable,
these polynomial matrix fractions are coprime.
Thus the matrices $zI_n - (A - BL)$ and $P(z) + LQ(z)$
have the *same invariant factors*.

Proof: Matrix Identity

Using the polynomial fraction matrices P and Q, the algebraic Riccati equation yields the identity

$$[P(z^{-1})+LQ(z^{-1})]^{\mathrm{T}}(D^T D+B^{\mathrm{T}}XB)[P(z)+LQ(z)]$$
$$=[CQ(z^{-1})+DP(z^{-1})]^{\mathrm{T}}[CQ(z)+DP(z)]$$

Define a polynomial matrix F by the equation

$$F^{\mathrm{T}}(z^{-1})F(z)=[CQ(z^{-1})+DP(z^{-1})]^{\mathrm{T}}[CQ(z)+DP(z)]$$

in such a way that F^{-1} is analytic in the domain $|z|\geq 1$.

This matrix F is referred to as the *spectral factor*.

Proof: Reachability Standard Form

Bring (A, B) to the *reachability standard form* using the similarity transformation matrix T.

The corresponding polynomial fraction matrices are related by

$$P'(z) = P(z), \quad Q'(z) = TQ(z)$$

and by Structure Theorem, Q' has the block-diagonal form

$$Q'(z) = \text{block-diag}\left[\begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_1 - 1} \end{bmatrix}, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_2 - 1} \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_m - 1} \end{bmatrix} \right].$$

Proof: Spectral Factorization

The spectral factorization reads

$$F^{\mathrm{T}}(z^{-1})F(z) = Q^{\mathrm{T}}(z^{-1})T^{\mathrm{T}}TQ(z)$$
$$= Q'^{T}(z^{-1})Q'(z) = \mathrm{diag}\,[r_1, r_2, \ldots, r_m]$$

so that

$$F(z) = \mathrm{diag}\,[\sqrt{r_1}z^{r_1}, \sqrt{r_2}z^{r_2}, \ldots, \sqrt{r_m}z^{r_m}].$$

The matrices $P(z) + LQ(z)$ and $zI_n - (A - BL)$ share the same invariant factors $z^{r_1}, z^{r_2}, \ldots, z^{r_m}$.

Therefore, $A - BL$ is *nilpotent* with Jordan structure comprising m nilpotent blocks of sizes $r_1, r_2, \ldots, r_m$.

This proves that L is a deadbeat gain.

Conclusions

- Deadbeat control and LQ regulation,
  two strategies different in nature, are in fact related.
- A deadbeat control law can be obtained
  by solving a particular LQ regulator problem,
  at least for reachable systems.
- The LQ optimal regulator gain is unique,
  whereas the deadbeat feedback gains are not.
  Only one deadbeat gain is LQ optimal.
- An alternative construction of such a gain
  is thus available, solving the Riccati equation.

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