Rotational motion with almost global stability

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Attitude observer for a rigid body

The estimated attitude relative to the true attitude of a rigid body can be described by a matrix $R(t)$ which is orthogonal: $R(t)^T R(t) = I$. The estimate is correct when $R(t) = I$.

Consider an observer with error dynamics

$$\dot{R}(t) = kR(t)[R(t)^T - R(t)] + R(t)E(t)$$

where $E(t)$ represents measurement noise. The condition $E(t) = -E(t)^T$ guarantees that $R(t)$ stays orthogonal.

Today's Theorem

If $\|E(t)\| \leq \epsilon < \sqrt{6}k$ then almost all solutions to

$$\dot{R}(t) = kR(t)[R(t)^T - R(t)] + R(t)E(t)$$

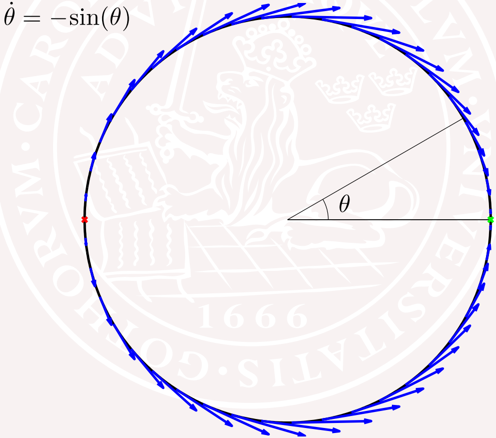
converge towards a ball around the identity matrix I with radius

$$\sqrt{4 - 4\sqrt{1 - \epsilon^2/(8k^2)}}$$

Remark: For $\epsilon = 0$, this proves almost global stability of $R = I$.

Why "almost all"?

For topological reasons, there will always be some trajectories that do not converge:



Outline

- Introduction and main result
- **Lyapunov analysis of the attitude observer**
- Review of density functions
- Deriving the main result using density functions

A Lyapunov Argument for Exact Measurements

The Lyapunov function $V(R) = \frac{1}{2}\|R - I\|^2$ satisfies $V \in [0, 4]$

$$\frac{d}{dt}V(R(t)) = -\frac{k}{2}\|R(t) - R(t)^T\|^2$$

An orthogonal 3×3 matrix $R(t) \neq I$ can be symmetric only if it has two eigenvalues at -1, that is when $V(R)$ takes its maximal value 4.

Hence the Lyapunov function is strictly decreasing once $V < 4$. This proves almost global stability of the equilibrium $R = I$.

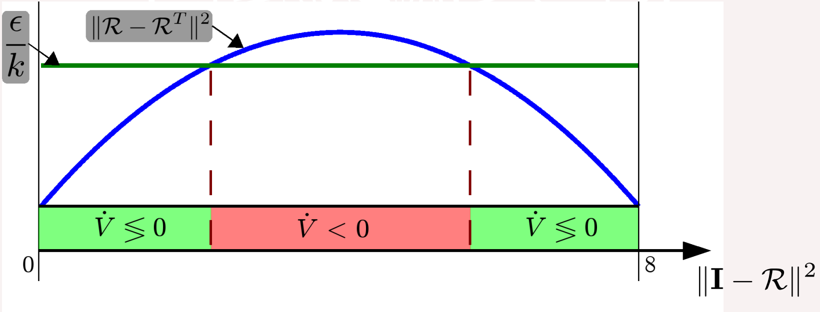
A Lyapunov Argument with Measurement Noise

For $\|E(t)\| \leq \epsilon$ the Lyapunov function satisfies

$$\frac{d}{dt}V(R(t)) \leq -\frac{1}{2}\|R(t) - R(t)^T\| \left(k\|R(t) - R(t)^T\| - \epsilon\right)$$

so V is decreasing except in intervals near $V = 0$ and $V = 4$, where $\|R - R^T\|$ is small.

Those intervals shrink as ϵ tends to zero.



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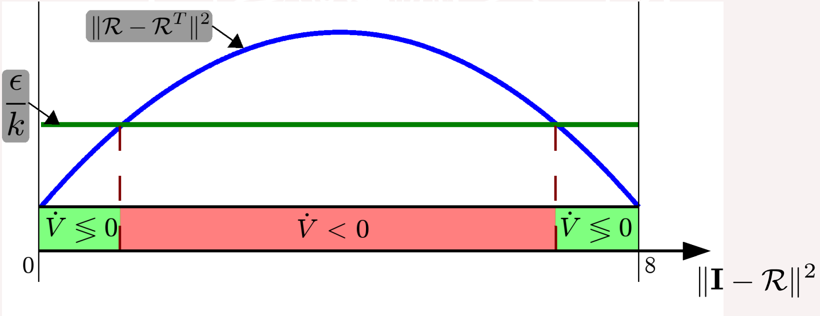

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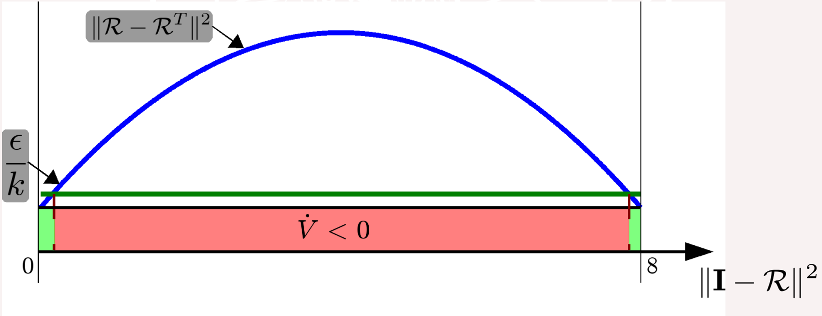

Stability by Cayley Parametrization

It is straightforward to verify that the transformation

$$R = (I + S)(I - S)^{-1} \qquad S = (R + I)^{-1}(R - I)$$

maps orthogonal R into skew-symmetric S and vice versa.

Using this transformation, the noise free observer dynamics

$$\dot{R} = kR(R^T - R)$$

can equivalently be written

$$\dot{S} = -kS$$

This proves $\lim_{t\to\infty} R(t) = I$ except when $R(0) + I$ is singular.

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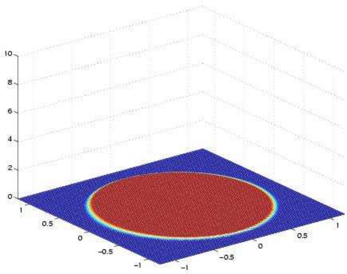
A criterion for almost global attractivity

Let $f \in \mathbf{C}^1(M, TM)$ while $\dot{x} = f(x)$ has a stable equilibrium in $x = 0$ and solutions exist for $t \in [0, \infty]$. Suppose there exists a non-negative $\rho \in \mathbf{C}^1(M \setminus \{0\}, \mathbf{R})$, integrable outside a neighborhood of zero, and

$$[\nabla \cdot (\rho f)](x) > 0 \qquad \text{for almost all } x \neq 0.$$

Then $\lim_{t\to\infty} x(t) = 0$ for almost all initial states $x(0)$.



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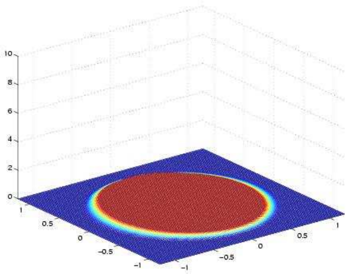

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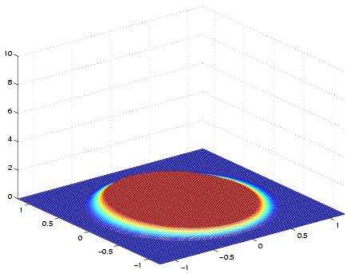

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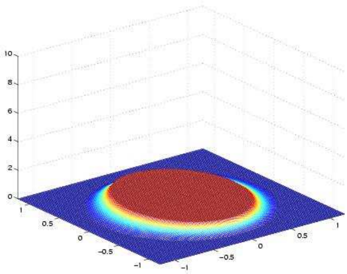

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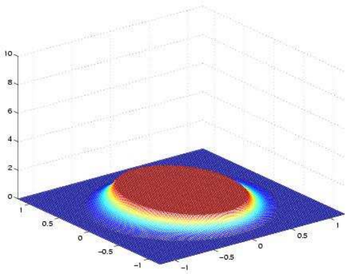

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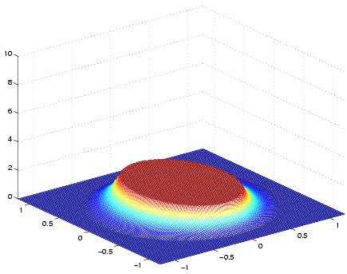

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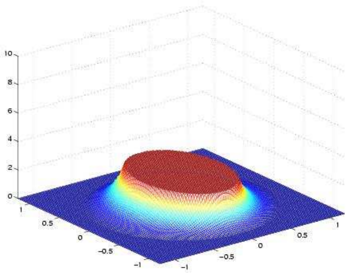

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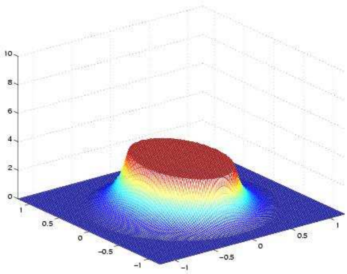

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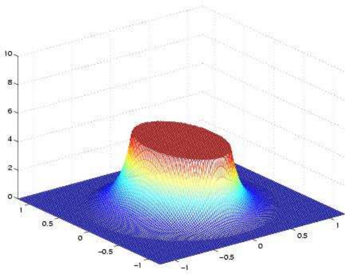

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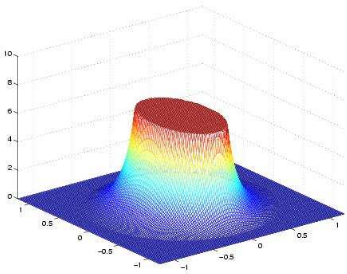

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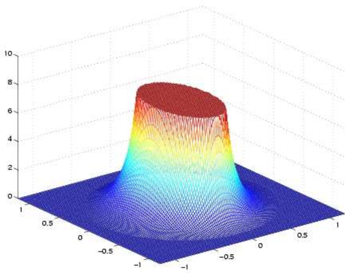

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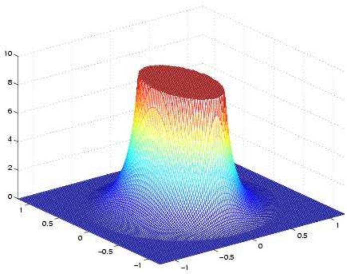

Weakly almost input-to-state stability

Consider $\dot{x} = f(x,u)$ with stable equilibrium in $x = 0$ for $u = 0$ and solutions for $t \in [0, \infty]$. Suppose ρ non-negative, integrable outside a neighborhood of zero, and

$$|x| \geq \gamma(|u|) \Rightarrow \nabla \cdot [\rho(x) f(x,u))] \geq Q(x)$$

with $Q(x) > 0$ for almost all $x \neq 0$. Then

$$\liminf_{t\to\infty} |x(t, x_0, u)| \leq \gamma(\|u\|_\infty)$$

for almost all initial states x_0.

[David Angeli 2004]

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Density functions for observer dynamics

Analyzing $\dot{R} = f(R)$ with the density function ρ where

$$f(R) = kR(R^T - R) \qquad \rho(R) = \frac{1}{\|I - R\|^4}$$

gives

$$\nabla \cdot (\rho f) = \frac{2k}{\|I - R\|^4} > 0$$

so

$$\lim_{t \to \infty} R(t) = I$$

for almost all initial states $R(0)$.

Notice: Strict inequality gives robustness to measurement noise

Density functions with measurement noise

$$f(R,E) = kR(R^T - R) + RE \qquad \rho(R) = \frac{1}{\|I-R\|^4}$$

gives

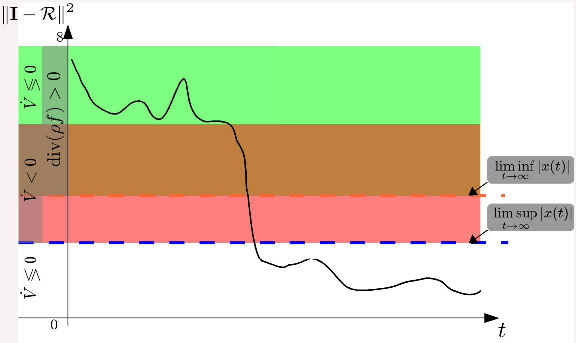
$$\nabla \cdot [\rho(R) f(R,E)] = \frac{2}{\|I-R\|^6}\left(k\|I-R\|^2 + \mathrm{tr}[(R^T - R)E]\right)$$

If $\|E\| \leq \epsilon$, the divergence test with $\gamma(x) = 8x^2/(2k^2 + x^2)$ gives

$$\liminf_{t\to\infty} \|R(t) - I\|^2 < \frac{8\epsilon^2}{2k^2 + \epsilon^2}$$

for almost all initial states $R(0)$. This brings the trajectory within a domain where the Lyapunov argument can be applied.

A picture summarizes the argument



If $\|E(t)\| \leq \epsilon < \sqrt{6}k$, the density function argument proves that almost all trajectories must enter the ball around I with radius

$\sqrt{4+4\sqrt{1-\epsilon^2/(8k^2)}}$ (below the orange dashed line)

Once there, the Lyapunov argument proves convergence to the ball with radius $\sqrt{4-4\sqrt{1-\epsilon^2/(8k^2)}}$ (below the blue line)

Conclusions

- Due to topological constraints in rigid body dynamics, every stability proof of Lyapunov type must involve nonstrict inequalities. Hence, *a Lyapunov argument is not robust*.
- On the contrary, we have shown for a rigid body attitude observer that *a density function argument can quantify the robustness* to measurement noise.